VOGAN'S ALGORITHM FOR COMPUTING COMPOSITION SERIES

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Building on joint work [8] with B. Speh, D. A. Vogan [9,10] has obtained an algorithm for computing composition series of the standard induced representations of a semisimple Lie group G. The algorithm takes a particularly simple form in the case that the representations are induced from a maximal parabolic subgroup.

On the one hand, this algorithm does not seem to be widely known, possibly because the emphasis in the Vogan papers is somewhat different. And on the other hand, the algorithm is particularly well suited to deciding some irreducibility questions that arise in our paper [1], which settles the contribution to the unitary dual of a linear connected G by Langlands quotients obtained from maximal parabolic subgroups. Thus it seems expedient to provide an exposition of Vogan's algorithm in the context of the examples needed for [1].

This approach to our irreducibility questions was suggested to us by Vogan, and it overlaps with the joint work of Barbasch and Vogan [3], which uses the algorithm to decide irreducibility in "critical cases" for classical groups. (In fact, the specific results that we give here for classical groups are some of the critical cases for Barbasch and Vogan, although our approach may be more direct.) Calculations with the algorithm appear in other papers as well, for example in [2].

The present paper is organized as follows. Section 1 collects some results from [8] and [9] and sketches the steps of the algorithm. Section 2 establishes a theorem that gives an efficient starting point

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for use of the algorithm. And Sections 3-7 show how the algorithm works in $SO(2n-1,2)$, $Sp(3,\mathbb{R})$, $SO^*(10)$, $SO^*(2n)$, and some exceptional groups of type E , respectively. The worked examples are exactly the ones needed for [1]. We proceed only in the generality needed for our examples——induction from a maximal parabolic subgroup, G linear and connected, rank G = rank K , integral infinitesimal character. Readers interested in relaxing any of these assumptions should consult [8,9,10], particularly pp. 293-294 of [8], §6 of [9], and all of [10].

We are indebted to Vogan for several valuable conversations on this topic and to Barbasch and Vogan for sharing with us the details of their work announced as [3].

1. Notation and algorithm

Let G be a linear connected semisimple group with a simply-connected complexification, let θ be a Cartan involution, let K be the corresponding maximal compact subgroup, and assume that rank G = rank K . Let TA be a θ-stable Cartan subgroup (with $T \subseteq K$ and A equal to a vector group), and let MA be the corresponding Levi factor of the associated cuspidal parabolic subgroups.

Let $P = MAN$ be one of these parabolic subgroups. If σ is a discrete series representation of M or nondegenerate limit of discrete series and if ν is a real-valued linear functional on the Lie algebra $\mathfrak{a}$ of A , we let $U(P,\sigma,\nu)$ be the standard induced representation given by unitary induction as

$$U(P,\sigma,\nu) = \operatorname{ind}_P^G(\sigma \otimes e^{\nu} \otimes 1) .$$

It is known that $U(P,\sigma,\nu)$ has a finite Jordan-Holder series, and Vogan's algorithm addresses the problem of finding the irreducible subquotients and their multiplicities. (We shall be addressing this problem only when $\dim A = 1$ and $U(P,\sigma,\nu)$ has integral infinitesimal character.)

The algorithm takes place in three stages. In the first two stages one considers the same problem for regular integral infinitesimal character. In the first stage we identify some $U(P,\sigma_k,\nu_k)$ with the same infinitesimal character for which the composition series is known. In the second stage we go through a succession of wall-crossing operations, following what happens to the decomposition. At the third stage we pass to our original parameters by means of Zuckerman's ψ functor [11].

Let us introduce more convenient notation. Let $\Delta(\mathfrak{m}^{\mathbb{C}},\mathfrak{t}^{\mathbb{C}})$ be the set of roots of M, and let M_0 be the identity component of M. Pick any irreducible constituent of $\sigma|_{M_0}$, and let $(\lambda, \Delta^+(\mathfrak{m}^{\mathbb{C}},\mathfrak{t}^{\mathbb{C}}))$ be its Harish-Chandra parameter. Corresponding to each real root β in $\Delta(\mathfrak{g}^{\mathbb{C}},(\mathfrak{t}\oplus\mathfrak{a})^{\mathbb{C}})$ is an element γ_β of G that is the image of $\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$ in $SL(2,\mathbb{R})$ under the homomorphism of $SL(2,\mathbb{R})$ into G built from β. Let $F(T)$ be the finite abelian group generated by the elements γ_β. The restriction of σ to the M-central subgroup $F(T)$ determines a character χ, and $(\lambda, \Delta^+(\mathfrak{m}^{\mathbb{C}},\mathfrak{t}^{\mathbb{C}}))$ and χ together determine σ. Introduce a positive system $\Delta^+(\mathfrak{g}^{\mathbb{C}},(\mathfrak{t}\oplus\mathfrak{a})^{\mathbb{C}})$ for the roots of G containing $\Delta^+(\mathfrak{m}^{\mathbb{C}},\mathfrak{t}^{\mathbb{C}})$ such that $\lambda+\nu$ is dominant. (Such a system always exists; if $\lambda+\nu$ is regular, it is unique.)

Adding to $\lambda+\nu$ a suitable parameter μ that is dominant integral for $\Delta^+(\mathfrak{g}^{\mathbb{C}},(\mathfrak{t}\oplus\mathfrak{a})^{\mathbb{C}})$ and adjusting χ compatibly, we are led to a standard induced representation $U(P,\sigma_0,\nu_0)$ whose infinitesimal character $\gamma_0 = \lambda+\nu+\mu$ is regular ([5], Appendix B). Moreover, the Zuckerman ψ functor [11] satisfies

$$U(P,\sigma,\nu) \cong \psi_{\lambda+\nu}^{\lambda+\nu+\mu}\, U(P,\sigma_0,\nu_0)\,.$$

We shall be working with several Cartan subgroups, and we need to match carefully the corresponding root systems. Thus let us fix a compact Cartan subgroup B of G contained in K, and let $\mathfrak{b}$ be its Lie algebra. We may assume that all other Cartan subgroups of interest are obtained by Cayley transform relative to a succession of roots. In particular, for $\mathfrak{a}$ as above, we can write $\mathfrak{a} \leftrightarrow \{\dots\}$, where $\{\dots\}$ is an ordered set of roots of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{b}^{\mathbb{C}})$ with respect to which we have formed Cayley transforms. Then there will be no confusion if we refer parameters like $\gamma_0 = \lambda + \nu + \mu$ to our compact Cartan subalgebra without introducing new notation for them.

It is important to note that γ_0, $\mathfrak{a} \leftrightarrow \{\dots\}$, and the adjusted χ determine the global character of $U(P, \sigma_0, \nu_0)$ completely. In fact, we build $\mathfrak{t} \oplus \mathfrak{a}$, transform γ_0 to it, decompose $\gamma_0 = \lambda_0 + \nu_0$, form σ_0, introduce any N' making ν_0 dominant, and induce $\sigma_0 \otimes e^{\nu_0} \otimes 1$ to obtain a representation $U(P', \sigma_0, \nu_0)$. This representation will have the same global character as $U(P, \sigma_0, \nu_0)$. We write $\pi(\gamma_0, \mathfrak{a} \leftrightarrow \{\dots\})$ for this global character, dropping the adjusted χ from the notation. (Vogan's notation differs slightly from this: He uses $\Theta(\cdot)$ for the character, suppressing the explicit mention of $\mathfrak{a}$ and instead carrying it in the domain of γ_0.) In our notation, note that $\pi(\gamma_0, \mathfrak{a} \leftrightarrow \emptyset)$ is a discrete series character of G.

Under our assumption that γ_0 is regular, it follows essentially from [4] that the representation $U(P', \sigma_0, \nu_0)$ has a unique irreducible quotient $J(P', \sigma_0, \nu_0)$. We write $\overline{\pi}(\gamma_0, \mathfrak{a} \leftrightarrow \{\dots\})$ for the global character of $J(P', \sigma_0, \nu_0)$. If N'' is a second choice of nilpotent subgroup such that ν_0 is dominant, then $J(P'', \sigma_0, \nu_0) \cong J(P', \sigma_0, \nu_0)$ and consequently the global character $\overline{\pi}(\gamma_0, \mathfrak{a} \leftrightarrow \{\dots\})$ is independent of the choice of N'.

Now let us work with a character $\pi(\gamma, \mathfrak{a} \leftrightarrow \{\dots\})$ or $\overline{\pi}(\gamma, \mathfrak{a} \leftrightarrow \{\dots\})$ such that γ is regular and integral. Let α be a real root (relative to the specified choice of $\mathfrak{a}$), and let χ be the suppressed character of $F(T)$ for this representation. We say that α is a cotangent case for this representation if

$$\chi(\gamma_\alpha) = (-1)^{2\langle \rho_\alpha, \alpha\rangle/|\alpha|^2}$$

or is a tangent case if

$$\chi(\gamma_\alpha) = -(-1)^{2\langle \rho_\alpha, \alpha\rangle/|\alpha|^2} .$$

Here ρ_α is half the sum of the roots having positive inner product with α. In either event, $n = 2\langle\gamma,\alpha\rangle/|\alpha|^2$ is an integer. We say that α satisfies the parity condition for the representation if either

α is a cotangent case and n is even

or

α is a tangent case and n is odd.

If $\mathfrak{g}$ is simple, it can be shown that the parity condition can fail for integral γ only if α is long and $\mathfrak{g} \cong \mathfrak{sp}(n,\mathbb{R})$ for some n.

One starting place for the algorithm is the following character identity due to Schmid [6,7]. See p. 271 of Speh-Vogan [8]; recall we are assuming G is connected.

Theorem 1.1 (Schmid's identity). Let γ be regular integral, and let α be a simple noncompact root (for the system $\Delta^+(\mathfrak{g}^{\mathbb{C}},\mathfrak{b}^{\mathbb{C}})$ that makes γ dominant). If α satisfies the parity condition for $\pi(\gamma, \mathfrak{a} \leftrightarrow \{\alpha\})$, then

$$\pi(\gamma, \mathfrak{a} \leftrightarrow \{\alpha\}) = \overline{\pi}(\gamma, \mathfrak{a} \leftrightarrow \{\alpha\}) + \overline{\pi}(\gamma, \mathfrak{a} \leftrightarrow \emptyset) + \overline{\pi}(s_\alpha\gamma, \mathfrak{a} \leftrightarrow \emptyset),$$

where s_α denotes reflection in α.

When the parity condition is not satisfied, the corresponding identity is as follows. (See Proposition 6.1 of Speh-Vogan [8].)

