MAT 324: Real Analysis, Fall 2017
Homework Assignment 10

This homework assignment covers 1.5 weeks.

Please read carefully Sections 6.3-6.6 in the textbook and pp164-170 in Rudin's book (see website), prove all propositions and do all exercises you encounter along the way, and write up clear solutions to the written assignment below.

Problem Set 10 (**due in class on Thursday, 11/30**): Problems 1-5 below and on next page

Please write your solutions legibly; the TA may disregard solutions that are not readily readable. All solutions must be stapled (no paper clips) and have your name (first name first) and HW number in the upper-right corner of the first page.

Problem 1

Suppose X is a set and $\mu^*\colon 2^X \longrightarrow [0,\infty]$ is a function such that

$$\mu^*(\emptyset)=0, \quad \mu^*(A)\le\mu^*(B) \text{ if } A\subset B, \quad \text{and} \quad \mu^*\bigg(\bigcup_{n=1}^{\infty}A_n\bigg) \le \sum_{n=1}^{\infty}\mu^*(A_n) \;\; \forall\, A_1,A_2,\ldots\subset X. \tag{1}$$

Define

$$\mathcal{M}_{\mu^*} = \big\{E\subset X\colon \mu^*(A)=\mu^*(A\cap E)+\mu^*(A\cap E^c)\ \forall\, A\subset X\big\}.$$

Show that

(a) $\mu^*(A\cup B)\le\mu^*(A)+\mu^*(B)$ for all $A,B\subset X$;

(b) $X\in\mathcal{M}_{\mu^*}$, $E^c\in\mathcal{M}_{\mu^*}$ if $E\in\mathcal{M}_{\mu^*}$, and $E\in\mathcal{M}_{\mu^*}$ if $\mu^*(E)=0$;

(c) if $E,F\in\mathcal{M}_{\mu^*}$ and $A\subset X$, then

$$\begin{aligned}\mu^*(A) &= \mu^*\big(A\cap E\cap F\big)+\mu^*\big(A\cap E\cap F^c\big)+\mu^*\big(A\cap E^c\cap F\big)+\mu^*\big(A\cap E^c\cap F^c\big)\\ &\ge \mu^*\big(A\cap(E\cup F)\big)+\mu^*\big(A\cap(E\cup F)^c\big);\end{aligned}$$

(d) $\mathcal{M}_{\mu^*}$ is a field (not σ-field yet) on X and

$$\mu^*\bigg(\bigcup_{n=1}^{k}E_n\bigg) = \sum_{n=1}^{k}\mu^*(E_n) \qquad \forall\ E_1,\ldots,E_k\in\mathcal{M}_{\mu^*} \text{ s.t. } E_n\cap E_{n'}=\emptyset\ \forall\, n\neq n';$$

(e) if $E_1,E_2,\ldots\in\mathcal{M}_{\mu^*}$ and $E_n\cap E_{n'}=\emptyset$ for all $n\neq n'$, then

$$\mu^*\bigg(A\cap\bigcup_{n=1}^{\infty}E_n\bigg) = \sum_{n=1}^{\infty}\mu^*(A\cap E_n)\ \ \forall\ A\subset X, \qquad \mu^*\bigg(\bigcup_{n=1}^{\infty}E_n\bigg) = \sum_{n=1}^{\infty}\mu^*(E_n);$$

(f) $(X,\mathcal{M}_{\mu^*},\mu\equiv\mu^*|_{\mathcal{M}_{\mu^*}})$ is a complete measure space.

Hint: parts of proof of Theorem 2.11 might be helpful for some of the above, in particular for (e).

Problem 2

Let X be a set, $\mathcal{A}\subset 2^X$, and $\ell\colon \mathcal{A}\longrightarrow[0,\infty]$ be a function such that $\emptyset\in\mathcal{A}$ and $\ell(\emptyset)=0$. For each $A\subset X$, define

$$Z_\ell(A)=\Big\{\sum_{n=1}^{\infty}\ell(I_n)\colon I_1,I_2,\ldots\in\mathcal{A},\ A\subset\bigcup_{n=1}^{\infty}I_n\Big\}\subset[0,\infty],\quad \mu_\ell^*(A)=\inf Z_\ell(A)\in[0,\infty].$$

(a) Show that $\mu^*\equiv\mu_\ell^*$ satisfies (1).

