POISSON INTEGRALS AND SEMISIMPLE GROUPS*

By

A. W. KNAPP AND R. E. WILLIAMSON

in Ithaca, N.Y., U.S.A. *in Hanover, N.H., U.S.A.*

1. Introduction. Our intention is to treat some classical theorems concerning harmonic functions in a setting in which the theorems can serve as an analytical tool for dealing with semisimple Lie groups.

A wide class of harmonic functions in the unit disc is obtained by means of the Poisson integral formula $h(re^{ix}) = \int P(re^{ix}, y)d\mu(y)$, where $d\mu(y)$ is a signed measure on the circle and where $P(re^{ix}, y)$ is the kernel

$$P(re^{ix}, y) = \frac{1 - r^2}{1 - 2r\cos(x - y) + r^2}.$$

For a given harmonic function h, such a measure $d\mu$ exists if and only if the functions h_r defined by $h_r(e^{ix}) = h(re^{ix})$ satisfy $\sup_{0<r<1} \| h_r \|_1 < \infty$. In addition, the measure $d\mu$ is of the form $f(x)\frac{1}{2\pi}dx$ with $f(x)$ a function of class L^p, $p > 1$, if and only if $\sup \| h_r \|_p < \infty$.

When the condition $\sup \| h_r \|_p < \infty$ is satisfied for some $p \geqq 1$, the measure $d\mu$ is assumed as boundary value in several senses: There are senses involving norm or weak-* convergence that follow directly from the fact that the Poisson kernel is an approximate identity. A deeper result is Fatou's theorem, which asserts convergence of h_r to a boundary function pointwise along almost every radius, or even almost everywhere non-tangentially.

Similar considerations apply to harmonic functions in the upper half-plane which we can identify with the disc by means of a linear fractional transforma-

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tion. The integral formula in the disc transforms to a formula in the half-plane with kernel

$$P(x + iy, t) = \frac{y}{(x-t)^2 + y^2},$$

provided we absorb a constant multiple of $1 + t^2$ into the image of the measure. This time we let $h_y(x) = h(x + iy)$, and the question that decides whether h is represented by a boundary function or measure is the question whether $\sup_{0<y<\infty} \| h_y \|_p < \infty$. If so, then there are senses (similar to those in the disc) in which the h_y converge to the boundary values. Fatou's theorem in this case concerns convergence to the real axis along verticals or non-tangentially.

The connection between the above situations and semisimple Lie groups is as follows: The semisimple group $G = SL(2, \mathbb{R})$ of 2-by-2 real matrices of determinant one operates transitively on the disc or half-plane by linear fractional transformations, and the isotropy subgroup is a maximal compact subgroup K. The harmonic functions in the disc or half-plane are exactly those functions on G/K that are annihilated by every G-invariant (linear) differential operator on G/K without constant term. The unit circle, the real line, the Poisson kernels, and the coordinate systems that we have used (polar in the disc, Cartesian in the half-plane) can all be given similar interpretations in terms of G. We can therefore ask whether the theorems that hold for $SL(2, \mathbb{R})$ remain valid for all semisimple groups.

The theorems about existence of functions or measures on the boundary we shall settle completely. These results are fairly simple consequences of a theorem of Furstenberg [2] concerning the representation of bounded harmonic functions as Poisson integrals. All that is needed is an appropriate passage to the limit.

Our main result, Theorem 4.1, is a generalization of Fatou's theorem. It is concerned with the analog of almost-everywhere convergence in the upper half-plane, and in order to distinguish it from some related theorems (see [9], [10], [15] and references listed in those papers), we shall mention what the result is for $G = SL(2, \mathbb{R}) \oplus SL(2, \mathbb{R})$. In this case, G/K is the product of two half-planes, and the Poisson kernel is the product of the kernels that go with each half-plane. Poisson integrals are thus defined for points $(x_1 + iy_1, x_2 + iy_2)$ in G/K, where $y_1 > 0$ and $y_2 > 0$. The mode of convergence in [10] for this group is that y_1 and y_2 tend to 0 in such a way that one of them is a fixed power of the other. Our present theorem will deal with bounded functions and will allow y_1 and y_2 to tend to 0 without any further restriction on them. As will be evident in §4, the present theorem for this group G is closely related to strong differentiation of double integrals in $\mathbb{R}^2$, in the sense of Jessen, Marcinkiewicz, and Zygmund [8].

In connection with the theorem of §4, we are very much indebted to A. Korányi for showing us the results of [10] before their publication.

2. Boundaries, Poisson kernels, and norm convergence.

We collect in this section a number of simple facts about harmonic functions that will be used in §3 and §4. Some of these were implicit in [7].

Our notation is as follows. We let $G = KAN$ be an Iwasawa decomposition of the connected semisimple Lie group G; since none of our results will depend on the center of G, we shall assume that the center of G is finite and hence that K is compact. If $g = kan$, we write $k(g)$ for k and we write $H(g)$ for the logarithm of a in the Lie algebra of A. Let θ be the corresponding Cartan involution of G, or of the Lie algebra of G. A typical restricted root is denoted λ, and the corresponding restricted-root space in the Lie algebra of G is $\mathfrak{g}_\lambda$; 2ρ is the sum of the positive restricted roots. If $\mathfrak{n}$ denotes the Lie algebra of N, we have

$$\mathfrak{n} = \sum_{\lambda > 0} \mathfrak{g}_\lambda \quad \text{and} \quad \theta\mathfrak{n} = \sum_{\lambda > 0} \mathfrak{g}_{-\lambda}.$$

We write M for the centralizer of A in K. Typical elements of K, A, and θN will be denoted k, a, and $\bar{n}$; dk will be normalized Haar measure on K. Let $D(G)$ be the set of left G-invariant differential operators on G, and let $D(G/K)$ be the set of left-invariant differential operators on G/K.

Lemma 2.1. *Let h be a C^∞ function on G, let $d\mu$ be a signed measure of compact support on G, and put*

$$d\mu * h(g_0) = \int_G h(g^{-1}g_0)d\mu(g).$$

Then h is in C^∞ and $D(d\mu * h) = d\mu * Dh$ for every $D \in D(G)$.

By [6, pp. 98, 391], it suffices to prove that $X(d\mu * h) = d\mu * Xh$ for every left-invariant vector field X. The proof of the lemma thereby becomes a straightforward real-variables problem.

A C^∞ function h on G/K is said to be *harmonic* if $Dh = 0$ for every $D \in D(G/K)$ satisfying $D1 = 0$. In dealing with harmonic functions, we shall consider two spaces as boundaries for G/K, namely θN and K/M. If we think of G/K as a generalization of the upper half-plane or unit disc, we can think of θN and K/M as generalizations of the real line and unit circle, respectively. There is a map between these boundaries that generalizes the Cayley transform. It is the map $\gamma: \theta N \to K/M$ given by $\gamma(\bar{n}) = k(\bar{n})M$. Harish-Chandra [5, pp. 284–287] showed that γ is a diffeomorphism of θN onto an open set in K/M whose complement is of lower dimension (and consequently of dk-measure 0). Moreover, he showed that the corresponding charge of variables formula is

$$(1) \qquad \int_{\theta N} f(\bar{n})e^{-2\rho H(\bar{n})}d\bar{n} = \int_{K/M} f(\gamma^{-1}(kM))dk$$

for a suitable normalization of Haar measure $d\bar{n}$. We shall assume from now on that $d\bar{n}$ is so normalized.