Theorem 1.2. Let γ be regular integral, and let α be a simple noncompact root. If α does not satisfy the parity condition for $\pi(\gamma, \mathfrak{a} \leftrightarrow \{\alpha\})$, then

$$\pi(\gamma\,,\,\mathfrak{a} \leftrightarrow \{\alpha\}) = \overline{\pi}(\gamma\,,\,\mathfrak{a} \leftrightarrow \{\alpha\})\,,$$

i.e., $\pi(\gamma\,,\,\mathfrak{a} \leftrightarrow \{\alpha\})$ is irreducible.

Next we recall the wall-crossing functors and the τ-invariant. Fix γ regular integral, let α be a simple root in the system that makes γ dominant, let Λ_α be the fundamental weight corresponding to α, and put $n = 2\langle\gamma,\alpha\rangle/|\alpha|^2$. In terms of Zuckerman's ψ and φ functors, define

$$\psi_\alpha = \psi^{\gamma}_{\gamma-n\alpha} \quad \text{and} \quad \varphi_\alpha = \varphi^{\gamma-n\alpha}_{\gamma}\,.$$

The wall-crossing functor is given by

$$s_\alpha\Theta = \varphi_\alpha\psi_\alpha\Theta - \Theta$$

for any virtual character Θ whose infinitesimal character is γ; s_α acts on the local expression for a global character by reflection in α (see Appendix C of [5]). We say that α is in the τ-invariant of $\overline{\pi}(\gamma\,,\,\mathfrak{a} \leftrightarrow \{\ldots\})$ if $\psi_\alpha\overline{\pi}(\gamma\,,\,\mathfrak{a} \leftrightarrow \{\ldots\}) = 0$.

For purposes of calculation, we shall want to regard the τ-invariant as a subset of integers, say of $\{1,\ldots,\ell\}$, where ℓ is the rank of $\mathfrak{g}$. To do so, we note that the only parameters of interest will be the Cartan subgroups and the various $w\gamma$ with γ fixed and w in the Weyl group of $\mathfrak{g}^{\mathbb{C}}$. The positive root systems for $w\gamma$ and γ are canonically identified via w, and the root systems for the different Cartan subgroups are all identified by our system of Cayley transforms. Thus we can number the roots in a single Dynkin diagram for $\mathfrak{g}^{\mathbb{C}}$ and obtain consistent numberings of the Dynkin diagrams of all the positive root systems we shall consider. In this way the τ-invariant of a character $\overline{\pi}$ can be regarded as a subset of $\{1,\ldots,\ell\}$. (This point will be clearer in the examples.)

If we replace γ by $w\gamma$ and if α is a simple root for the system that makes γ dominant, then $w\alpha$ is simple for the system making $w\gamma$ dominant. Moreover, as observed in [11], the functor $\psi_\alpha = \psi^{\gamma}_{\gamma-n\alpha}$ is the same as the functor $\psi_{w\alpha} = \psi^{w\gamma}_{w\gamma-nw\alpha}$. Similar remarks apply to φ, and thus s_α is the same functor as $s_{w\alpha}$. In keeping with the notation of the previous paragraph, we can thus denote our wall-crossing functors unambiguously by $s_1, \ldots, s_\ell$.

The observations about ψ_α in the previous paragraph make it clear that the τ-invariant is an invariant of the Langlands quotient in question. In particular, we can detect inequivalence of two Langlands quotients by seeing that their τ-invariants are different.

The τ-invariant controls what happens at the third stage of the algorithm when we pass back to our original parameters by means of the ψ functor $\psi_{\lambda+\nu}^{\lambda+\nu+\mu}$. Recall that $\lambda+\nu+\mu$ is dominant for the system $\Delta^+(\mathfrak{g}^{\mathbb{C}}, (\mathfrak{t}\oplus\mathfrak{a})^{\mathbb{C}})$. From this positive system we obtain a set of <u>singular roots</u>, namely those simple roots orthogonal to $\lambda+\nu$. As usual, we may canonically identify the singular roots with a subset of $\{1,\ldots,\ell\}$.

<u>Theorem 1.3</u> ([8], Theorems 5.15, 6.16, 6.18). Let Θ be an irreducible character with regular integral infinitesimal character γ conjugate to $\lambda+\nu+\mu$. Then $\psi_{\lambda+\nu}^{\lambda+\nu+\mu}\Theta$ is irreducible or 0. It is 0 if and only if the set of singular roots for $\lambda+\nu$ has nonempty intersection with the τ-invariant of Θ.

Computation of τ-invariants is a routine matter because of the next theorem.

<u>Theorem 1.4</u> ([8], Theorem 6.16). Let γ be regular integral, let $\overline{\pi}(\gamma, \mathfrak{a} \leftrightarrow \{\ldots\})$ be given, and let $\Delta^+(\mathfrak{g}^{\mathbb{C}}, (\mathfrak{t}\oplus\mathfrak{a})^{\mathbb{C}})$ be the corresponding positive system. Then a simple root α for this system is in the τ-invariant of $\overline{\pi}(\gamma, \mathfrak{a} \leftrightarrow \{\ldots\})$ if and only if α satisfies one of the following:

(a) α is imaginary and $\mathfrak{m}$-compact

(b) α is complex and $\theta\alpha$ is negative

(c) α is real and satisfies the parity condition for $\overline{\pi}(\gamma, \mathfrak{a} \leftrightarrow \{\ldots\})$.

Now we turn to computation of the effect of the wall-crossing functors. Theorem 1.5 will show the effect on a full induced character. Then we consider the constituents. If $\psi_\alpha \Theta = 0$, then it follows from the definition that $s_\alpha \Theta = -\Theta$. Thus we have only to know the effect of s_α on Θ in the case that α is not in the τ-invariant; this we write down in Theorem 1.6.

Theorem 1.5 ([8], Corollary 5.12). Let γ be regular integral, let $\pi(\gamma, \mathfrak{a} \leftrightarrow \{\ldots\})$ be given, and let $\Delta^+(\mathfrak{g}^{\mathbb{C}}, (\mathfrak{t}\oplus\mathfrak{a})^{\mathbb{C}})$ be the corresponding positive system. If α is a complex simple root for this system, then

$$s_\alpha \pi(\gamma, \mathfrak{a} \leftrightarrow \{\ldots\}) = \pi(s_\alpha \gamma, \mathfrak{a} \leftrightarrow \{\ldots\}) .$$

Remark. Here α complex and simple makes the positive roots vanishing on $\mathfrak{a}$ be the same for γ and $s_\alpha \gamma$. Hence this theorem is indeed implied by Corollary 5.12 and the sentence before Lemma 5.8 in [8].

Theorem 1.6 ([8], Theorem 6.16, and [9], Theorem 4.12). Let γ be regular integral, let $\overline{\pi}(\gamma, \mathfrak{a} \leftrightarrow \{\ldots\})$ be given, and let $\Delta^+(\mathfrak{g}^{\mathbb{C}}, (\mathfrak{t}\oplus\mathfrak{a})^{\mathbb{C}})$ be the corresponding positive system. Suppose that α is a simple root for this system that is not in the τ-invariant of $\overline{\pi}(\gamma, \mathfrak{a} \leftrightarrow \{\ldots\})$. Then the wall-crossing functor s_α satisfies

$$s_\alpha \overline{\pi}(\gamma, \mathfrak{a} \leftrightarrow \{\ldots\}) = \overline{\pi}(\gamma, \mathfrak{a} \leftrightarrow \{\ldots\}) + U_\alpha(\overline{\pi}(\gamma, \mathfrak{a} \leftrightarrow \{\ldots\})) ,$$

where $U_\alpha(\overline{\pi}(\gamma, \mathfrak{a} \leftrightarrow \{\ldots\}))$ is a sum of true characters as follows:

(a) If α is imaginary and $\mathfrak{m}$-noncompact, then

$$U_\alpha(\overline{\pi}(\gamma, \mathfrak{a} \leftrightarrow \{\ldots\})) = \begin{cases} \overline{\pi}(\gamma, \mathfrak{a} \leftrightarrow \{\ldots,\alpha\}) + \Theta_0 & \text{if } s_\alpha \notin W(M:T) \\ \overline{\pi}_1(\gamma, \mathfrak{a} \leftrightarrow \{\ldots,\alpha\}) + \overline{\pi}_2(\gamma, \mathfrak{a} \leftrightarrow \{\ldots,\alpha\}) + \Theta_0 & \\ & \text{if } s_\alpha \in W(M:T) \end{cases}$$

with $\overline{\pi}_1$ and $\overline{\pi}_2$ differing in how χ is defined (see p. 264 of [8]).

(b) If α is complex and $\theta\alpha$ is positive, then

$$U_\alpha(\overline{\pi}(\gamma, \mathfrak{a} \leftrightarrow \{\ldots\})) = \overline{\pi}(s_\alpha\gamma, \mathfrak{a} \leftrightarrow \{\ldots\}) + \Theta_0$$

with $\mathfrak{a}$ unchanged.

(c) If α is real and does not satisfy a parity condition for $\overline{\pi}(\gamma, \mathfrak{a} \leftrightarrow \{\ldots\})$, then

$$U_\alpha(\overline{\pi}(\gamma, \mathfrak{a} \leftrightarrow \{\ldots\})) = \Theta_0 .$$

Moreover, in all cases Θ_0 is a finite sum of irreducible characters, each of which has (the index corresponding to) α in its τ-invariant and each of which occurs in $\pi(\gamma, \mathfrak{a} \leftrightarrow \{\ldots\})$.

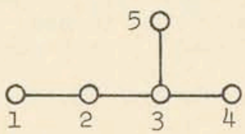
Remark. The main terms of U_α, as well, have α in their τ-invariant, by Lemma 3.11b of [9].

Finally there is a reciprocity theorem that is helpful in computing Θ_0.

Theorem 1.7 ([9], Theorem 4.14). Let γ be regular integral, and suppose Θ and Θ' are irreducible characters with infinitesimal character γ. If i is in $\tau(\Theta)$ but j is not in $\tau(\Theta)$, and if j is in $\tau(\Theta')$ but i is not in $\tau(\Theta')$ (and if indices i and j do not span a group G_2), then the multiplicity of Θ' in $U_j(\Theta)$ equals the multiplicity of Θ in $U_i(\Theta')$, and this common multiplicity is at most one.