(b) Suppose in addition that $\mathcal{A}$ is a field (not necessarily σ-field) on X and

$$\ell\bigg(\bigcup_{n=1}^{\infty}I_n\bigg)=\sum_{n=1}^{\infty}\ell(I_n)\quad\forall\ I_1,I_2,\ldots\in\mathcal{A}\ \text{ s.t. }\ I_n\cap I_{n'}=\emptyset\ \forall\, n\neq n'\ \text{and}\ \bigcup_{n=1}^{\infty}I_n\in\mathcal{A}\,.$$

Show that $\mu_\ell^*|_{\mathcal{A}}=\ell$ and $\mathcal{A}\subset\mathcal{M}_{\mu_\ell^*}$.

(c) What should $\mathcal{A}$ and ℓ be taken to construct the Lebesgue measure m_n on $\mathbb{R}^n$? Justify your answer.

Problem 3

Let $(X,\mathcal{F}_1,\mu_1)$ and $(Y,\mathcal{F}_2,\mu_2)$ be σ-finite measure spaces and $\sigma(\mathcal{F}_1\times\mathcal{F}_2)$ be the σ-field generated by

$$\mathcal{F}_1\times\mathcal{F}_2\equiv\big\{A\times B\colon A\in\mathcal{F}_1,\ A\in\mathcal{F}_2\big\}.$$

Suppose that μ is a measure on $\sigma(\mathcal{F}_1\times\mathcal{F}_2)$ such that

$$\mu(A\times B)=\mu_1(A)\mu_2(B)\qquad\forall\ A\times B\in\mathcal{F}_1\times\mathcal{F}_2.$$

Show that μ is the product measure $\mu_1\times\mu_2$ on $\sigma(\mathcal{F}_1\times\mathcal{F}_2)$.

Problem 4

Let $\mathbb{I}=[0,1]$, $(\mathbb{I},\mathcal{M}_{\mathbb{I}},m_{\mathbb{I}})$ be the usual Lebesgue measure space, and $(\mathbb{I},2^{\mathbb{I}},\mu)$ be the measure space so that μ is the counting measure. Let

$$E=\big\{(x,x)\colon x\in\mathbb{I}\big\}\subset\mathbb{I}\times\mathbb{I}$$

be the diagonal. Show that

$$E\in\sigma\big(\mathcal{M}_{\mathbb{I}}\times 2^{\mathbb{I}}\big),\qquad \varphi_E(x)\equiv\mu(E_x)=1\ \ \forall\,x\in\mathbb{I},\qquad \psi_E(y)\equiv m_{\mathbb{I}}(E^y)=0\ \ \forall\,y\in\mathbb{I}.$$

Why doesn't this contradict equation (6.3) in the book?

Problem 5

Let $\mathbb{I}=[0,1]$ and $(\mathbb{I},\mathcal{M}_{\mathbb{I}},m_{\mathbb{I}})$ be the usual Lebesgue measure space. For each $n\in\mathbb{Z}^+$, let $f_n\colon\mathbb{I}\longrightarrow\mathbb{R}$ be a continuous function such that

$$f_n(x)=0\ \ \forall\,x\notin\big[2^{-n},2^{-n+1}\big],\qquad \int_0^1 f_n(x)\,\mathrm{d}x=1.$$

Show that

(a) the sum $f(x,y)=\sum\limits_{n=1}^{\infty}\big(f_n(x)-f_{n+1}(x)\big)f_n(y)$ converges for all $(x,y)\in\mathbb{I}^2$ and the function $f\colon\mathbb{I}^2\longrightarrow\mathbb{R}$ is continuous except at $(0,0)$;

(b) $\displaystyle\int_0^1\bigg(\int_0^1 f(x,y)\,\mathrm{d}y\bigg)\mathrm{d}x=1$ and $\displaystyle\int_0^1 f(x,y)\,\mathrm{d}x=0$ for all $y\in\mathbb{I}$. Why doesn't this contradict any of the Fubini theorems?