From (1) we see that $e^{-2\rho H(\bar{n})}$ is integrable (with integral 1). Harish-Chandra [5, p. 287] further showed that $e^{-2\rho H(\bar{n})} \leqq 1$ everywhere. (See also Proposition 5.1 below.)

Relative to each of the boundaries θN and K/M, Poisson kernels are defined by

$$(2) \qquad P(gK, \bar{n}) = e^{-2\rho H(g^{-1}\bar{n})} \quad \text{and} \quad P(gK, kM) = e^{-2\rho H(g^{-1}k)}.$$

Notice that $P(K, \bar{n}) = e^{-2\rho H(\bar{n})}$.

Lemma 2.2. *Let $1 \leqq q \leqq \infty$ and let $g \in G$. Then $P(gK, \bar{n})$ is in $L^q(\theta N)$ and is in $C_0(\theta N)$, the space of continuous functions vanishing at infinity. Moreover,*

$$\int_{\theta N} P(gK, \bar{n})d\bar{n} = \int_{K/M} P(gK, kM)dk = 1. \tag{3}$$

The proof that $P(gK, \bar{n})$ vanishes at infinity is quite technical, and so the proof of Lemma 2.2 is postponed to §5.

From Lemma 2.2 it follows that we can use $P(gK, \bar{n})$ or $P(gK, kM)$ as kernel to define the Poisson integral of any L^p function, $1 \leqq p \leqq \infty$, or of any finite signed measure on θN or K/M. For example, the Poisson integral of a function f on θN is the function h on G/K given by

$$h(gK) = \int_{\theta N} P(gK, \bar{n})f(\bar{n})d\bar{n}.$$

We shall occasionally write $PI(\cdot)$ for Poisson integrals.

If $f(\bar{n})$ is a function in some L^p class on θN and f_1 is the function on K/M defined to be $f \circ \gamma^{-1}$ on $\gamma(\theta N)$, then formula (1) and the fact that $e^{-2\rho H(\bar{n})} \in L^1 \cap L^\infty$ show that f_1 is integrable on K/M. Moreover, (1) shows that the Poisson integrals of f and f_1 are the same. Similar statements apply to signed measures, and therefore *every Poisson integral defined relative to θN is a Poisson integral defined relative to K/M.*

Godement [4] proved that harmonic functions are characterized by a mean value property: A continuous function h on G/K is harmonic if and only if when h is lifted to a function h' on G (defined by $h'(g) = h(gK)$), then h' satisfies

$$h'(g) = \int_K h'(gkg')dk$$

or all g and g' in G.

Proposition 2.3. *Every Poisson integral is harmonic. In particular, $P(gK, \bar{n})$ and $P(gK, kM)$ are harmonic for each fixed $\bar{n}$ and k.*

Proof. Since Poisson integrals over θN are Poisson integrals over K/M, it is enough to prove that Poisson integrals over K/M are harmonic. By Lemma 2.2 of [6, p. 390] and by Lemma 2.1 above, it is enough to prove that $P(gK, k_0M)$ is harmonic for fixed k_0. (See the proof of Lemma 3.1 for a similar argument.) From the identity

$$H(g_1 g_2) = H(g_1 k(g_2)) + H(g_2)$$

with $g_1 = h^{-1}k^{-1}$ and $g_2 = g^{-1}k_0$, we see that

$$\int_K P(gkhK, k_0 M)dk = P(gK, k_0 M) \int_K \exp[-2\rho H(h^{-1}k^{-1}k(g^{-1}k_0))]dk.$$

With a change of variables, the integral on the right reduces to $\int_K P(h, kM)dk$, or to $\int_{K/M} P(h, kM)dk$, which is 1 by Lemma 2.2. Hence the lift of $P(gK, k_0 M)$ satisfies the mean value property, and $P(gK, k_0 M)$ is harmonic.

The rest of this section will deal with elementary convergence properties of Poisson integrals. The convergence will be as $a \in A$ tends to ∞ out the exponential of the positive Weyl chamber. More specifically, we use the notation $\lim_{a \to \infty}$ to refer to convergence as $a \in A$ tends to ∞ in such a way that $\lambda(H(a)) \to +\infty$ for every positive restricted root λ. Limits of the form $\lim_{a \to \infty} h(kaK)$, with $k \in K$, generalize radial limits in the unit disc, and limits of the form $\lim_{a \to \infty} h(\bar{n}aK)$, with $\bar{n} \in \theta N$, generalize vertical limits in the upper half-plane. The next lemma was implicit in [7].

Lemma 2.4. *If f is bounded on θN and continuous at $\bar{n}_0$ and if h is the Poisson integral of f, then $\lim_{a \to \infty} h(\bar{n}_0 aK) = f(\bar{n}_0)$.*

Proof. We have

$$h(\bar{n}_0 aK) = \int_{\theta N} P(aK, \bar{n}) f(\bar{n}_0 \bar{n}) d\bar{n} = \int_{\theta N} e^{-2\rho H(a^{-1}\bar{n}aa^{-1})} f(\bar{n}_0 \bar{n}) d\bar{n}$$

$$= \int_{\theta N} e^{-2\rho H(\bar{n}' a^{-1})} f(\bar{n}_0 a \bar{n}' a^{-1}) e^{-2\rho H(a)} d\bar{n}'$$

under the change of variables $a^{-1}\bar{n}a = \bar{n}'$ and $d\bar{n} = e^{-2\rho H(a)} d\bar{n}'$. Hence

$$h(\bar{n}_0 aK) = \int_{\theta N} e^{-2\rho H(\bar{n})} f(\bar{n}_0 a \bar{n} a^{-1}) d\bar{n}.$$

As $a \to \infty$, $a\bar{n}a^{-1} \to e$. Hence we have dominated convergence, with the continuity of f at $\bar{n}_0$ giving

$$\lim_{a \to \infty} h(\bar{n}_0 aK) = f(\bar{n}_0) \int_{\theta N} e^{-2\rho H(\bar{n})} d\bar{n} = f(\bar{n}_0).$$

Proposition 2.5. *The Poisson kernel $P(aK, \bar{n})$, for $a \in A$, satisfies*

(a) $P(aK, \bar{n}) \geqq 0$

(b) $\int_{\theta N} P(aK, \bar{n}) d\bar{n} = 1$

(c) *If U is any open neighborhood of e in θN, then* $\lim_{a \to \infty} \int_{\theta N - U} P(aK, \bar{n}) d\bar{n} = 0.$

Proof. (a) is clear from the definition, and (b) is a special case of formula (3). For (c), let f be the characteristic function of $\theta N - U$ and apply Lemma 2.4 with $\bar{n}_0 = e$.

Proposition 2.5′. *The Poisson kernel $P(aK, kM)$, for $a \in A$, satisfies*

(a) $P(aK, kM) \geqq 0$

(b) $\int_{K/M} P(aK, kM) dk = 1$

(c) *If U is any open neighborhood of eM in K/M, then* $\lim_{a \to \infty} \int_{K/M - U} P(aK, kM) dk = 0.$

Proof. (a) is clear from the definition, and (b) is a special case of formula (3). Result (c) follows by applying Proposition 2.5 to $\gamma^{-1}(U)$ and then using formula (1) to transform to K/M.