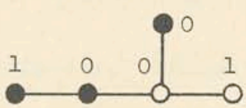
Now we can state the algorithm roughly. We begin with the regular integral parameter γ_0 constructed earlier. Since $\dim \mathfrak{a} = 1$, we can write the character of $U(P', \sigma_0, \nu_0)$ as $\pi(\gamma_0, \mathfrak{a} \leftrightarrow \{\alpha\})$ for some α. By a succession of reflections in complex roots, we pass from γ_0 to $\gamma_1, \gamma_2, \ldots, \gamma_k$ to a point where we know how $\pi(\gamma_k, \mathfrak{a} \leftrightarrow \{\alpha\})$ decomposes. (For example, if α is simple in the system for γ_k, then either a Schmid identity (Theorem 1.1) or Theorem 1.2 will be available. In Section 2 we shall establish a more efficient starting point.) We write the decomposition of $\pi(\gamma_k, \mathfrak{a} \leftrightarrow \{\alpha\})$ and apply to the whole identity the reflection that takes us to $\pi(\gamma_{k-1}, \mathfrak{a} \leftrightarrow \{\alpha\})$, computing the individual terms by Theorems 1.6 and 1.7. (It is not clear whether these tools will be sufficient in general. However, they will suffice in our examples, and the additional tools in [10] will suffice in general.) Then we reflect again to pass to $\pi(\gamma_{k-2}, \mathfrak{a} \leftrightarrow \{\alpha\})$, and so on, until we have a decomposition of $\pi(\gamma_0, \mathfrak{a} \leftrightarrow \{\alpha\})$. Finally we use Theorems 1.3 and 1.4 to pass to our original parameters. If only one nonzero term survives, our original representation $U(P,\sigma,\nu)$ was irreducible.

Setting matters up with the initial reflections requires some further explanation. We illustrate matters for an example with $SO^*(10)$. The numbering of the Dynkin diagram will be



Our given data will be a positive system for $\mathfrak{g}^{\mathbb{C}}$ containing $\Delta^+(\mathfrak{m}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$ such that the root defining $\mathfrak{a}$ is simple. This root α will be root 2. The white dots are the compact roots (when referred to $\mathfrak{b}^{\mathbb{C}}$), while the black dots are noncompact. We specify λ by attaching $2\langle\lambda,\beta\rangle/|\beta|^2$ to each simple root β. Then the diagram is

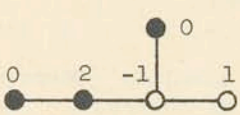
λ :



$\alpha =$ ②

and we investigate reducibility/irreducibility at $\nu = \alpha$. The diagram for $\lambda + \nu$ is

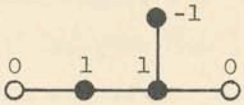
$\lambda+\nu$:



$\alpha =$ ②

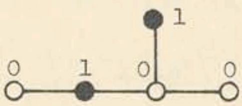
Δ_2^+

We apply a succession of reflections in roots nonorthogonal to α in an effort to make $\lambda+\nu$ dominant. If $\Delta_1^+ = s_3\Delta_2^+$, we obtain



$\alpha = ② + ③ \qquad \Delta_1^+$

Finally $\Delta_0^+ = s_5\Delta_1^+$ gives us



$\alpha = ② + ③ + ⑤ \qquad \Delta_0^+$

This Δ_0^+ is a system compatible with $\Delta^+(\mathfrak{m}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$ that makes $\lambda+\nu$ dominant. We can take it as the system in which γ_0 is to be dominant. The set of singular roots is $\{1,3,4\}$.

There is no need to carry along an explicit value of γ_0; having Δ_0^+ and the expression for α will be enough. Then we can define $\gamma_1 = s_5\gamma_0$, which will be dominant for Δ_1^+, and $\gamma_2 = s_3\gamma_1$, which will be dominant for Δ_2^+. Since α is simple for Δ_2^+ and satisfies the parity condition (our group is not $Sp(n,\mathbb{R})$), $\pi(\gamma_2, \mathfrak{a} \leftrightarrow \{\alpha\})$ is given by a Schmid identity. So we have a starting place for the algorithm.

2. An inductive application

When we set up matters as at the end of Section 1 and then proceed with the wall crossings, we typically find that the first few wall-crossing steps are independent of our example. What is happening is that the first few steps take place in a common real rank one example. The theorem below formalizes this process and its result. Because of this theorem, we shall find that the set-up at the end of Section 1 should be done in such a way as to minimize the number of steps that are outside a real rank one subgroup.

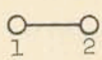
Theorem 2.1. Let γ be a regular integral infinitesimal character dominant for Δ^+. Suppose that α is a $\mathfrak{g}$-noncompact root that is the sum of all the simple roots in a single-line real rank one subgroup with positive system $\Delta_0^+ \subseteq \Delta^+$, and suppose α satisfies the parity condition for $\pi(\gamma, a \leftrightarrow \{\alpha\})$. Let n be the number of simple roots in Δ_0^+, and suppose $n \geq 2$. Let ε_1 and ε_2 be the nodes among these simple roots, say with ε_1 compact and ε_2 noncompact. If $n = 2$ and if $\{\alpha\}$ is abbreviated as α, then

$$\pi(\gamma, a \leftrightarrow \alpha) = \overline{\pi}(\gamma, a \leftrightarrow \alpha) + \overline{\pi}(s_{\varepsilon_1}\gamma, a \leftrightarrow \alpha) + \overline{\pi}(s_{\varepsilon_2}\gamma, a \leftrightarrow \alpha) + \overline{\pi}(s_{\varepsilon_2}\gamma, a \leftrightarrow \emptyset), \tag{2.1}$$

while if $n \geq 3$, then

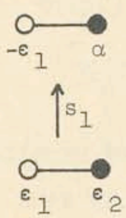
$$\pi(\gamma, a \leftrightarrow \alpha) = \overline{\pi}(\gamma, a \leftrightarrow \alpha) + \overline{\pi}(s_{\varepsilon_1}\gamma, a \leftrightarrow \alpha) + \overline{\pi}(s_{\varepsilon_2}\gamma, a \leftrightarrow \alpha) + \overline{\pi}(s_{\varepsilon_1}s_{\varepsilon_2}\gamma, a \leftrightarrow \alpha). \tag{2.2}$$

Proof. We proceed by induction on n, treating $n=2$ and $n=3$ separately. First let $n=2$. The relevant part of the Dynkin diagram is



and we introduce

$\Delta_1^+ = s_{\varepsilon_1}\Delta_0^+$ $\quad$ $\alpha = ②$ $\quad$ $s_{\varepsilon_1}\gamma$ dominant



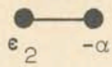
$\Delta_0^+ \subseteq \Delta^+$ $\quad$ $\alpha = ① + ②$ $\quad$ γ dominant

By the Schmid identity (Theorem 1.1) applied to $s_{\varepsilon_1}\Delta^+$,

$$\pi(s_{\varepsilon_1}\gamma, a \leftrightarrow \alpha) = \overline{\pi}(s_{\varepsilon_1}\gamma, a \leftrightarrow \alpha) + \overline{\pi}(s_{\varepsilon_1}\gamma, a \leftrightarrow \emptyset) + \overline{\pi}(s_\alpha s_{\varepsilon_1}\gamma, a \leftrightarrow \emptyset). \tag{2.3}$$

We shall apply the functor s_1. To compute the τ-invariants of the right side of (2.3), we need one more diagram:

$s_\alpha s_{\epsilon_1}\Delta_0^+$



$s_\alpha s_{\epsilon_1}\gamma$ dominant

The τ-invariants within the set $\{1,2\}$ are

$$\tau(\overline{\pi}(s_{\epsilon_1}\gamma\,,\, a \leftrightarrow \alpha)) = \{2\}$$

$$\tau(\overline{\pi}(s_{\epsilon_1}\gamma\,,\, a \leftrightarrow \emptyset)) = \{1\}$$

$$\tau(\overline{\pi}(s_\alpha s_{\epsilon_1}\,,\, a \leftrightarrow \emptyset)) = \emptyset$$

by Theorem 1.4. Then Theorem 1.6 gives

$$s_1\overline{\pi}(s_{\epsilon_1}\gamma\,,\, a\leftrightarrow\alpha) = \overline{\pi}(s_{\epsilon_1}\gamma\,,\, a\leftrightarrow\alpha) + \overline{\pi}(\gamma\,,\, a\leftrightarrow\alpha) + \Theta_1$$

$$s_1\overline{\pi}(s_{\epsilon_1}\gamma\,,\, a\leftrightarrow\emptyset) = -\overline{\pi}(s_{\epsilon_1}\gamma\,,\, a\leftrightarrow\emptyset)$$

$$s_1\overline{\pi}(s_\alpha s_{\epsilon_1}\gamma\,,\, a\leftrightarrow\emptyset) = \overline{\pi}(s_\alpha s_{\epsilon_1}\gamma\,,\, a\leftrightarrow\emptyset) + \overline{\pi}(s_\alpha s_{\epsilon_1}\gamma\,,\, a\leftrightarrow\epsilon_2) + \Theta_2\,.$$

Here Θ_2 is the sum of constituents of $\pi(s_\alpha s_{\epsilon_1}\gamma\,,\, a\leftrightarrow\emptyset) = \overline{\pi}(s_\alpha s_{\epsilon_1}\gamma\,,\, a\leftrightarrow\emptyset)$ having 1 in the τ-invariant, and so $\Theta_2 = 0$. Also Θ_1 is the sum of constituents of $\pi(s_{\epsilon_1}\gamma\,,\, a\leftrightarrow\alpha)$ having 1 in the τ-invariant, and so (2.1) shows $\Theta_1 = c\overline{\pi}(s_{\epsilon_1}\gamma\,,\, a\leftrightarrow\emptyset)$. Now Theorem 1.7 gives

$$c = \text{mult } \overline{\pi}(s_{\epsilon_1}\gamma\,,\, a\leftrightarrow\emptyset) \text{ in } U_1(\overline{\pi}(s_{\epsilon_1}\gamma\,,\, a\leftrightarrow\alpha))\,, \qquad 1\notin\tau=\{2\}\,,$$

$$= \text{mult } \overline{\pi}(s_{\epsilon_1}\gamma\,,\, a\leftrightarrow\alpha) \text{ in } U_2(\overline{\pi}(s_{\epsilon_1}\gamma\,,\, a\leftrightarrow\emptyset))\,, \qquad 2\notin\tau=\{1\}\,.$$

For the latter we write

$$U_2(\overline{\pi}(s_{\epsilon_1}\gamma\,,\, a\leftrightarrow\emptyset)) = \overline{\pi}(s_{\epsilon_1}\gamma\,,\, a\leftrightarrow\alpha) + \Theta_3\,.$$

