21.11 Theorem.
No nondegenerate interval I in $\mathbb{R}$ can be written as the union of two or more disjoint nondegenerate closed intervals. (An interval is "nondegenerate" if it has more than one point.)

We first note that I cannot be written as the union of finitely many disjoint nondegenerate closed intervals,

$$I = [a_1, b_1] \sqcup [a_2, b_2] \sqcup \ldots \sqcup [a_k, a_k], \tag{1}$$

with $k \geq 2$. If it were possible to do so, we could assume that

$$a_1 < b_1 < a_2 < b_2 < \ldots < a_k < b_k.$$

The open interval interval (b_1, a_2) would then be missing from the right side of (1), and so this union would not be an interval. The same reasoning would apply if the first interval in (1) were of the form $(-\infty, b_1]$ or the last were of the form $[a_k, +\infty)$.

It remains to consider a decomposition of I into infinitely many disjoint nondegenerate closed intervals. Since every such interval contains a rational number, the number of these intervals must be countable. Thus, suppose that

$$I = \bigsqcup_{n \in \mathbb{N}} [a_n, b_n]\,. \tag{2}$$

Let $a = \inf I \in \mathbb{R} \sqcup \{-\infty\}$ and $b = \sup I \in \mathbb{R} \sqcup \{+\infty\}$. If $a \in I$, then $a_m = a$ for some $m \in \mathbb{N}$. Dropping the interval $[a_m, b_m]$ from both sides of (2), we would get a similar decomposition for an interval without the left endpoint. By the same reasoning, we can drop the right endpoint from I. We can thus assume that $a, b \notin I$, i.e. I is open in $\mathbb{R}$.

Let E be the set of the endpoints a_n, b_n of all intervals in (2) along with a if $a \in \mathbb{R}$ and b if $b \in \mathbb{R}$. Since

$$E = \mathbb{R} - (-\infty, a) - (b, +\infty) - \bigcup_{n=1}^{\infty} (a_n, b_n),$$

the subset E is closed in $\mathbb{R}$. We show that it is also perfect, i.e. for every $x \in E$ and $\delta \in \mathbb{R}^+$ the open ball $(x - \delta, x + \delta)$ around x in $\mathbb{R}$ contains a point $x' \in E$ different from x. Since $x \in I$ and $I \subset \mathbb{R}$ is open,

$$I \cap (x - \delta, x), I \cap (x, x + \delta) \neq \emptyset.$$

By (2), there thus exist $k, m \in \mathbb{N}$ such that

$$[a_k, b_k] \cap (x - \delta, x), [a_m, b_m] \cap (x, x + \delta) \neq \emptyset.$$

If $x = a_n$ for some n, then $x \notin [a_k, b_k]$ and thus $x' \equiv b_k \in E \cap (x - \delta, x)$. If $x = b_n$ for some n, then $x \notin [a_m, b_m]$ and thus $x' \equiv a_m \in E \cap (x, x + \delta)$. In either case, we find a point $x' \in E \cap (x - \delta, x + \delta)$ different from x. The same reasoning applies to decompositions as in (2) containing an interval form $(-\infty, b_n]$ or an interval of the form $[a_n, +\infty)$.

In summary, we find that $E \subset \mathbb{R}$ is a perfect subset. Since E is countable, this contradicts the conclusion of Discussion 21.10 and so a decomposition (2) does not exist.

21.11 Theorem.
No interval I in $\mathbb{R}$ can be written as the union of two or more disjoint nondegenerate closed intervals. (An interval is "nondegenerate" if it has more than one point.)

A closed nondegenerate interval is all of $\mathbb{R}$, or a closed half-infinite interval $(-\infty, b]$ with $b \in \mathbb{R}$ or $[a, +\infty)$ with $a \in \mathbb{R}$, or a closed bounded interval with $[a, b]$ with $a, b \in \mathbb{R}$ and $a < b$. Suppose there is a decomposition of $\mathbb{R}$ of the form

$$\mathbb{R} = (-\infty, b] \sqcup [a, +\infty) \bigsqcup_{\alpha \in \mathcal{A}} [a_\alpha, b_\alpha] \tag{1}$$

for some indexing set $\mathcal{A}$ and $a, b, a_\alpha, b_\alpha \in \mathbb{R}$ with $a_\alpha < b_\alpha$. Since every interval $[a_\alpha, b_\alpha]$ contains a rational number, the set $\mathcal{A}$ is at most countable. Let E be the set of the endpoints a, b, a_α, b_α of all intervals in (1). Since

$$E = \mathbb{R} - (-\infty, b) - (a, +\infty) - \bigcup_{\alpha \in \mathcal{A}} (a_\alpha, b_\alpha),$$

the subset E is closed in $\mathbb{R}$. We show in the next paragraph that E is also perfect. Since E is nonempty, but at most countable, this contradicts the conclusion of Discussion 21.10, and so a decomposition (1) does not exist.

We now show that the set E of endpoints of the intervals in (1) is perfect, i.e. for every $x \in E$ and $\delta \in \mathbb{R}^+$ the open ball $(x-\delta, x+\delta)$ around x in $\mathbb{R}$ contains a point $x' \in E$ different from x. By (1), there exist $\beta, \gamma \in \mathcal{A}$ such that

$$[a_\beta, b_\beta] \cap (x-\delta, x), [a_\gamma, b_\gamma] \cap (x, x+\delta) \neq \emptyset.$$

If x is a left endpoint, i.e. $x = a$ or $x = a_\alpha$ for some $\alpha \in \mathcal{A}$, then $x \notin [a_\beta, b_\beta]$ and thus

$$x' \equiv b_\beta \in E \cap (x-\delta, x).$$

If x is a right endpoint, i.e. $x = b$ or $x = b_\alpha$ for some $\alpha \in \mathcal{A}$, then $x \notin [a_\gamma, b_\gamma]$ and thus

$$x' \equiv a_\gamma \in E \cap (x, x+\delta).$$

In either case, we find a point $x' \in E \cap (x-\delta, x+\delta)$ different from x.

Suppose I is any interval decomposed as the union of two or more disjoint nondegenerate closed intervals. Let b_β be the right endpoint of an interval in the decomposition and a_γ be the left endpoint of another interval in the decomposition so that $b_\beta < a_\gamma$. The intervals $[a_\alpha, b_\alpha]$ in the decomposition of I so that $a_\alpha, b_\alpha \in (b_\beta, a_\gamma)$ then decompose the open interval (b_β, a_γ) into disjoint closed intervals. Adding $(-\infty, b_\beta]$ and $[a_\alpha, +\infty)$ to this decomposition of (b_β, a_γ), we obtain a decomposition of $\mathbb{R}$ as in (1). Since the latter does not exist, we conclude that no interval I in $\mathbb{R}$ can be written as the union of two or more disjoint nondegenerate closed intervals.