Proposition 2.6. *Let $f(\bar{n})$ be a function on θN, let $h(\bar{n}aK)$ be its Poisson integral, and let $h_a(\bar{n}) = h(\bar{n}aK)$.*

(a) *If* $1 \leqq p < \infty$ *and if* f *is in* L^p, *then* $\lim_{a\to\infty} h_a = f$ in L^p.

(b) *If* f *is in* L^∞, *then* $\lim_{a\to\infty} h_a = f$ *weak-∗ against* L^1.

(c) *If* f *is bounded on* θN *and uniformly continuous on* $E \subseteq \theta N$, *then* $\lim_{a\to\infty} h_a = f$ *uniformly on* E.

(d) *If* f *is replaced by a finite signed measure* $d\mu$, *then*

$$\lim_{a\to\infty} h_a(\bar{n})d\bar{n} = d\mu(\bar{n}) \text{ weak-}* \text{ against } C_0(\theta N).$$

Proof. This is a routine consequence of Proposition 2.5.

Proposition 2.6′. *Let* $f(kM)$ *be a function on* K/M, *let* $h(kaK)$ *be its Poisson integral, and let* $h_a(kM) = h(kaK)$.

(a) *If* $1 \leqq p < \infty$ *and if* f *is in* L^p, *then* $\lim_{a\to\infty} h_a = f$ *in* L^p.

(b) *If* f *is in* L^∞, *then* $\lim_{a\to\infty} h_a = f$ *weak-∗ against* L^1.

(c) *If* f *is bounded on* K/M *and uniformly continuous on* $E \subseteq K/M$, *then* $\lim_{a\to\infty} h_a = f$ *uniformly on* E.

(d) *If* f *is replaced by a finite signed measure* $d\mu$, *then* $\lim_{a\to\infty} h_a(kM)dk = d\mu(kM)$ *weak-∗ against* $C(K/M)$.

Proof. This is a routine consequence of Proposition 2.5′.

Proposition 2.7. *If* $e^{-2\rho H(\bar{n})}d\mu(\bar{n})$ *and* $e^{-2\rho H(\bar{n})}d\nu(\bar{n})$ *are distinct finite signed measures on* θN, *then the Poisson integrals of* $d\mu$ *and* $d\nu$ *are distinct. Consequently if* $f_1 \in L^{p_1}(\theta N)$ *and* $f_2 \in L^{p_2}(\theta N)$ *are functions with the same Poisson integral, then* $f_1 = f_2$ *almost everywhere.*

Proof. The first statement is a consequence of formula (1) and Proposition 2.7′ below. For the second statement, if $f \in L^p(\theta N)$, then $e^{-2\rho H(\bar{n})}f(\bar{n})d\bar{n}$ is a finite signed measure on θN; this is because it is true for L^1 and L^∞ separately. Therefore the second statement is a special case of the first.

Proposition 2.7′. *If $d\mu(kM)$ and $d\nu(kM)$ are distinct finite signed measures on K/M, then their Poisson integrals are distinct.*

Proof. Apply Proposition 2.6′d.

3. Existence of boundary values. The symmetric space G/K has a natural Riemannian structure inherited from the Killing form on G, and the corresponding Laplacian operator is in $D(G/K)$ since the metric is G-invariant. We shall say that a C^∞ function on G/K is *weakly harmonic* if it is annihilated by this Laplacian.

Lemma 3.1. *If h is weakly harmonic on G/K and if ϕ is a continuous function of compact support on θN, then the convolution $\phi * h$ on G/K defined by*

$$\phi * h(gK) = \int_{\theta N} \phi(\bar{n})h(\bar{n}^{-1}gK)d\bar{n}$$

is weakly harmonic.

Proof. Lift h to a C^∞ function $h'(g)$ defined on G, and let

$$\phi * h'(g) = \int_{\theta N} \phi(\bar{n})h'(\bar{n}^{-1}g)d\bar{n}.$$

By Lemma 2.1, $\phi * h'$ is a C^∞ function and $D(\phi * h') = \phi * D(h')$ for all $D \in D(G)$. By Lemma 2.2 of [6, p. 390], the Laplacian Δ in $D(G/K)$ lifts to an operator Δ' in $D(G)$. Hence $\Delta'(\phi * h') = \phi * \Delta'(h')$. Since $\Delta(h) = 0$, $\Delta'(h') = 0$. Thus $\Delta'(\phi * h') = 0$. But $\phi * h'$ is right K-invariant and equals $(\phi * h)'$. Thus $\Delta(\phi * h) = 0$.

Remarks.

1. Lemma 3.1 is not a special property of the Laplacian, and there are obvious generalizations of the lemma.

2. Furstenberg showed in Theorem 4.4 of [2] that every bounded weakly harmonic function on G/K is harmonic, and he showed in Theorem 4.2 of [2] that every bounded harmonic function on G/K is the Poisson integral of an L^∞ function on a boundary that was later recognized by Moore [11] as K/M. Furstenberg characterized the Poisson kernels as Radon-Nikodym derivatives, and it follows from the decomposition of dg according to the Iwasawa decomposition that $P(gK, kM)$ is the correct formula. The details were given in [7]; see also [10, §1].

3. Because of the correspondence between the Poisson integral formula for K/M and the one for θN, we see from Remark 2 that every bounded weakly harmonic function on G/K is the Poisson integral of an L^∞ function on θN.

4. A general weakly harmonic function defined everywhere on G/K need not be harmonic, as one can see from examples when $G = SL(2, \mathbb{R}) \oplus SL(2, \mathbb{R})$.

Theorem 3.2. *Let $1 \leqq p \leqq \infty$. If h is weakly harmonic on G/K and if $\sup_{a \in N} \| h(\bar{n}aK) \|_{p,\bar{n}} < \infty$, then h is harmonic and is representable as the Poisson integral of*

(1) *a finite signed measure on θN if $p = 1$*

(2) *an L^p function on θN if $p > 1$.*

Proof. This theorem in the case $p = \infty$ is due to Furstenberg and Moore (see Remark 3). Thus we may assume $p < \infty$.