This shows $c \geq 1$, and Theorem 1.7 says $c = 1$. Applying s_1 to (2.3) and using Theorem 1.5, we obtain

$$\pi(\gamma, \alpha \leftrightarrow \alpha) = s_1 \pi(s_{\varepsilon_1}\gamma, \alpha \leftrightarrow \alpha)$$

$$= s_1\overline{\pi}(s_{\varepsilon_1}\gamma, \alpha \leftrightarrow \alpha) + s_1\overline{\pi}(s_{\varepsilon_1}\gamma, \alpha \leftrightarrow \emptyset) + s_1\overline{\pi}(s_\alpha s_{\varepsilon_1}\gamma, \alpha \leftrightarrow \emptyset)$$

$$= \overline{\pi}(s_{\varepsilon_1}\gamma, \alpha \leftrightarrow \alpha) + \overline{\pi}(\gamma, \alpha \leftrightarrow \emptyset) + \overline{\pi}(s_{\varepsilon_1}\gamma, \alpha \leftrightarrow \emptyset)$$

$$- \overline{\pi}(s_{\varepsilon_1}\gamma, \alpha \leftrightarrow \emptyset)$$

$$+ \overline{\pi}(s_\alpha s_{\varepsilon_1}\gamma, \alpha \leftrightarrow \emptyset) + \overline{\pi}(s_\alpha s_{\varepsilon_1}\gamma, \alpha \leftrightarrow \varepsilon_2) . \qquad (2.4)$$

Since ε_1 is compact and $s_{\varepsilon_1} s_\alpha s_{\varepsilon_1} = s_{\varepsilon_2}$, we have

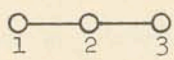
$$\overline{\pi}(s_\alpha s_{\varepsilon_1}\gamma, \alpha \leftrightarrow \emptyset) = \overline{\pi}(s_{\varepsilon_1} s_\alpha s_{\varepsilon_1}\gamma, \alpha \leftrightarrow \emptyset) = \overline{\pi}(s_{\varepsilon_2}\gamma, \alpha \leftrightarrow \emptyset)$$

and

$$\overline{\pi}(s_\alpha s_{\varepsilon_1}\gamma, \alpha \leftrightarrow \varepsilon_2) = \overline{\pi}(s_{\varepsilon_1} s_\alpha s_{\varepsilon_1}\gamma, \alpha \leftrightarrow s_{\varepsilon_1}\varepsilon_2) = \overline{\pi}(s_{\varepsilon_2}\gamma, \alpha \leftrightarrow \alpha) .$$

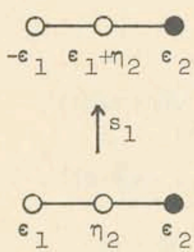
Substitution into (2.4) gives the desired result (2.1).

Next let $n = 3$. The corresponding diagrams are



$\Delta_1^+ = s_{\varepsilon_1}\Delta_0^+$

$\Delta_0^+ \subseteq \Delta^+$



α = ② + ③ $s_{\varepsilon_1}\gamma$ dominant

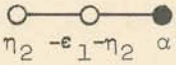
α = ① + ② + ③ γ dominant

By (2.1) for $s_{\varepsilon_1}\gamma$,

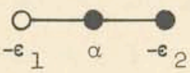
$$\pi(s_{\varepsilon_1}\gamma, \alpha \leftrightarrow \alpha) = \overline{\pi}(s_{\varepsilon_1}\gamma, \alpha \leftrightarrow \alpha) + \overline{\pi}(s_{\varepsilon_1+\eta_2} s_{\varepsilon_1}\gamma, \alpha \leftrightarrow \alpha)$$

$$+ \overline{\pi}(s_{\varepsilon_2\varepsilon_1}\gamma, \alpha \leftrightarrow \alpha) + \overline{\pi}(s_{\varepsilon_2\varepsilon_1}\gamma, \alpha \leftrightarrow \emptyset) . \qquad (2.5)$$

We shall apply the functor s_1. To compute the τ-invariants of the right side of (2.5), we need the diagrams

$s_{\epsilon_1+\eta_2} s_{\epsilon_1} \Delta_0^+$ 

$\alpha = ③$ $s_{\epsilon_1+\eta_2} s_{\epsilon_1} \gamma$ dominant

$s_{\epsilon_2} s_{\epsilon_1} \Delta_0^+$ 

$\alpha = ②$ $s_{\epsilon_2} s_{\epsilon_1} \gamma$ dominant

The τ-invariants within $\{1,2,3\}$ are

$$\begin{aligned}
\tau(\overline{\pi}(s_{\epsilon_1}\gamma, \alpha \leftrightarrow \alpha)) &= \{2,3\} \\
\tau(\overline{\pi}(s_{\epsilon_1+\eta_2} s_{\epsilon_1}\gamma, \alpha \leftrightarrow \alpha)) &= \{1,3\} \\
\tau(\overline{\pi}(s_{\epsilon_2} s_{\epsilon_1}\gamma, \alpha \leftrightarrow \alpha)) &= \{2\} \\
\tau(\overline{\pi}(s_{\epsilon_2} s_{\epsilon_1}\gamma, \alpha \leftrightarrow \emptyset)) &= \{1\}\,. \qquad (2.6)
\end{aligned}$$

Then

$$\begin{aligned}
s_1\overline{\pi}(s_{\epsilon_1}\gamma, \alpha \leftrightarrow \alpha) &= \overline{\pi}(s_{\epsilon_1}\gamma, \alpha \leftrightarrow \alpha) + \overline{\pi}(\gamma, \alpha \leftrightarrow \alpha) + \Theta_1 \\
s_1\overline{\pi}(s_{\epsilon_1+\eta_2} s_{\epsilon_1}\gamma, \alpha \leftrightarrow \alpha) &= -\,\overline{\pi}(s_{\epsilon_1+\eta_2} s_{\epsilon_1}\gamma, \alpha \leftrightarrow \alpha) \\
s_1\overline{\pi}(s_{\epsilon_2} s_{\epsilon_1}\gamma, \alpha \leftrightarrow \alpha) &= \overline{\pi}(s_{\epsilon_2} s_{\epsilon_1}\gamma, \alpha \leftrightarrow \alpha) + \overline{\pi}(s_{\epsilon_2}\gamma, \alpha \leftrightarrow \alpha) + \Theta_2 \\
s_1\overline{\pi}(s_{\epsilon_2} s_{\epsilon_1}\gamma, \alpha \leftrightarrow \emptyset) &= -\,\overline{\pi}(s_{\epsilon_2} s_{\epsilon_1}\gamma, \alpha \leftrightarrow \emptyset)\,. \qquad (2.7)
\end{aligned}$$

From (2.5) and (2.6)

$$\Theta_1 = c_1\overline{\pi}(s_{\epsilon_1+\eta_2} s_{\epsilon_1}\gamma, \alpha \leftrightarrow \alpha) + c_2\overline{\pi}(s_{\epsilon_2} s_{\epsilon_1}\gamma, \alpha \leftrightarrow \emptyset)\,.$$

Now Theorem 1.7 gives

$$\begin{aligned}
c_1 &= \text{mult}\ \overline{\pi}(s_{\epsilon_1+\eta_2} s_{\epsilon_1}\gamma, \alpha \leftrightarrow \alpha) \ \text{ in } \ U_1(\overline{\pi}(s_{\epsilon_1}\gamma, \alpha \leftrightarrow \alpha)),\ 1 \notin \tau = \{2,3\}, \\
&= \text{mult}\ \overline{\pi}(s_{\epsilon_1}\gamma, \alpha \leftrightarrow \alpha) \ \text{ in } \ U_2(\overline{\pi}(s_{\epsilon_1+\eta_2} s_{\epsilon_1}\gamma, \alpha \leftrightarrow \alpha)),\ 2 \notin \tau = \{1,3\}.
\end{aligned}$$

For the latter, we write

$$U_2\overline{\pi}(s_{\epsilon_1+\eta_2}s_{\epsilon_1}\gamma\,,\,\alpha\leftrightarrow\alpha) = \overline{\pi}(s_{\epsilon_1}\gamma\,,\,\alpha\leftrightarrow\alpha) + \Theta_3\,.$$

This shows $c_1 \geq 1$, and hence $c_1 = 1$. Next,

$$c_2 = \text{mult}\ \overline{\pi}(s_{\epsilon_2}s_{\epsilon_1}\gamma\,,\,\alpha\leftrightarrow\emptyset)\ \text{ in }\ U_1\overline{\pi}(s_{\epsilon_1}\gamma\,,\,\alpha\leftrightarrow\alpha)\,, \qquad 1\notin\tau=\{2,3\},$$

$$= \text{mult}\ \overline{\pi}(s_{\epsilon_1}\gamma\,,\,\alpha\leftrightarrow\alpha)\ \text{ in }\ U_2\overline{\pi}(s_{\epsilon_2}s_{\epsilon_1}\gamma\,,\,\alpha\leftrightarrow\emptyset)\,, \qquad 2\notin\tau=\{1\}.$$

Now

$$U_2\overline{\pi}(s_{\epsilon_2}s_{\epsilon_1}\gamma\,,\,\alpha\leftrightarrow\emptyset) = \overline{\pi}(s_{\epsilon_2}s_{\epsilon_1}\gamma\,,\,\alpha\leftrightarrow\alpha) + \Theta_4\,.$$

Here Θ_4 is the sum of constituents of $\pi(s_{\epsilon_2}s_{\epsilon_1}\gamma\,,\,\alpha\leftrightarrow\emptyset)$ $= \overline{\pi}(s_{\epsilon_2}s_{\epsilon_1}\gamma\,,\,\alpha\leftrightarrow\emptyset)$ having 2 in the τ-invariant, and so $\Theta_4 = 0$. Since $\overline{\pi}(s_{\epsilon_1}\gamma\,,\,\alpha\leftrightarrow\alpha)$ and $\overline{\pi}(s_{\epsilon_2}s_{\epsilon_1}\gamma\,,\,\alpha\leftrightarrow\alpha)$ have respective τ-invariants $\{2,3\}$ and $\{2\}$, they are unequal characters, and thus $c_2 = 0$. Hence

$$\Theta_1 = \overline{\pi}(s_{\epsilon_1+\eta_2}s_{\epsilon_1}\gamma\,,\,\alpha\leftrightarrow\alpha)\,. \tag{2.8}$$

To compute Θ_2, we use the Schmid identity

$$\pi(s_{\epsilon_2}s_{\epsilon_1}\gamma\,,\,\alpha\leftrightarrow\alpha) = \overline{\pi}(s_{\epsilon_2}s_{\epsilon_1}\gamma\,,\,\alpha\leftrightarrow\alpha) + \overline{\pi}(s_{\epsilon_2}s_{\epsilon_1}\gamma\,,\,\alpha\leftrightarrow\emptyset) + \overline{\pi}(s_\alpha s_{\epsilon_2}s_{\epsilon_1}\gamma\,,\,\alpha\leftrightarrow\emptyset)$$

and compute that the τ-invariants for the terms on the right are $\{2\}$, $\{1\}$, and $\{3\}$. Then it follows that

$$\Theta_2 = c\overline{\pi}(s_{\epsilon_2}s_{\epsilon_1}\gamma\,,\,\alpha\leftrightarrow\emptyset)\,.$$

