MAT 324: Real Analysis, Fall 2017
Homework Assignment 3

Please read carefully Sections 3.1-3.4 in the textbook, prove all propositions and do all exercises you encounter along the way, and write up clear solutions to the written assignment below. The exams in this class will be based to a large extent on these propositions, exercises, and assignments.

Problem Set 3 (**due in class on Thursday, 9/28**): Problems 1-5 on the next page

Please write your solutions legibly; the TA may disregard solutions that are not readily readable. All solutions must be stapled (no paper clips) and have your name (first name first) and HW number in the upper-right corner of the first page.

Problem 1

Let $(X,\mathcal{F},\mu)$ be a measure space and $\overline{\mathcal{F}}$ be the completion of $\mathcal{F}$ with respect to μ as defined in Section 2.5. Show that

(a) $\overline{\mathcal{F}} = \big\{E\cup A\colon E\in\mathcal{F},\ A\subset F \text{ for some } F\in\mathcal{F} \text{ with } \mu(F)=0\big\}$;

(b) there exists a unique measure $\overline{\mu}\colon \overline{\mathcal{F}} \longrightarrow [0,\infty]$ so that $\overline{\mu}|_{\mathcal{F}}=\mu$;

(c) the measure space $(X,\overline{\mathcal{F}},\overline{\mu})$ is complete.

Problem 2

Give an example of a *non*-measurable function $f\colon \mathbb{R}\longrightarrow\mathbb{R}$ such that $f^2\colon \mathbb{R}\longrightarrow\mathbb{R}$ is measurable.

Problem 3

Let $(X,\mathcal{F},\mu)$ be a measure space. A function $f\colon X\longrightarrow\mathbb{R}$ is called measurable if $f^{-1}(I)\in\mathcal{F}$ for every interval $I\subset\mathbb{R}$. Show that this condition is equivalent to each of the four conditions (ii)-(v) in Theorem 3.3 in the book with $a\in\mathbb{Q}$ separately.
Note: Theorem 3.3 establishes the equivalence for $(X,\mathcal{F},\mu)=(\mathbb{R},\mathcal{M},m)$ with each the four conditions (ii)-(v) with $a\in\mathbb{R}$. The proof for an arbitrary measure space $(X,\mathcal{F},\mu)$ and $a\in\mathbb{R}$ is identical (replace $\mathcal{M}$ by $\mathcal{F}$ everywhere). So, you only need to check that it is enough to take $a\in\mathbb{Q}$.

Problem 4

Let $A\subset\mathbb{R}$. Show that the function

$$f\colon \mathbb{R}\longrightarrow\mathbb{R}, \qquad f(x)=\begin{cases} x, & \text{if } x\in A;\\ -x, & \text{if } x\notin A;\end{cases}$$

is measurable if and only if $A\in\mathcal{M}$.

Problem 5

Let $(X,\mathcal{F},\mu)$ be a measure space.

(a) Suppose (Y,d) is a metric space, $g: Y\longrightarrow\mathbb{R}$ is a continuous function, and $h: X\longrightarrow Y$ is a function such that $h^{-1}(\mathcal{O})\in\mathcal{F}$ for every open subset $\mathcal{O}\subset Y$. Show that the function $g\circ h\colon X\longrightarrow\mathbb{R}$ is measurable.

(b) Suppose $h_1,\ldots,h_k\colon X\longrightarrow\mathbb{R}$ are measurable functions and $h=(h_1,\ldots,h_k)\colon X\longrightarrow\mathbb{R}^k$. Show that $h^{-1}(\mathcal{O})\in\mathcal{F}$ for every open subset $\mathcal{O}\subset\mathbb{R}^k$.

(c) Show that a function $f\colon X\longrightarrow\mathbb{R}$ is measurable if and only if the function f^3 is measurable.

Hint: proof of Lemma 3.7 might be helpful