Put $\| h \| = \sup_a \| h(\bar{n}aK) \|_{p,\bar{n}}$. Let $\{U_j\}$ be a neighborhood base at the identity of θN such that each U_j is open and has compact closure. Let V_j be an open neighborhood of the identity such that $V_j V_j \subseteq U_j$, and let ψ_j be a continuous function $\geqq 0$ with support in V_j and with $\int_{\theta N} \psi_j d\bar{n} = 1$. Then $\psi_j * h$ is weakly harmonic by Lemma 3.1, and it is bounded because, if q is the conjugate index of p, we have

$$|\psi_j * h(gK)| \leqq \| \psi_j \|_q \| h(\bar{n}^{-1} gK) \|_{\bar{n},p} \leqq \| \psi_j \|_q \| h \| .$$

The case $p = \infty$ therefore shows that

(4) $$\psi_j * h = PI(f_j)$$

for some L^∞ function f_j on θN. Put $\phi_j = \psi_j * \psi_j$ and convolve ψ_j with the two sides of (4). Since ψ_j is continuous and has compact support (contained in U_j), we obtain

(5) $$\phi_j * h(gK) = \psi_j * PI(f_j)(gK)$$

for all g. The right side of (5) is

$$= \int_{\theta N} \psi_j(\bar{n}_0) PI(f_j)(\bar{n}_0^{-1} gK) d\bar{n}_0$$

$$= \int_{\theta N} \int_{\theta N} \psi_j(\bar{n}_0) e^{-2\rho H(g^{-1}\bar{n}_0\bar{n})} f_j(\bar{n}) d\bar{n}\, d\bar{n}_0$$

$$= \int_{\theta N} \int_{\theta N} \psi_j(\bar{n}_0) e^{-2\rho H(g^{-1}\bar{n})} f_j(\bar{n}_0^{-1}\bar{n}) d\bar{n}\, d\bar{n}_0 .$$

In the last expression we can interchange the order of integration because the iterated integral is finite when f_j is replaced with $|f_j|$. The result is that

(6) $$\psi_j * PI(f_j) = PI(\psi_j * f_j) .$$

Put $F_j = \psi_j * f_j$. Combining (5) and (6), we see that $\phi_j * h$ is the Poisson integral of the continuous bounded function F_j.

By Lemma 2.4 we have

$$\lim_{a \to \infty} \phi_j * h(\bar{n}aK) = F_j(\bar{n})$$

for all $\bar{n}$. Thus if we apply Fatou's Lemma to $|\phi_j * h|^p$, we obtain

$$\| F_j \|_p \leqq \liminf_{a \to \infty} \| \phi_j * h(\bar{n}aK) \|_{\bar{n}, p} \leqq \| \phi_j \|_1 \| h \| = \| h \| .$$

Consequently the functions F_j are in L^p and are bounded in norm. Find a weak-$*$ convergent subsequence (and write it as F_j) against the appropriate space: $C_0(\theta N)$ if $p = 1$ or $L^q(\theta N)$ if $1 < p < \infty$. The limit is a finite signed measure in the first case, but for uniformity of notation we shall write it as a function $F(\bar{n})$ in every case.

By Proposition 2.3 the proof will be complete if we show that h is the Poisson integral of $F(\bar{n})$. For fixed g, Lemma 2.2 shows that the Poisson kernel $P(gK, \bar{n})$ is in $C_0(\theta N)$ and $L^q(\theta N)$ for $1 \leqq q \leqq \infty$. Hence

$$\int_{\theta N} P(gK, \bar{n})F(\bar{n})d\bar{n} = \lim_j \int_{\theta N} P(gK, \bar{n})F_j(\bar{n})d\bar{n} = \lim_j \phi_j * h(gK) = h(gK),$$

the last step holding because $\{\phi_j\}$ is an approximate identity.

Theorem 3.2′. *Let $1 \leqq p \leqq \infty$. If h is weakly harmonic on G/K and if $\sup_{a \in A} \| h(kaK) \|_{p\, k} < \infty$, then h is harmonic and is representable as the Poisson integral of*

(1) *a finite signed measure on K/M if $p = 1$*

(2) *an L^p function on K/M if $p > 1$.*

If h is harmonic and $\geqq 0$, then h is representable as the Poisson integral of

(3) *a finite positive measure on K/M.*

Proof. The proofs of (1) and (2) are similar to those in Theorem 3.2. All that is needed is the definition of convolution

$$\phi * h(gK) = \int_K \phi(k)h(k^{-1}gK)dk$$

and the content of Remark 2. The analog of the first statement of Lemma 2.2 is trivial since K/M is compact.

For (3), we know from [4] that h, when viewed as a function h' on G, satisfies the mean-value property

$$h'(g) = \int_K h'(gkg')dk$$

for all g and g' in G. Putting $g = e$ and $g' = a$, we see that $\sup_a \| h(kaK) \|_{1.k} < \infty$. By (1), h is the Poisson integral of a finite signed measure. This signed measure must be positive, because Proposition 2.6′d shows it is the weak-$*$ limit of positive measures.

4. A Fatou theorem. Following Korányi [10], we say that the function h on G/K converges to the function f on θN at $\bar{n}_0$ *admissibly and unrestrictedly* if, for each non-empty compact set $C \subseteq \theta N$,

$$\lim_{a \to \infty} h(\bar{n}_0 a \bar{n} K) = f(\bar{n}_0)$$

uniformly for $\bar{n} \in C$. (The notation $a \to \infty$ is explained just before Lemma 2.4.)

Theorem 4.1. *If f is in $L^{\infty}(\theta N)$ and if h is its Poisson integral, then h converges to f admissibly and unrestrictedly almost everywhere.*

This theorem is one generalization of the classical theorem of Fatou about non-tangential convergence of Poisson integrals in the upper half-plane. When G is the direct sum of n copies of $SL(2, \mathbb{R})$, Theorem 4.1 is closely related to the results of Jessen, Marcinkiewicz, and Zygmund [8] concerning strong differentiation of multiple integrals in $\mathbb{R}^n$. In fact, the dependence of the proof of Theorem 4.1 on techniques developed in [8] will be apparent.

Let $|E|$ denote the measure of the set E. Korányi's Proposition 4.3 of [10] shows that our Theorem 4.1 follows from the case $p = \infty$ of

Theorem 4.2. *Let V be a bounded open neighborhood of e in θN. If $1 < p \leqq \infty$ and if $f \in L^p(\theta N)$, then*

$$\lim_{a \to \infty} |aVa^{-1}|^{-1} \int_{aVa^{-1}} |f(\bar{n}_0 \bar{n}) - f(\bar{n}_0)| \, d\bar{n} = 0$$

for almost every $\bar{n}_0$.

With V fixed as in Theorem 4.2, define

$$f^*(\bar{n}_0) = \sup_{a \in A} |aVa^{-1}|^{-1} \int_{aVa^{-1}} |f(\bar{n}_0 \bar{n})| \, d\bar{n}$$

for each measurable function f on θN. Most of the proof of Theorem 4.2 consists in proving Theorem 4.3 below.

Theorem 4.3. *If $1 < p \leqq \infty$, then there is a constant C (depending on p and V) such that*

$$\| f^* \|_p \leqq C \| f \|_p$$

for all $f \in L^p(\theta N)$.

We precede the proof of Theorem 4.3 with two lemmas. Let Σ^+ be the set of positive restricted roots, and let r be the number of members λ of Σ^+ such that $\frac{1}{2}\lambda$ is not in Σ^+. The first lemma is due to Gindikin and Karpelevič [3].

Lemma 4.4. *The r members λ of Σ^+ such that $\frac{1}{2}\lambda$ is not in Σ^+ can be arranged in an order $\lambda_1, \cdots, \lambda_r$ in such a way that if N_j is the analytic subgroup of N with Lie algebra $\mathfrak{g}_{\lambda_j} \oplus \mathfrak{g}_{2\lambda_j}$, then the following things happen:*

(a) *The map $N_1 \times \cdots \times N_r \to N$ given by multiplication is a diffeomorphism onto.*

(b) *For $2 \leqq j \leqq r$, $N_j N_{j+1} \cdots N_r$ is a closed subgroup, and it is a normal subgroup of $N_{j-1} N_j N_{j+1} \cdots N_r$.*

We shall use the notation of Lemma 4.4 throughout the rest of this section. The symbols $\bar{n}, \bar{n}'$, etc., will refer to θN, and corresponding symbols with subscripts j will refer to θN_j, where N_j is defined as in Lemma 4.4.