By Theorem 1.7,

$$c = \text{mult}\ \overline{\pi}(s_{\epsilon_2}s_{\epsilon_1}\gamma\,,\,\alpha\leftrightarrow\emptyset)\ \text{ in }\ U_1(\overline{\pi}(s_{\epsilon_2}s_{\epsilon_1}\gamma\,,\,\alpha\leftrightarrow\alpha))\,, \qquad 1\notin\tau=\{2\},$$

$$= \text{mult}\ \overline{\pi}(s_{\epsilon_2}s_{\epsilon_1}\gamma\,,\,\alpha\leftrightarrow\alpha)\ \text{ in }\ U_2(\overline{\pi}(s_{\epsilon_2}s_{\epsilon_1}\gamma\,,\,\alpha\leftrightarrow\emptyset))\,, \qquad 2\notin\tau=\{1\}.$$

Since

$$U_2(\overline{\pi}(s_{\epsilon_2}s_{\epsilon_1}\gamma\,,\,\alpha\leftrightarrow\emptyset)) = \overline{\pi}(s_{\epsilon_2}s_{\epsilon_1}\gamma\,,\,\alpha\leftrightarrow\alpha) + \Theta_5\,,$$

we have $c \geq 1$. Thus $c = 1$ and

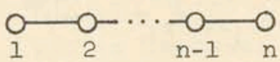
$$\Theta_2 = \overline{\pi}(s_{\epsilon_2} s_{\epsilon_1} \gamma, \alpha \leftrightarrow \emptyset). \tag{2.9}$$

Finally we apply s_1 to both sides of (2.5), use the identity

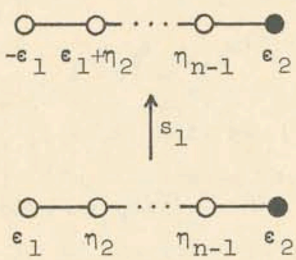
$$s_1 \pi(s_{\epsilon_1} \gamma, \alpha \leftrightarrow \alpha) = \pi(\gamma, \alpha \leftrightarrow \alpha)$$

given in Theorem 1.5, and substitute from (2.7), (2.8), and (2.9) to obtain (2.2) for $n = 3$.

Now let $n \geq 4$, and assume inductively that (2.1) and (2.2) have been proved for all cases $\leq n-1$. The starting diagrams are



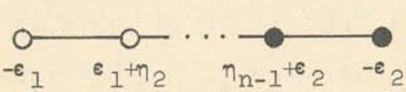
$\Delta_1^+ = s_{\epsilon_1} \Delta_0^+$ $\quad$ $\Delta_0^+ \subseteq \Delta^+$



$\alpha = ② + \ldots + ⓝ$ $\quad$ $s_{\epsilon_1}\gamma$ dominant

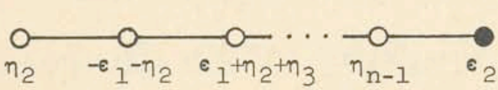
$\alpha = ① + \ldots + ⓝ$ $\quad$ γ dominant

We shall use also the diagrams



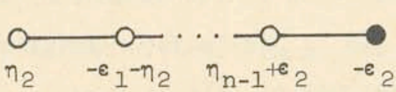
$\alpha = ② + \ldots + (n-1)$ (2.10)

$s_{\epsilon_1} s_{\epsilon_2} \gamma$ dominant



$\alpha = ③ + \ldots + ⓝ$ (2.11)

$s_{\epsilon_1+\eta_2} s_{\epsilon_1} \gamma$ dominant



$\alpha = ③ + \ldots + (n-1)$ (2.12)

$s_{\epsilon_1+\eta_2} s_{\epsilon_2} s_{\epsilon_1} \gamma$ dominant

By inductive hypothesis, (2.2) for $n-1$ and $s_{\epsilon_1}\gamma$ gives us

$$\pi(s_{\epsilon_1}\gamma, \alpha\leftrightarrow\alpha) = \bar{\pi}(s_{\epsilon_1}\gamma, \alpha\leftrightarrow\alpha) + \bar{\pi}(s_{\epsilon_1}s_{\epsilon_2}, \alpha\leftrightarrow\alpha)$$
$$+ \bar{\pi}(s_{\epsilon_1+\eta_2}s_{\epsilon_1}\gamma, \alpha\leftrightarrow\alpha) + \bar{\pi}(s_{\epsilon_1+\eta_2}s_{\epsilon_2}s_{\epsilon_1}\gamma, \alpha\leftrightarrow\alpha), \qquad (2.13)$$

and the respective τ-invariants for the terms on the right are $\{2,\dots,n\}$, $\{2,\dots,n-1\}$, $\{1,3,4,\dots,n\}$, and $\{1,3,4,\dots,n-1\}$. Then

$$\begin{aligned}
s_1\bar{\pi}(s_{\epsilon_1}\gamma, \alpha\leftrightarrow\alpha) &= \bar{\pi}(s_{\epsilon_1}\gamma, \alpha\leftrightarrow\alpha) + \bar{\pi}(\gamma, \alpha\leftrightarrow\alpha) + \Theta_1 \\
s_1\bar{\pi}(s_{\epsilon_1}s_{\epsilon_2}\gamma, \alpha\leftrightarrow\alpha) &= \bar{\pi}(s_{\epsilon_1}s_{\epsilon_2}\gamma, \alpha\leftrightarrow\alpha) + \bar{\pi}(s_{\epsilon_2}\gamma, \alpha\leftrightarrow\alpha) + \Theta_2 \\
s_1\bar{\pi}(s_{\epsilon_1+\eta_2}s_{\epsilon_1}\gamma, \alpha\leftrightarrow\alpha) &= -\bar{\pi}(s_{\epsilon_1+\eta_2}s_{\epsilon_1}\gamma, \alpha\leftrightarrow\alpha) \\
s_1\bar{\pi}(s_{\epsilon_1+\eta_2}s_{\epsilon_2}s_{\epsilon_1}\gamma, \alpha\leftrightarrow\alpha) &= -\bar{\pi}(s_{\epsilon_1+\eta_2}s_{\epsilon_2}s_{\epsilon_1}\gamma, \alpha\leftrightarrow\alpha). \qquad (2.14)
\end{aligned}$$

From (2.13),

$$\Theta_1 = c_1\bar{\pi}(s_{\epsilon_1+\eta_2}s_{\epsilon_1}\gamma, \alpha\leftrightarrow\alpha) + c_2\bar{\pi}(s_{\epsilon_1+\eta_2}s_{\epsilon_2}s_{\epsilon_1}\gamma, \alpha\leftrightarrow\alpha).$$

By the same argument as when $n = 3$, we find $c_1 = 1$. Also

$$\begin{aligned}
c_2 &= \text{mult of } \bar{\pi}(s_{\epsilon_1+\eta_2}s_{\epsilon_2}s_{\epsilon_1}\gamma, \alpha\leftrightarrow\alpha) \text{ in } U_1(\bar{\pi}(s_{\epsilon_1}\gamma, \alpha\leftrightarrow\alpha)), \\
&\qquad\qquad 1 \notin \tau = \{2,\dots,n-1\}, \\
&= \text{mult of } \bar{\pi}(s_{\epsilon_1}\gamma, \alpha\leftrightarrow\alpha) \text{ in } U_2(\bar{\pi}(s_{\epsilon_1+\eta_2}s_{\epsilon_2}s_{\epsilon_1}\gamma, \alpha\leftrightarrow\alpha)), \\
&\qquad\qquad 2 \notin \tau = \{1,3,4,\dots,n-1\}.
\end{aligned}$$

Now

$$U_2(\bar{\pi}(s_{\epsilon_1+\eta_2}s_{\epsilon_2}s_{\epsilon_1}\gamma, \alpha\leftrightarrow\alpha)) = \bar{\pi}(s_{\epsilon_2}s_{\epsilon_1}\gamma, \alpha\leftrightarrow\alpha) + \Theta_3,$$

with each constituent of Θ_3 in $\pi(s_{\epsilon_1+\eta_2}s_{\epsilon_2}s_{\epsilon_1}\gamma, \alpha\leftrightarrow\alpha)$. By inductive assumption for $n-3$ (if $n > 4$) or by a Schmid identity if $n=4$, this π has three or four terms in its expansion, all computed within the diagram (2.12), and 1 is in the τ-invariant of each. Since 1 is not in the τ-invariant of $\bar{\pi}(s_{\epsilon_1}\gamma, \alpha\leftrightarrow\alpha)$,

$\overline{\pi}(s_{\epsilon_1}\gamma, \alpha \leftrightarrow \alpha)$ does not occur in Θ_3. But nor do we have $\overline{\pi}(s_{\epsilon_1}\gamma, \alpha \leftrightarrow \alpha) = \overline{\pi}(s_{\epsilon_2}s_{\epsilon_1}\gamma, \alpha \leftrightarrow \alpha)$, since n is in the τ-invariant of the left side but not the right side. Thus $c_2 = 0$, and

$$\Theta_1 = \overline{\pi}(s_{\epsilon_1+\eta_2}s_{\epsilon_1}\gamma, \alpha \leftrightarrow \alpha). \tag{2.15}$$

To compute Θ_2, we note that each constituent of Θ_2 must occur in $\pi(s_{\epsilon_1}s_{\epsilon_2}\gamma, \alpha \leftrightarrow \alpha)$ and have 1 in its τ-invariant. By inductive assumption for $n-2$, this π has four terms in its expansion, all computed within the diagram (2.10). A term in which α includes root 2 will not have 1 in its τ-invariant, while a term with α not including root 2 (or in the case $n=4$—in which $\alpha \leftrightarrow \emptyset$) will have 1 in its τ-invariant. There are two terms of the latter kind,

$$\overline{\pi}(s_{\epsilon_1+\eta_2}s_{\epsilon_2}s_{\epsilon_1}\gamma, \alpha \leftrightarrow \alpha)$$

and

$$\begin{cases} \overline{\pi}(s_{\eta_3+\epsilon_2}s_{\epsilon_1}s_{\epsilon_2}\gamma, \alpha \leftrightarrow \emptyset) & \text{if } n = 4 \\ \overline{\pi}(s_{\eta_{n-1}+\epsilon_2}s_{\epsilon_1+\eta_2}s_{\epsilon_1}s_{\epsilon_2}\gamma, \alpha \leftrightarrow \alpha) & \text{if } n > 4. \end{cases}$$

Let c_3 and c_4 be the respective coefficients of these terms in Θ_2. A familiar argument shows $c_3 = 1$. For the computation of c_4, let us treat $n = 4$ and $n > 4$ separately.

