MAT 324: Real Analysis, Fall 2017
Homework Assignment 2

This homework assignment covers 1.5 weeks.

Please read carefully Sections 2.3-2.7 and page 302 in the textbook, prove all propositions and do all exercises you encounter along the way, and write up clear solutions to the written assignment below. The exams in this class will be based to a large extent on these propositions, exercises, and assignments.

Problem Set 2 (**due in class on Thursday, 9/21**): Problems 1-6 on the next page

Please write your solutions legibly; the TA may disregard solutions that are not readily readable. All solutions must be stapled (no paper clips) and have your name (first name first) and HW number in the upper-right corner of the first page.

Problem 1

Suppose $A \subset B \subset C \subset \mathbb{R}$ are such that $A, C \in \mathcal{M}$ and $m(A) = m(C) < \infty$. Show that $B \in \mathcal{M}$ and $m(A) = m(B) = m(C)$.

Problem 2

Let $E_1, E_2, \ldots$ be disjoint measurable sets and $A \subset \mathbb{R}$ be any subset. Show that

$$m^*\left(A \cap \bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} m^*\big(A \cap E_n\big).$$

Problem 3

Let $E_1, E_2, \ldots, E_{20} \subset [0,1]$ be measurable subsets. Show that

$$m\left(\bigcap_{n=1}^{20} E_n\right) \geq \sum_{n=1}^{20} m(E_n) - 19.$$

Problem 4

Show that there exist $A, B \subset \mathbb{R}$ such that

$$A \cap B = \emptyset \qquad \text{and} \qquad m^*(A \cup B) < m^*(A) + m^*(B).$$

Hint: p302 might be helpful

Problem 5

Let $\ell_1, \ell_2, \ldots \in (0,1)$ be a sequence such that $\sum_{n=1}^{\infty} 2^{n-1} \ell_n < 1$. Starting with $C_0 \equiv [0,1]$, let $C_n \subset [0,1]$ for $n \in \mathbb{Z}^+$ be the subset obtained from C_{n-1} by removing the open middle interval of length ℓ_n from each of the 2^{n-1} disjoint closed intervals making up C_{n-1}. Show that

$$C \equiv \bigcap_{n=1}^{\infty} C_n \subset [0,1]$$

is a closed Borel subset. Find its measure.

Problem 6

For $X \subset \mathbb{R}$, let

$$\mathcal{M}_X = \big\{ E \cap X : E \in \mathcal{M} \big\}, \qquad \mu_X = m^*\big|_{\mathcal{M}_X}.$$

(a) Show that $(X, \mathcal{M}_X, \mu_X)$ is a complete measure space if $X \subset \mathbb{R}$ is measurable.

(b) Which properties of a complete measure space $(X, \mathcal{M}_X, \mu_X)$ may not satisfy if X is not assumed to be measurable? Give an example.