Lemma 4.5. *Let $1 \leqq j \leqq r$ and let R_j be a bounded open neighborhood of the identity in θN_j. Then there exists a constant C_j (depending on R_j) such that whenever f is a measurable function on θN_j, then the function f_j^* on θN_j defined by*

$$f_j^*(\bar{n}_j) = \sup_{a \in A} \left| a R_j a^{-1} \right|^{-1} \int_{a R_j a^{-1}} \left| f(\bar{n}_j \bar{n}_j') \right| d\bar{n}_j'$$

satisfies

$$\left| \{ \bar{n}_j \,|\, f_j^*(\bar{n}_j) > \xi \} \right| \leqq C_j \xi^{-1} \| f \|_{1, \theta N_j}$$

for all $\xi > 0$.

Proof. Let λ be defined by $\theta\mathfrak{n}_j = \mathfrak{g}_{-\lambda} \oplus \mathfrak{g}_{-2\lambda}$, fix a vector H_0 in the positive Weyl chamber, and let $a_t = \exp tH_0$. The crucial observation is that to each $a \in A$ there is associated a real number t such that $aEa^{-1} = a_t E a_t^{-1}$ for every subset E of θN_j. (Namely, take $t = \lambda(\log a)/\lambda(H_0)$.) This means that we can replace the a in the definition of f_j^* by a_t and take the supremum over t.

For an open set $E \subseteq \theta N_j$, let hull(E) be the set of all $xy^{-1}z$ such that $x \in E$ and there is a $t \geqq -1$ with y and z in $a_t E a_t^{-1}$. It is easy to check that $\operatorname{hull}(a_t R_j a_t^{-1}) = a_t \operatorname{hull}(R_j) a_t^{-1}$, and it follows that

$$\sup_{-\infty<t<\infty} \frac{|\operatorname{hull}(a_t R_j a_t^{-1})|}{|a_t R_j a_t^{-1}|} = \frac{|\operatorname{hull}(R_j)|}{|R_j|}.$$

By [13] or [1], the left side of this equality will serve as C_j. On the other hand, the right side of the equality is finite since hull(R_j) is bounded. The lemma follows.

Proof of Theorem 4.3. We may assume $p < \infty$. We shall prove the theorem first for the special case that V is an open set $R = R_1 R_2 \cdots R_r$, where R_j is an open neighborhood of the identity in θN_j and $R_1 R_2 \cdots R_r$ denotes the set of products from these sets in the given order. With respect to the decomposition

$$\theta N = (\theta N_1)(\theta N_2) \cdots (\theta N_r) = (\theta N_1)(\theta N_c)$$

of Lemma 4.4, let π be the projection of θN onto θN_1 and let σ be the projection of θN onto θN_c. We write $R = (\pi R)(\sigma R)$ and $\bar{n}' = \bar{n}_1' \bar{n}_c'$, and we decompose $\bar{n}$ in the other order as $\bar{n} = \bar{n}_c \bar{n}_1$. With suitable normalizations of Haar measures, we have $d\bar{n} = d\bar{n}_1 d\bar{n}_c = d\bar{n}_c d\bar{n}_1$ by [6, p. 372]. Thus

$$f^*(\bar{n}) = \sup_a |aRa^{-1}|^{-1} \int_{aRa^{-1}} |f(\bar{n}_c \bar{n}_1 \bar{n}')|\, d\bar{n}'$$

$$\leqq \sup_a |a\pi Ra^{-1}|^{-1} \int_{a\pi Ra^{-1}} \left\{ \sup_a |a\sigma Ra^{-1}|^{-1} \int_{a\sigma Ra^{-1}} |f(\bar{n}_c \bar{n}_1 \bar{n}_1' \bar{n}_c')|\, d\bar{n}_c' \right\} d\bar{n}_1'.$$

Put

$$f_1(\bar{n}) = \sup_a \left| a\sigma R a^{-1} \right|^{-1} \int_{a\sigma R a^{-1}} \left| f(\bar{n}\, \bar{n}_c') \right| d\bar{n}_c',$$

so that the function in braces above is $f_1(\bar{n}_c \bar{n}_1 \bar{n}_1')$. With $\bar{n}_1$ as the variable, Lemma 4.5 shows that the mapping of $f_1(\bar{n}_c \bar{n}_1)$ to $f^*(\bar{n}_c \bar{n}_1)$ is of weak type $(1,1)$ with bound independent of $\bar{n}_c$. Clearly the mapping is of type (∞, ∞) with bound 1. By the Marcinkiewicz Interpolation Theorem, it is of type (p,p) with bound independent of $\bar{n}_c$. That is,

$$\| f^*(\bar{n}_c \bar{n}_1) \|_{p,\bar{n}_1} \leqq C' \| f_1(\bar{n}_c \bar{n}_1) \|_{p,\bar{n}_1}. \tag{7}$$

By raising this inequality to the p^{th} power and integrating with respect to n_c, we get

$$\| f^*(\bar{n}) \|_{p,\bar{n}} \leqq C' \| f_1(\bar{n}) \|_{p,\bar{n}}. \tag{8}$$

Suppose we can prove that

$$\| f_1(\bar{n}_1 \bar{n}_c) \|_{p,\bar{n}_c} \leqq C'' \| f(\bar{n}_1 \bar{n}_c) \|_{p,\bar{n}} \tag{9}$$

with C'' independent of f and $\bar{n}_1$. Then

$$\| f_1(\bar{n}) \|_{p,\bar{n}} \leqq C'' \| f(\bar{n}) \|_{p,\bar{n}}.$$

Combining this result with (8), we obtain

$$\| f^* \|_p \leqq C'C'' \| f \|_p. \tag{10}$$

This conclusion means that we have reduced the problem of proving (10) to proving inequality (9), in which the variable $\bar{n}_1$ plays no essential role. That is, we have replaced the problem concerning $f^*(\bar{n})$ and the group θN with a problem concerning the function $f_1(\bar{n}_1 \bar{n}_c)$ (with $\bar{n}_1$ fixed) and the group θN_c, with the variable $\bar{n}_1$ eliminated. It is clear that by iterating this procedure and using Lemmas 4.4 and 4.5 at each stage, we can split off one variable $\bar{n}_j$ at a time, until we have exhausted them all. The proof of the special case is complete.

Now let V be a general bounded open neighborhood of e in θN. Let R be a set of the special kind just considered such that $V \subseteq R$. Then

$$|aVa^{-1}|^{-1} \int_{aVa^{-1}} |f(\bar{n}_0\bar{n})|d\bar{n} \leqq \frac{|aRa^{-1}|}{|aVa^{-1}|} \left\{|aRa^{-1}|^{-1} \int_{aRa^{-1}} |f(\bar{n}_0\bar{n})|\, d\bar{n}\right\}$$
$$= \frac{|R|}{|V|} \left\{|aRa^{-1}|^{-1} \int_{aRa^{-1}} |f(\bar{n}_0\bar{n})|\, d\bar{n}\right\}.$$

Since $|R|/|V| < \infty$, the f^* defined relative to V is dominated by a multiple of the f^* defined relative to R. Thus Theorem 4.3 is proved.