First suppose $n = 4$. Then

$$c_4 = \text{mult } \overline{\pi}(s_{\eta_3+\epsilon_2}s_{\epsilon_1}s_{\epsilon_2}\gamma, \alpha \leftrightarrow \emptyset) \text{ in } U_1(\overline{\pi}(s_{\epsilon_1}s_{\epsilon_2}\gamma, \alpha \leftrightarrow \alpha)), 1 \notin \tau = \{2,3\},$$

$$= \text{mult } \overline{\pi}(s_{\epsilon_1}s_{\epsilon_2}\gamma, \alpha \leftrightarrow \alpha) \text{ in } U_3(\overline{\pi}(s_{\eta_3+\epsilon_2}s_{\epsilon_1}s_{\epsilon_2}\gamma, \alpha \leftrightarrow \emptyset)), 3 \notin \tau = \{1,2\},$$

and we find that

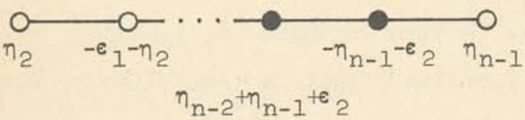
$$U_3(\overline{\pi}(s_{\eta_3+\epsilon_2}s_{\epsilon_1}s_{\epsilon_2}\gamma, \alpha \leftrightarrow \emptyset)) = \overline{\pi}(s_{\eta_3+\epsilon_2}s_{\epsilon_1}s_{\epsilon_2}\gamma, \alpha \leftrightarrow \eta_3+\epsilon_2).$$

The τ-invariant of the term on the right turns out to be $\{1,3\}$, while $\tau(\overline{\pi}(s_{\epsilon_1}s_{\epsilon_2}\gamma, \alpha \leftrightarrow \alpha))$ is $\{2,3\}$. Thus $c_4 = 0$ when $n = 4$.

Now suppose $n > 4$. Then

$$c_4 = \text{mult } \overline{\pi}(s_{\eta_{n-1}+\epsilon_2} s_{\epsilon_1+\eta_2} s_{\epsilon_1} s_{\epsilon_2} \gamma, \alpha \leftrightarrow \alpha) \text{ in } U_1(\overline{\pi}(s_{\epsilon_1} s_{\epsilon_2} \gamma, \alpha \leftrightarrow \alpha)),$$
$$1 \notin \tau = \{2,3,\ldots,n-1\},$$
$$= \text{mult } \overline{\pi}(s_{\epsilon_1} s_{\epsilon_2} \gamma, \alpha \leftrightarrow \alpha) \text{ in } U_2(\overline{\pi}(s_{\eta_{n-1}+\epsilon_2} s_{\epsilon_1+\eta_2} s_{\epsilon_1} s_{\epsilon_2} \gamma, \alpha \leftrightarrow \alpha)),$$
$$2 \notin \tau = \{1,3,\ldots,n-2,n\},$$

the diagram for the latter τ-invariant being



$\alpha = (3) + \ldots + (n-2)$

Here

$$U_2(\overline{\pi}(s_{\eta_{n-1}+\epsilon_2} s_{\epsilon_1+\eta_2} s_{\epsilon_1} s_{\epsilon_2} \gamma, \alpha \leftrightarrow \alpha) = \overline{\pi}(s_{\eta_{n-1}+\epsilon_2} s_{\epsilon_1} s_{\epsilon_2} \gamma, \alpha \leftrightarrow \alpha) + \Theta_3 \tag{2.16}$$

with each term of Θ_3 contained in

$$\pi(s_{\eta_{n-1}+\epsilon_2} s_{\epsilon_1+\eta_2} s_{\epsilon_1} s_{\epsilon_2} \gamma, \alpha \leftrightarrow \alpha). \tag{2.17}$$

Our inductive assumption for $n-4$ (or a Schmid identity if $n=5$) shows that 1 is in the τ-invariant of each term of (2.17), while 1 is not in the τ-invariant of $\overline{\pi}(s_{\epsilon_1} s_{\epsilon_2} \gamma, \alpha \leftrightarrow \alpha)$. So $\overline{\pi}(s_{\epsilon_1} s_{\epsilon_2} \gamma, \alpha \leftrightarrow \alpha)$ does not occur in Θ_3. Finally n is in the τ-invariant of the first term on the right of (2.16) but is not in the τ-invariant of $\overline{\pi}(s_{\epsilon_1} s_{\epsilon_2} \gamma, \alpha \leftrightarrow \alpha)$. Thus $c_4 = 0$ when $n > 4$.

So all cases $n \geq 4$ have

$$\Theta_2 = \overline{\pi}(s_{\epsilon_1+\eta_2} s_{\epsilon_2} s_{\epsilon_1} \gamma, \alpha \leftrightarrow \alpha). \tag{2.18}$$

Finally we apply s_1 to both sides of (2.13), use the identity

$$s_1 \pi(s_{\epsilon_1} \gamma, \alpha \leftrightarrow \alpha) = \pi(\gamma, \alpha \leftrightarrow \alpha)$$

given in Theorem 1.5, and substitute from (2.14), (2.15), and (2.18) to obtain (2.2) for n . This completes the induction and the proof of the theorem.

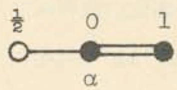
3. SO(2n-1,2)

For SO(2n-1,2) we shall consider two sets of examples. For each we specify λ by attaching $2\langle\lambda,\beta\rangle/|\beta|^2$ to each simple root β in the Dynkin diagram of type B_n. The black simple roots are the noncompact ones, and P is built from α. We number the simple roots from 1 to n, with n denoting the short simple root. The theorem in each case is that $U(P,\sigma,\nu)$ is irreducible for the indicated value of ν.

a. First set of examples

$n = 3$

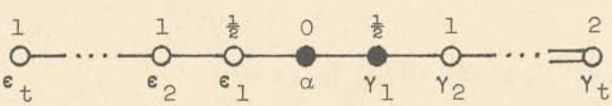
λ :



tangent case at $\nu = \frac{1}{2}\alpha$

n odd ≥ 5, $t = \frac{1}{2}(n-1)$

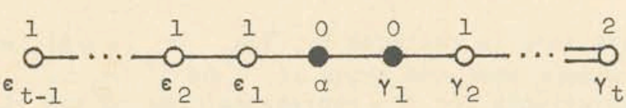
λ :



tangent case at $\nu = (t - \frac{1}{2})\alpha$

n even ≥ 4, $t = \frac{1}{2}n$

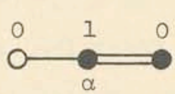
λ :



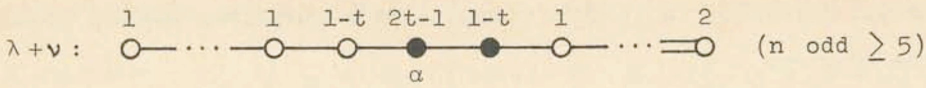
cotangent case at $\nu = (t-1)\alpha$

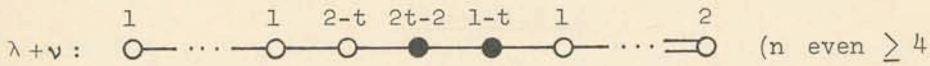
The diagrams for $\lambda + \nu$ are

$\lambda + \nu$:

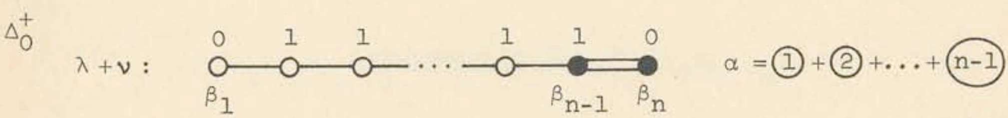


$(n = 3)$





For $n = 3$, we let $t = 1$. In the same way as at the end of Section 1, we apply a sequence of complex root reflections to each of these diagrams. When n is odd, the sequence is $s_{2t} \cdots s_{t+2} s_1 \cdots s_{t-1} s_t$. When n is even, the sequence is $s_{2t-1} \cdots s_{t+1} s_1 \cdots s_{t-1}$. In all cases the resulting diagram is



This Δ_0^+ is a system compatible with $\Delta^+(\mathfrak{m}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$ that makes $\lambda + \nu$ dominant. We take it as the system in which γ_0 is to be dominant. The set of singular roots is $\{1, n\}$.

The nature of α allows us to apply Theorem 2.1 immediately to obtain

$$\pi(\gamma_0, \alpha \leftrightarrow \alpha) = \bar{\pi}(\gamma_0, \alpha \leftrightarrow \alpha) + \bar{\pi}(s_{\beta_1} \gamma_0, \alpha \leftrightarrow \alpha)$$

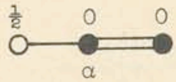
$$+ \bar{\pi}(s_{\beta_{n-1}} \gamma_0, \alpha \leftrightarrow \alpha) + \bar{\pi}(s_{\beta_1} s_{\beta_{n-1}} \gamma_0, \alpha \leftrightarrow \alpha), \qquad (3.1)$$

except that the last term is replaced by $\bar{\pi}(s_{\beta_{n-1}} \gamma_0, \alpha \leftrightarrow \emptyset)$ when $n = 3$. The respective τ-invariants for the representations on the right side are $\{1, \ldots, n-1\}$, $\{2, \ldots, n-1\}$, $\{1, \ldots, n-2, n\}$, and $\{2, \ldots, n-2, n\}$. The only one of these that is disjoint from the set $\{1, n\}$ of singular roots is the second one. By Theorem 1.3, the only term on the right side of (3.1) with nonzero image under the ψ functor is the second one, and its image is irreducible. Therefore $U(P, \sigma, \nu)$ is irreducible.

b. Second set of examples

$n = 3$

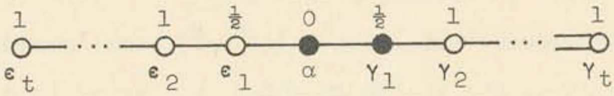
λ :



tangent case at $\nu = \frac{1}{2}\alpha$

n odd ≥ 5, $t = \frac{1}{2}(n-1)$

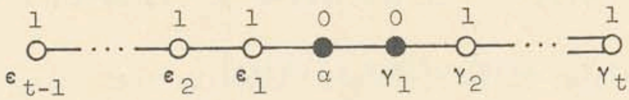
λ :



tangent case at $\nu = (t - \frac{1}{2})\alpha$

n even ≥ 4, $t = \frac{1}{2}n$

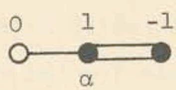
λ :



cotangent case at $\nu = (t-1)\alpha$

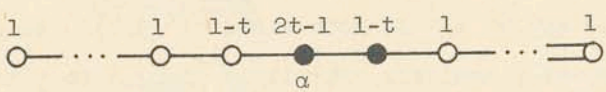
The diagrams for $\lambda + \nu$ are

$\lambda + \nu$:



