MAT 324: Real Analysis, Fall 2017
Homework Assignment 6

Please read carefully Sections 4.4-4.6 in the textbook, prove all propositions and do all exercises you encounter along the way, and write up clear solutions to the written assignment below. The exams in this class will be based to a large extent on these propositions, exercises, and assignments.

Problem Set 6 (**due in class on Thursday, 10/19**): Problems 1-4 on the next page

Please write your solutions legibly; the TA may disregard solutions that are not readily readable. All solutions must be stapled (no paper clips) and have your name (first name first) and HW number in the upper-right corner of the first page.

Problem 1

Let $(X, \mathcal{F}, \mu)$ be a measure space and $f_n : X \longrightarrow \overline{\mathbb{R}}$ be a sequence of measurable functions converging almost everywhere to a function f. Suppose

$$\limsup_{m \longrightarrow \infty} \left(\int_X \big(\sup_{n \leq m} |f_n| \big) \mathrm{d}\mu \right) < \infty.$$

Show that $\displaystyle\int f \mathrm{d}\mu = \lim_{n \longrightarrow \infty} \left(\int f_n \mathrm{d}\mu \right).$

Problem 2

Let $(X, \mathcal{F}, \mu)$ be a measure space. Suppose $f_1, f_2, \ldots : X \longrightarrow \overline{\mathbb{R}}$ is a sequence of measurable functions converging a.e. to a measurable function f and $g_1, g_2, \ldots : X \longrightarrow \overline{\mathbb{R}}$ is a sequence of integrable functions converging a.e. to an integrable function g such that

$$|f_n| \leq g_n \text{ a.e.} \qquad \text{and} \qquad \int_X g \mathrm{d}\mu = \lim_{n \longrightarrow \infty} \left(\int_X g_n \mathrm{d}\mu \right).$$

Show that

$$\int_X f \mathrm{d}\mu = \lim_{n \longrightarrow \infty} \left(\int_X f_n \mathrm{d}\mu \right).$$

Hint: proof of Theorem 4.26 might be helpful

Problem 3

For each $n \in \mathbb{Z}^+$, define

$$f_n, g_n : [0, \infty) \longrightarrow \mathbb{R}, \qquad f_n(x) = \frac{n^2 x \mathrm{e}^{-n^2 x^2}}{1+x}, \quad g_n(x) = \frac{x \mathrm{e}^{-x^2}}{1+x/n}.$$

(a) Find $\displaystyle\int_0^\infty \big(\lim_{n \longrightarrow \infty} f_n \big) \mathrm{d}x$ and $\displaystyle\int_0^\infty \big(\lim_{n \longrightarrow \infty} g_n \big) \mathrm{d}x$.

(b) Show that

$$\lim_{n \longrightarrow \infty} \left(\int_0^\infty f_n \mathrm{d}x \right) = \lim_{n \longrightarrow \infty} \left(\int_0^\infty g_n \mathrm{d}x \right)$$

and find this limit.

(c) Show that there exists no Lebesgue integrable function $F : [0, \infty) \longrightarrow [0, \infty]$ such that $f_n \leq F$ a.e. on $[0, \infty]$ for all $n \in \mathbb{Z}^+$.

Problem 4

Show that the function

$$f : [0, \infty) \longrightarrow \mathbb{R}, \qquad f(x) = \begin{cases} \frac{\sin x}{x}, & \text{if } x \in \mathbb{R}^+; \\ 1, & \text{if } x = 0; \end{cases}$$

has an improper Riemann integral over $[0, \infty)$, but is not Lebesgue integrable on $[0, \infty)$.
Hint: Exercise 4.10 might be helpful