Proof of Theorem 4.2. Let $1 < p < \infty$. Let f be in $L^p(\theta N)$, and let $\varepsilon > 0$ be given. Choose a continuous function f_0 of compact support such that $\| f - f_0 \|_p \leqq \varepsilon$, and let $f^a(\bar{n})$ and $f_0^a(\bar{n})$ denote the average values of f and f_0 over $\bar{n}aVa^{-1}$. Then

$$\limsup_{a\to\infty} |f^a - f| \leqq \limsup_{a\to\infty} |(f-f_0)^a| + |f - f_0| + \limsup_{a\to\infty} |f_0^a - f_0|.$$

The last term is 0 trivially. Therefore

$$\| \limsup_{a\to\infty} |f^a - f| \|_p \leqq \|(f-f_0)^*\|_p + \|f - f_0\|_p \leqq (C+1)\|f - f_0\|_p \leqq (C+1)\varepsilon,$$

and it follows that $\lim f^a = f$ almost everywhere.

Next, to prove that $\lim f^a = f$ almost everywhere for $f \in L^\infty$, write f as the sum of a function with support in a compact set E and a function vanishing on E. The result for L^2 functions applies to the first function, and the corresponding result for functions continuous at a point (a trivial result) applies to the second function. The result is that f^a converges to f at almost every point of the interior of E. Since E is arbitrary and θN is σ-compact, the theorem follows.

Finally we have seen that $\lim f^a = f$ almost everywhere whenever f is the sum of an L^∞ function and an L^p function for some $p > 1$. Lebesgue's well known argument [14, p. 65] can therefore be used to complete the proof of Theorem 4.2.

5. Proof of Lemma 2.2. Let $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{a}$ and $\mathfrak{m}$ denote the Lie algebras of G, K, A, and M, respectively. Let $H \in \mathfrak{a}$ and $\bar{n} \in \theta N$, and put $a = \exp H$. Expressions like the one for $P(aK, \bar{n})$ were examined in §2 of [9], and first we recall the results obtained there. Let κ be the dimension of K and let ν be the dimension of θN. Pick an orthogonal basis $\{X_i\}$ of $\mathfrak{n} \oplus \mathfrak{m}$ compatible with the decomposition into root spaces. (The orthogonality is relative to $B_\theta(X, Y) = -B(X, \theta Y)$, where B is the Killing form.) The basis $\{X_i\}$ will be fixed once and for all, and whenever a basis for $\theta\mathfrak{n}$ or $\theta\mathfrak{n} + \mathfrak{m}$ is used, it will always be taken to be the appropriate θX_i's. If X_i is a member of the basis, we write λ_i for the restricted root $\geqq 0$ such that $X_i \in \mathfrak{g}_{\lambda_i}$. By Theorem 2.1 of [9], we have

$$P(aK, \bar{n})^{-1} = \det[\{e^{\lambda_i((H)}\mathrm{Ad}(\bar{n}^{-1})X_i + e^{-\lambda_i(H)}\mathrm{Ad}(\bar{n}^{-1})\theta X_i\}_{\mathfrak{k}}], \tag{11}$$

where the subscript $\mathfrak{k}$ indicates the column vector of coordinates of the $\mathfrak{k}$-component (relative to $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$) in the basis $\{X_i + \theta X_i\}$ of $\mathfrak{k}$.

View the X_i as arranged in order in such a way that the members of $\mathfrak{m}$ are listed last. For each subset s of the integers $1, \cdots, \kappa$, form the determinant of the matrix whose i^{th} column is

$$\begin{cases} \{e^{\lambda_i(H)}\mathrm{Ad}(\bar{n}^{-1})X_i\}_{\mathfrak{k}} & \text{if } i \in s \\ \{e^{-\lambda_i(H)}\mathrm{Ad}(\bar{n}^{-1})\theta X_i\}_{\mathfrak{k}} & \text{if } i \notin s. \end{cases} \tag{12}$$

The sum of the resulting 2^κ determinants is $P(aK, \bar{n})^{-1}$ by (11). If s and s' are two subsets of $\{1, \cdots, \kappa\}$ whose intersections with $\{1, \cdots, \nu\}$ are identical, then the corresponding determinants are equal, because $X_i = \theta X_i$ for $i > \nu$. Hence $P(aK, \bar{n})^{-1}$ is $2^{\kappa-\nu}$ times the sum of the determinants (12) corresponding just to subsets s of $\{1 \cdots \nu\}$.

Let $\{\text{——}\}_{\theta\mathfrak{n}\oplus\mathfrak{m}}$ denote the column vector of coordinates of the $(\theta\mathfrak{n} \oplus \mathfrak{m})$-component of $\{\text{——}\}$ (relative to $\mathfrak{g} = \theta\mathfrak{n} \oplus \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$) in the basis $\{\theta X_i\}$ of $\theta\mathfrak{n} \oplus \mathfrak{m}$. For each subset s of $\{1, \cdots, \nu\}$ form the determinant of the matrix whose i^{th} column is

$$\begin{array}{ll} \{e^{\lambda_i(H)}\mathrm{Ad}(\bar{n}^{-1})X_i\}_{\theta\mathfrak{n}\oplus\mathfrak{m}} & \text{if } i \in s \\ \{e^{-\lambda_i(H)}\mathrm{Ad}(\bar{n}^{-1})\theta X_i\}_{\theta\mathfrak{n}\oplus\mathfrak{m}} & \text{if } i \notin s. \end{array}$$

Each column of this matrix has an exponential as a common factor, and we let $G_s(a)$ be the product of all the exponentials. The determinant can therefore be written as

$$G_s(a)D_s(\bar{n}),$$

which we take as a definition of $D_s(\bar{n})$. This definition is slightly different from the one in §2 of [9]; it is a simple matter to check that the connection is that $G_s(a)D_s(\bar{n})$ is $2^{\kappa-\nu}$ times the determinant of (12). It follows that $P(aK,\bar{n})^{-1} = \sum G_s(a)D_s(\bar{n})$ and that, by Theorem 2.2 of [9], $D_s(\bar{n}) \geqq 0$ for all s.

By a polynomial on θN, we mean a function that is a polynomial on the vector space $\theta\mathfrak{n}$ if we identify θN with $\theta\mathfrak{n}$ *via* the exponential map.

Proposition 5.1. *As s varies through the set of subsets of $\{1,\cdots,\nu\}$, the functions $D_s(\bar{n})$ have the following properties:*

(a) *Each $D_s(\bar{n})$ is a polynomial $\geqq 0$, and*

$$G_s(a) = \exp\left(\sum_{j\in s}\lambda_j(H) - \sum_{j\notin s}\lambda_j(H)\right).$$

(b) $D_\phi(\bar{n}) \equiv 1$ for $\bar{n}\in\theta N$, where ϕ is the empty set.