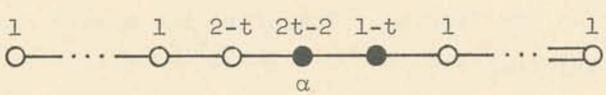
$(n = 3)$

$\lambda + \nu$:



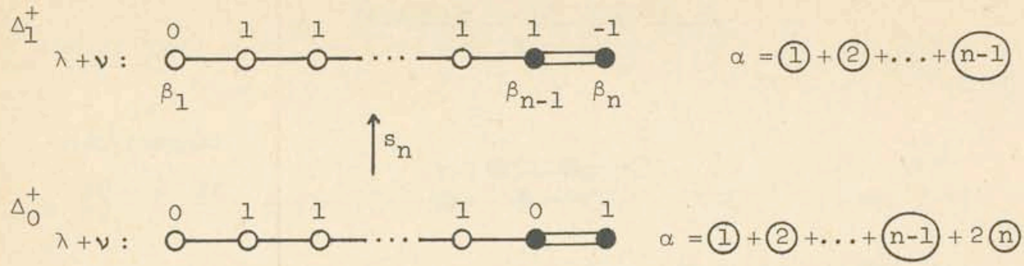
(n odd ≥ 5)

$\lambda + \nu$:



(n even ≥ 4)

For $n = 3$, we let $t = 1$. We apply the same sequence of complex root reflections as in Section 3a, obtaining a system Δ_1^+, and then we apply the reflection s_n to Δ_1^+ to obtain Δ_0^+. The diagrams in all cases are



This Δ_0^+ is a system compatible with $\Delta^+(\mathfrak{m}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$ that makes $\lambda+\nu$ dominant. We take it as the system in which γ_0 is to be dominant. The set of singular roots is $\{1,n-1\}$.

We can apply Theorem 2.1 in the system Δ_1^+ to obtain

$$\pi(s_n\gamma_0, \mathfrak{a} \leftrightarrow \alpha) = \bar{\pi}(s_n\gamma_0, \mathfrak{a} \leftrightarrow \alpha) + \bar{\pi}(s_1 s_n\gamma_0, \mathfrak{a} \leftrightarrow \alpha)$$
$$+ \bar{\pi}(s_{n-1}s_n\gamma_0, \mathfrak{a} \leftrightarrow \alpha) + \bar{\pi}(s_1 s_{n-1}s_n\gamma_0, \mathfrak{a} \leftrightarrow \alpha) , \qquad (3.2)$$

except that the last term is replaced by $\bar{\pi}(s_{n-1}s_n\gamma_0, \mathfrak{a} \leftrightarrow \emptyset)$ when $n=3$. The respective τ-invariants for the representations on the right side are $\{1,\ldots,n-1\}$, $\{2,\ldots,n-1\}$, $\{1,\ldots,n-2,n\}$, and $\{2,\ldots,n-2,n\}$. We apply s_n to both sides of (3.2). We could go through the step-by-step analysis, but it is simpler to observe that everything that happens in the computation is oblivious to the presence of the double line in the diagram. Therefore the answer has to be of the form given in Theorem 2.1:

$$\pi(\gamma_0, \mathfrak{a} \leftrightarrow \alpha) = \bar{\pi}(\gamma_0, \mathfrak{a} \leftrightarrow \alpha) + \bar{\pi}(s_1\gamma_0, \mathfrak{a} \leftrightarrow \alpha)$$
$$+ \bar{\pi}(s_n\gamma_0, \mathfrak{a} \leftrightarrow \alpha) + \bar{\pi}(s_1 s_n\gamma_0, \mathfrak{a} \leftrightarrow \alpha) .$$

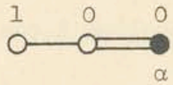
The respective τ-invariants for the representations on the right side are $\{1,\ldots,n-2,n\}$, $\{2,\ldots,n-2,n\}$, $\{1,\ldots,n-1\}$, and $\{2,\ldots,n-1\}$. The only one of these that is disjoint from the set $\{1,n-1\}$ of singular roots is the second one. Thus only the image of

$\overline{\pi}(s_1\gamma_0, \alpha \leftrightarrow \alpha)$ is nonzero when we apply the ψ functor, and its image is irreducible. Therefore $U(P,\sigma,\nu)$ is irreducible.

4. $Sp(3,\mathbb{R})$

For $Sp(3,\mathbb{R})$ we shall specify λ by attaching $2\langle\lambda,\beta\rangle/|\beta|^2$ to each simple root β in the Dynkin diagram of type C_3. The black simple roots are the noncompact ones, and P is built from α.

λ :

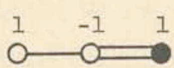


cotangent case
at $\nu = \frac{1}{2}\alpha$

We shall prove that $U(P,\sigma,\nu)$ is irreducible for the indicated value of ν.

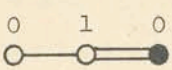
We number the roots as $\{1,2,3\}$ from left to right. The diagram of $\lambda+\nu$ is

$\lambda+\nu$:



$\alpha = ③$ Δ_1^+

Put $\Delta_0^+ = s_2\Delta_1^+$. The picture is



$\alpha = 2② + ③$ $\Delta_0^+ = s_2\Delta_1^+$

We take Δ_0^+ as the system in which γ_0 is dominant. The set of singular roots is $\{1,3\}$. We do not have an immediately available character identity in Δ_0^+ but have one in Δ_1^+. Here α does not satisfy the parity condition, and Theorem 1.2 says that

$$\pi(s_2\gamma_0, \alpha \leftrightarrow \alpha) = \overline{\pi}(s_2\gamma_0, \alpha \leftrightarrow \alpha). \tag{4.1}$$

The τ-invariant for the right side is $\{1\}$, and we find

$$s_2\overline{\pi}(s_2\gamma_0, \mathfrak{a} \leftrightarrow \alpha) = \overline{\pi}(s_2\gamma_0, \mathfrak{a} \leftrightarrow \alpha) + \overline{\pi}(\gamma_0, \mathfrak{a} \leftrightarrow \alpha) + \Theta .$$

Here the constituents of Θ must occur on the right side of (4.1) and must have 2 in their τ-invariants. So $\Theta = 0$. Therefore

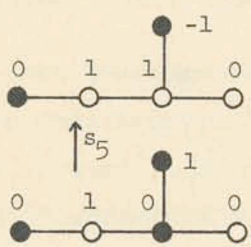
$$\pi(\gamma_0, \mathfrak{a} \leftrightarrow \alpha) = s_2\pi(s_2\gamma_0, \mathfrak{a} \leftrightarrow \alpha) = s_2\overline{\pi}(s_2\gamma_0, \mathfrak{a} \leftrightarrow \alpha)$$
$$= \overline{\pi}(s_2\gamma_0, \mathfrak{a} \leftrightarrow \alpha) + \overline{\pi}(\gamma_0, \mathfrak{a} \leftrightarrow \alpha) .$$

The τ-invariants for the two terms on the right side are $\{1\}$ and $\{2\}$. Only the second of these is disjoint from the set $\{1,3\}$ of singular roots, and it follows just as in Section 3 that $U(P,\sigma,\nu)$ is irreducible.

5. $SO^*(10)$

For $SO^*(10)$, we specify λ as at the end of Section 1. This is a cotangent case and we treat $\nu = \alpha$. We prove that $U(P,\sigma,\nu)$ is irreducible for this value of ν.

We number the roots as at the end of Section 1, and the diagram for $\lambda + \nu$ is what is called Δ_2^+ there. It is a little more convenient to define Δ_1^+ as $s_3 s_1 \Delta_2^+$. We continue with $\Delta_0^+ = s_5 \Delta_1^+$. Then our diagrams are



Δ_1^+ $\alpha =$ ① + ② + ③

Δ_0^+ $\alpha =$ ① + ② + ③ + ⑤

We use Δ_0^+ to define γ_0. The set of singular roots is $\{1,3,4\}$.

We can apply Theorem 2.1 in the system Δ_1^+ to obtain

$$\pi(s_5\gamma_0, \mathfrak{a} \leftrightarrow \alpha) = \overline{\pi}(s_5\gamma_0, \mathfrak{a} \leftrightarrow \alpha) + \overline{\pi}(s_1 s_5\gamma_0, \mathfrak{a} \leftrightarrow \alpha)$$

$$+ \overline{\pi}(s_3 s_5\gamma_0, \mathfrak{a} \leftrightarrow \alpha) + \overline{\pi}(s_1 s_3 s_5\gamma_0, \mathfrak{a} \leftrightarrow \alpha) . \quad (5.1)$$

The respective τ-invariants of the terms on the right are $\{1,2,3\}$, $\{2,3\}$, $\{1,2,4\}$, and $\{2,4\}$, and 5 is not in any of these. But even more, we see from Theorem 2.1 that 5 is not in the τ-invariant of any constituent of the corresponding representations $\pi(\ldots, \mathfrak{a} \leftrightarrow \alpha)$ for the members of the right side of (5.1). This says that all the extra Θ terms are 0 when we apply s_5 to the terms on the right side of (5.1). Consequently application of s_5 to (5.1) gives exactly

$$\pi(\gamma_0, \mathfrak{a} \leftrightarrow \alpha) = (\text{same 4 terms as in (5.1)})$$

$$+ \overline{\pi}(\gamma_0, \mathfrak{a} \leftrightarrow \alpha) + \overline{\pi}(s_1\gamma_0, \mathfrak{a} \leftrightarrow \alpha)$$

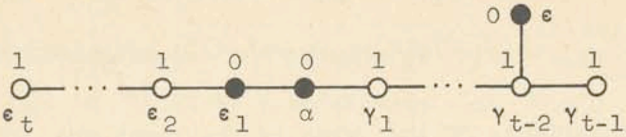
$$+ \overline{\pi}(s_5 s_3 s_5\gamma_0, \mathfrak{a} \leftrightarrow \alpha, \beta) + \overline{\pi}(s_5 s_1 s_3 s_5\gamma_0, \mathfrak{a} \leftrightarrow \alpha, \beta) .$$

Here β denotes the noncompact root in position 5 for the positive system in question. We readily compute that the respective τ-invariants of our four new terms are $\{1,2,5\}$, $\{2,5\}$, $\{1,2,4,5\}$, and $\{2,4,5\}$. Of our eight τ-invariants, the only one that fails to meet the set $\{1,3,4\}$ of singular roots is $\{2,5\}$. Thus only the image of $\overline{\pi}(s_1\gamma_0, \mathfrak{a} \leftrightarrow \alpha)$ is nonzero when we apply the ψ functor, and it follows as in Sections 3 and 4 that $U(P,\sigma,\nu)$ is irreducible.