(c) $P(aK,\bar{n})^{-1} = \sum_s G_s(a)D_s(\bar{n})$.

(d) $e^{2\rho H(\bar{n})} = \sum_s D_s(\bar{n})$, *and $e^{2\rho H(\bar{n})}$ is a polynomial $\geqq 1$.*

Proof. For (a), write $\bar{n} = \exp X = \exp \sum x_j\theta X_j$. If $\{Z_l\}$ is any basis of $\mathfrak{g}$, then it is easy to check that

$$[\mathrm{ad}(-X)]^k Z_l = \sum P_{klm}Z_m,$$

where P_{klm} is a polynomial in the x_j's with coefficients depending on k,l,m, and the structural constants of $\mathfrak{g}$. Since $\theta\mathfrak{n}$ is nilpotent, $\mathrm{Ad}(\bar{n}^{-1})Z_l = \sum P_{lm}Z_m$ with each P_{lm} a polynomial, and it follows readily that each $D_s(\bar{n})$ is a polynomial. Each $D_s(\bar{n})$ is $\geqq 0$ by Theorem 2.2 of [9], and the formula for $G_s(a)$ is by direct calculation. Result (b) holds because the matrix with determinant $D_\phi(\bar{n})$ is triangular with ones on the diagonal, and (c) we know. Result (d) is the special case of (a) through (c) obtained by putting $a = e$.

For proving Lemma 2.2, we shall need a lower bound on the rate of growth of $e^{2\rho H(\bar{n})}$ for $\bar{n}$ tending to infinity. For this purpose it will be sufficient to consider only some special terms in the sum $\sum_s D_s(\bar{n}) = e^{2\rho H(\bar{n})}$, and these we now describe.

Let M' be the normalizer of A in K, and let $w = m'M$ be an element of the Weyl group M'/M. If λ is a restricted root, we denote by λ^w the restricted root such that

$$\mathfrak{g}_{\lambda^w} = \mathrm{Ad}(m')\mathfrak{g}_\lambda .$$

To w we associate a subset s of the integers $1, \cdots, v$ by requiring that i be in s if and only if λ_i^w is negative. (Recall the λ_i for $1 \leqq i \leqq v$ are positive, by convention.) Define $D_w(\bar{n})$ to be the associated function $D_s(\bar{n})$. Thus, for example, $D_e(\bar{n}) = D_\phi(\bar{n})$. As in Theorem 2.2 of [9], we can consider the function $D_w(g)$, defined for $g \in G$ by replacing $\bar{n}$ by g in the determinant that defines $D_w(\bar{n})$.

Lemma 5.2. *If w is in the Weyl group and m' is a representative in M', then $D_w(g) = D_e(m'g)$ for all $g \in G$.*

Proof. Neither side of the equality in the conclusion is changed if we adjust the orthogonal vectors θX_i so that they each have norm one relative to B_θ. The determinant for $D_w(g)$ involves $\mathrm{Ad}(g^{-1})$ of vectors in a set that is the union of orthonormal bases for the spaces ,

$$\begin{cases} \mathfrak{g}_\lambda & \text{for those } \lambda \geqq 0 \text{ for which } \lambda^w < 0 \\ \mathfrak{g}_{-\lambda} & \text{for those } \lambda \geqq 0 \text{ for which } \lambda^w \geqq 0 . \end{cases} \tag{13}$$

The determinant for $D_e(m'g)$ involves $\mathrm{Ad}(g^{-1})$ of the vectors $\mathrm{Ad}(m'^{-1})\theta X_i$. The set $\{\mathrm{Ad}(m'^{-1})\theta X_i\}$ is the union of orthonormal bases for the spaces

$$\mathrm{Ad}(m'^{-1})\mathfrak{g}_{-\mu} \text{ for } \mu \geqq 0 .$$

If we put $\mu = \lambda^w$, we see that these spaces are the same as the spaces

$$\mathfrak{g}_{-\lambda} \text{ for } \lambda^w \geqq 0 ,$$

and the latter spaces are the same as the spaces (13). Hence by first performing an orthogonal transformation in each $\mathfrak{g}_{-\lambda}$ $(\lambda > 0)$ and then applying $\mathrm{Ad}(m'^{-1})$, we get, in some order, the set of vectors that $\mathrm{Ad}(g^{-1})$ operates on in the determinant for $D_w(g)$. Therefore $D_w(g) = \pm D_e(m'g)$ with the sign determined by the determinants of the orthogonal transformations and by the number of transpositions of columns. But the sign must be $+$, since Theorem 2.2 of [9] shows both sides are $\geqq 0$. The lemma is proved.

Lemma 5.3. *If w is in the Weyl group and m' is a representative in M', then $D_w(\gamma_1 m'^{-1}\gamma_2) \neq 0$ for all γ_1 and γ_2 in MAN.*

Proof. Lemma 5.2 shows that

$$D_w(g) = D_e(m'g) = \det[P_{\mathfrak{m}\oplus\theta\mathfrak{n}}\mathrm{Ad}(g^{-1}m'^{-1})\,|\,\mathfrak{m}\oplus\theta\mathfrak{n}],$$

where $P_{\mathfrak{m}\oplus\theta\mathfrak{n}}$ is the projection of $\mathfrak{g}$ on $\mathfrak{m}\oplus\theta\mathfrak{n}$ along $\mathfrak{a}\oplus\mathfrak{n}$ and where the vertical bar means "restricted to." Hence $D_w(g) = 0$ if and only if there is a vector $X \neq 0$ in $\mathfrak{m}\oplus\theta\mathfrak{n}$ such that $\mathrm{Ad}(g^{-1}m'^{-1})X$ is in $\mathfrak{a}\oplus\mathfrak{n}$. The existence of such an X is not affected if g is multiplied on the right by a member of MAN or on the left by a member of MA.

Now $D_w(m'^{-1}) = D_e(e) = 1 \neq 0$. Thus there is a neighborhood U of the identity in N such that $D_w(nm'^{-1}) \neq 0$ for $n \in U$. We can multiply nm'^{-1} on the left and right by members of A and not change the nonzero character of $D_w(nm'^{-1})$, according to the first paragraph. Since M' normalizes A and since every element of N is of the form ana^{-1} with $n \in U$ and $a \in A$, we conclude $D_w(nm'^{-1}) \neq 0$ for all $n \in N$. The conclusion of the lemma then follows from the result of the first paragraph.

The spaces θN and $\theta\mathfrak{n}$ have a box norm induced by the basis $\{\theta X_i\}$. If $\bar{n} = \exp(\sum_i c_i\theta X_i)$, we define $|\bar{n}| = \max|c_i|$.

Lemma 5.4. $\min_{|\bar{n}|=1} (e^{2\rho H(\bar{n})} - 1) > 0.$

Proof. By parts (a), (b), and (d) of Proposition 5.1, it is enough to prove that for each $\bar{n} \in \theta N$ other than $\bar{n} = e$, there is some $D_s(\bar{n})$ with $s \neq \varnothing$ for which $D_s(\bar{n}) \neq 0$. In view of Lemma 5.3 and the Bruhat Decomposition

Theorem (i.e., $G = \cup MAN m' MAN$ over a system of Weyl group representatives), it is enough to remark that $\bar{n} \notin MAN$, a fact that is well-known (see [5], p. 284). The proof is complete.

Proposition 5.5. *There exist positive constants* α *and* β *such that* $e^{2\rho H(\bar{n})} \geqq \alpha |\bar{n}|^{\beta}$ *for* $|\bar{n}| \geqq 1$.

Proof. Fix H_0 in the positive Weyl chamber of $\mathfrak{a}$, let $a_t = \exp tH_0$, and view the additive group of reals as acting on θN by conjugation by a_t^{-1}. By parts (a) to (d) of Proposition 5.1, we have

$$\begin{aligned} e^{2\rho H(a_t^{-1}\bar{n}a_t)} &= 1 + e^{2\rho H(a_t)} \sum_{s \neq \phi} G_s(a_t) D_s(\bar{n}) \\ &\geqq 1 + e^{2\rho H(a_t)} [\min_{s \neq \phi} G_s(a_t)] \sum_{s \neq \phi} D_s(\bar{n}) \\ &= 1 + e^{2\rho H(a_t)} [\min_{s \neq \phi} G_s(a_t)] (e^{2\rho H(\bar{n})} - 1). \end{aligned}$$

If j_0 is chosen so that $\lambda_{j_0}(H_0)$ is the smallest value that a positive restricted root assumes on H_0, then Proposition 5.1a shows that, for $t \geqq 0$,

$$e^{2\rho H(a_t^{-1}\bar{n}a_t)} \geqq 1 + \exp(t\lambda_{j_0}(H_0))(e^{2\rho H(\bar{n})} - 1). \tag{14}$$

Inequality (14) shows that $e^{2\rho H(\bar{n})}$ is nondecreasing along orbits. Hence if S_1 and S_2 are two shells about the identity, each intersecting each orbit once, with S_1 inside S_2, then

$$\inf_{S_1} e^{2\rho H(\bar{n})} \leqq \inf_{S_2} e^{2\rho H(\bar{n})}. \tag{15}$$

Choose k_0 so that $\lambda_{k_0}(H_0)$ is as large as possible. The most general element of θN of norm $\geqq 1$ is of the form $a_t^{-1}\bar{n}a_t$ with $|\bar{n}| = 1$ and $t \geqq 0$, and this element has norm $\leqq \exp(t\lambda_{k_0}(H_0))$. Combining (15) and (14), we obtain, for $t \geqq 0$,

$$\begin{aligned} \min_{|\bar{n}| = \exp(t\lambda_{k_0}(H_0))} e^{2\rho H(\bar{n})} &\geqq \min_{|\bar{n}|=1} e^{2\rho H(a_t^{-1}\bar{n}a_t)} \\ &\geqq \alpha \exp(t\lambda_{j_0}(H_0)), \end{aligned} \tag{16}$$

where α is the left-hand side in the statement of Lemma 5.4. Inequality (16) is the statement of the proposition with $\beta = \lambda_{j_0}(H_0)/\lambda_{k_0}(H_0)$.

Corollary 5.6. *$e^{-2\rho H(\bar{n})}$ vanishes at infinity.*

Proof of Lemma 2.2. For the statements concerning θN, it is sufficient to consider $P(aK, \bar{n})$ for $a \in A$, because any $g \in G$ is of the form $g = \bar{n}_0 a k_0$ and therefore satisfies

$$P(gK, \bar{n}) = e^{-2\rho H(a^{-1}\bar{n}_0 \bar{n})} .$$

We have

$$\int_{\theta N} P(aK, \bar{n}) d\bar{n} = \int_{\theta N} e^{-2\rho H(a^{-1}\bar{n}a)} e^{2\rho H(a)} d\bar{n} = \int_{\theta N} e^{-2\rho H(\bar{n}')} d\bar{n}' = 1,$$

the middle equality following from the change of variables $\bar{n}' = a^{-1}\bar{n}a$. Hence $P(aK, \bar{n})$ is in $L^1(\theta N)$. To see that $P(aK, \bar{n})$ vanishes at infinity (and hence is in L^∞), we use parts (a), (c), and (d) of Proposition 5.1 to write

$$P(aK, \bar{n})^{-1} = \sum_s G_s(a) D_s(\bar{n}) \geqq [\min_s G_s(a)] \sum_s D_s(\bar{n}) = [\min_s G_s(a)] e^{2\rho H(\bar{n})} .$$

Thus

$$P(aK, \bar{n}) \leqq [\min_s G_s(a)]^{-1} e^{-2\rho H(\bar{n})},$$

and the result follows from the vanishing at infinity of $e^{-2\rho H(\bar{n})}$ (see Corollary 5.6). Finally the identity $\int_{K/M} P(gK, kM) dk = 1$ follows from formula (1) and the identity $\int_{\theta N} P(gK, \bar{n}) d\bar{n} = 1$. The proof is complete.

References

1. R. E. Edwards and E. Hewitt, Pointwise limits for sequences of convolution operators, *Acta Math.*, **113** (1965), 181–218.

2. H. Furstenberg, A Poisson formula for semi-simple Lie groups, *Ann. of Math.*, 77 (1963), 335–386.

3. S. G. Gindikin and F. I. Karpelevič, Plancherel measure for Riemann symmetric spaces of nonpositive curvature, *Soviet Math.*, **3** (1962), 962–965.

4. R. Godement, Une généralisation du théorème de la moyenne pour les fonctions harmoniques, *C. R. Acad. Sci., Paris*, **234** (1952), 2137–2139.

5. Harish-Chandra, Spherical functions on a semisimple Lie group, I, *Amer. J. of Math.*, **80** (1958), 241–310.

6. S. Helgason, *Differential Geometry and Symmetric Spaces*, Academic Press, New York, 1962.

7. ———, Lecture notes, Massachusetts Institute of Technology, fall, 1966.

8. B. Jessen, J. Marcinkiewicz, and A. Zygmund, Note on the differentiability of multiple integrals, *Fund. Math.*, **25** (1935), 217–234.

9. A. W. Knapp, Fatou's theorem for symmetric spaces, I, *Ann. of Math.*, **88** (1968), 106–127.

10. A. Korányi, Boundary behavior of Poisson integrals on symmetric spaces, *Trans. Amer. Math. Soc.*, **140** (1969), 393–409.

11. C. C. Moore, Compactifications of symmetric spaces, *Amer. J. of Math.*, **86** (1964), 201–218.

12. H. E. Rauch, Harmonic and analytic functions of several variables and the maximal theorem of Hardy and Littlewood, *Canad. J. Math.*, **8** (1956), 171–183. The author of this paper would like to point out that his Theorem 9 is false.

13. E. M. Stein, Maximal functions and Fatou's theorem, *C.I.M.E. Summer course on bounded homogeneous domains*, Cremonese, Roma, 1967.

14. A. Zygmund, *Trigonometric Series*, I, Cambridge U. Press, Cambridge, 1959.

15. S. Helgason and A. Korányi, A Fatou-type theorem for harmonic functions on symmetric spaces, *Bull. Amer. Math. Soc.*, **74** (1968), 258–263.

CORNELL UNIVERSITY, ITHACA, N. Y., U.S.A.

AND

DARTMOUTH COLLEGE, HANOVER, N.H., U.S.A.

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