6. $SO^*(2n)$

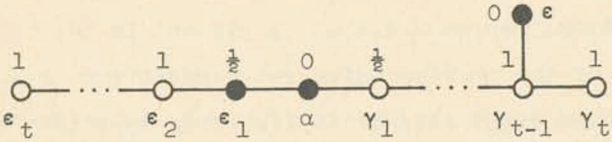
For $SO^*(2n)$ with $n \geq 6$, we shall specify λ by attaching $2\langle \lambda, \beta\rangle / |\beta|^2$ to each simple root β in the Dynkin diagram of type D_n. The black simple roots are the noncompact ones, and P is built from α.

n odd , $t = \frac{1}{2}(n-1)$

λ :


cotangent case at $\nu = (t-1)\alpha$

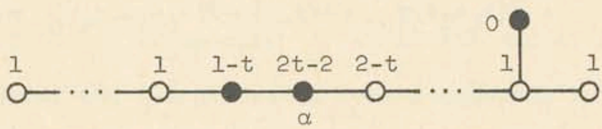
n even , $t = \frac{1}{2}(n-2)$

λ :


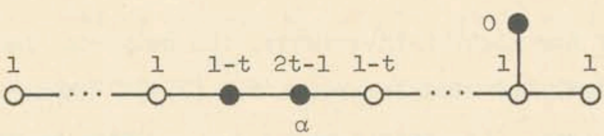
tangent case at $\nu = (t-\frac{1}{2})\alpha$

We shall prove that $U(P,\sigma,\nu)$ is irreducible for the indicated values of ν.

We number the roots on the horizontal as $\{1,\ldots,n-1\}$ and denote by n the root extending upward. The diagrams for $\lambda+\nu$ are

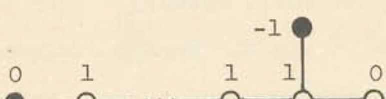
$\lambda+\nu$:


(n odd)

$\lambda+\nu$:


(n even)

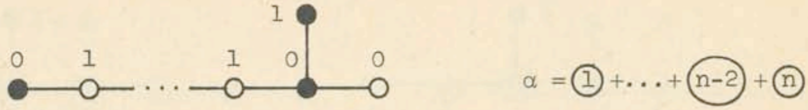
We apply a sequence of complex root reflections to one or the other of these diagrams. When n is odd, the sequence is $s_{2t-1}\cdots s_{t+2}s_1\cdots s_t$. When n is even the sequence is $s_{2t}\cdots s_{t+2}s_1\cdots s_t$. In both cases the resulting diagram is

Δ_1^+


$\alpha = (1) + \ldots + (n-2)$

We let $\Delta_0^+ = s_n \Delta_1^+$ and use Δ_0^+ to define γ_0. The diagram is

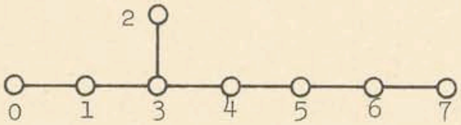
Δ_0^+



and the set of singular roots is $\{1, n-2, n-1\}$. The argument now proceeds just as with $SO^*(10)$ in Section 5, and the result is that $U(P,\sigma,\nu)$ is irreducible.

7. Groups of type E

For groups of type E, we shall consider 13 specific examples, of which 2 are in E_6, 5 are in E_7, and 6 are in E_8. The theorem is that $U(P,\sigma,\nu)$ is irreducible in all 13 cases. Our numbering of the roots in E_8 is



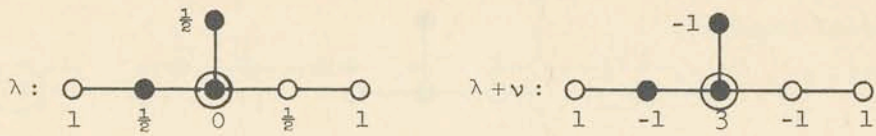
and we drop root 7 or roots 6 and 7 in E_7 or E_6.

For each example, we state what ν is, and we give two diagrams, the left one for the m parameter λ (with $2\langle\lambda,\beta\rangle/|\beta|^2$ attached to the simple root β) and the right one for the infinitesimal character $\lambda+\nu$. The black simple roots are the noncompact ones, α is circled, and P is built from α.

In addition, we give the sequence of reflections used to pass to Δ_0^+. A vertical line in the middle indicates the stage at which we apply Theorem 2.1. (Thus effectively we have only to implement wall crossings for reflections to the left of this line.) With each example we list the set of singular roots.

The 13 examples are listed below. After giving the list, we shall discuss the proof of irreducibility.

(a) E_6 with $\nu = \frac{3}{2}\alpha$.

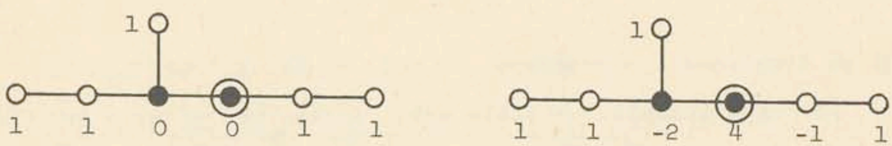


Reflections: $s_2 | s_0 s_1 s_5 s_4$. Singular set: $\{1,3,5\}$.

(b) E_6 with $\nu = 2\alpha$.

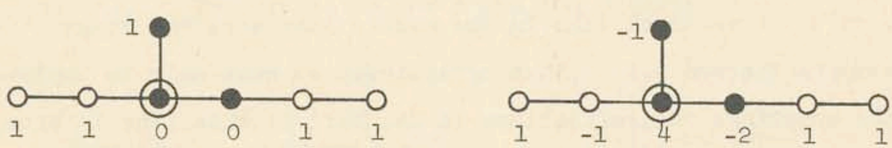
Reflections: $s_3 s_2 | s_0 s_1 s_5 s_4$. Singular set: $\{1,4,5\}$.

(c) E_7 with $\nu = 2\alpha$.



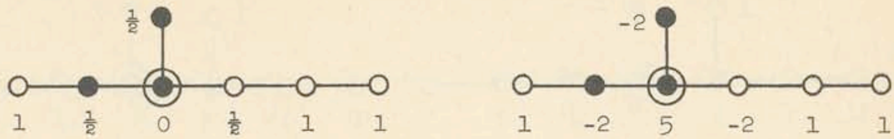
Reflections: $s_2 | s_0 s_1 s_3 s_5$. Singular set: $\{0,3,6\}$.

(d) E_7 with $\nu = 2\alpha$.



Reflections: $s_2 | s_0 s_1 s_5 s_4$. Singular set: $\{0,3,6\}$.

(e) E_7 with $\nu = \frac{5}{2}\alpha$.

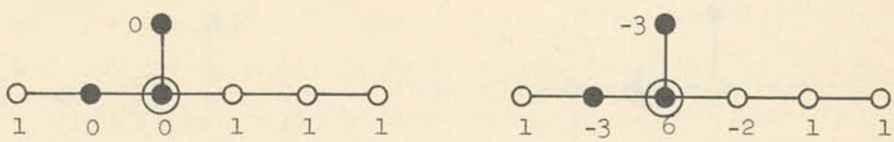


Reflections: $s_3 s_2 | s_0 s_1 s_5 s_4$. Singular set: $\{1,4,6\}$.

(f) E_7 with $\nu = 2\alpha$.

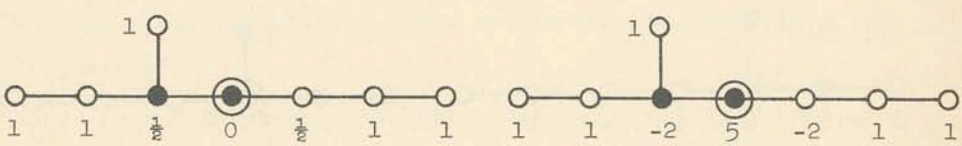
Reflections: $s_6 s_3 s_2 | s_5 s_1 s_3$. Singular set: $\{0,1,4,5\}$.

(g) E_7 with $\nu = 3\alpha$.



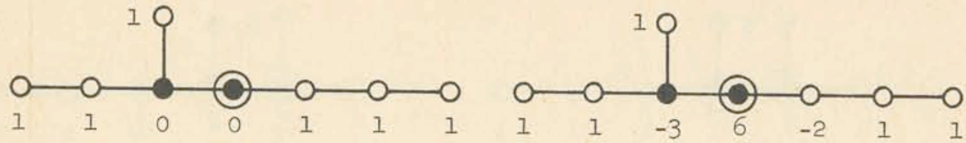
Reflections: $s_4 s_1 s_3 s_2 | s_0 s_1 s_5 s_4$. Singular set: $\{3,5,6\}$.

(h) E_8 with $\nu = \frac{5}{2}\alpha$.



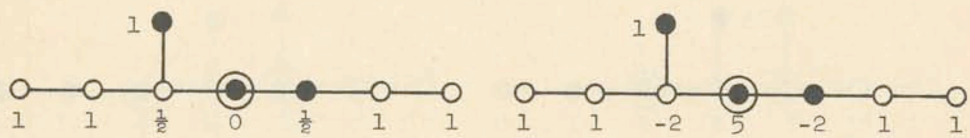
Reflections: $s_2 | s_0 s_1 s_3 s_6 s_5$. Singular set: $\{0,3,7\}$.

(i) E_8 with $\nu = 3\alpha$.



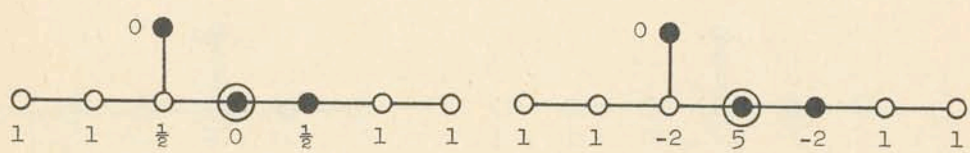
Reflections: $s_3 s_2 | s_0 s_1 s_3 s_6 s_5$. Singular set: $\{1,4,7\}$.

(j) E_8 with $\nu = \frac{5}{2}\alpha$.



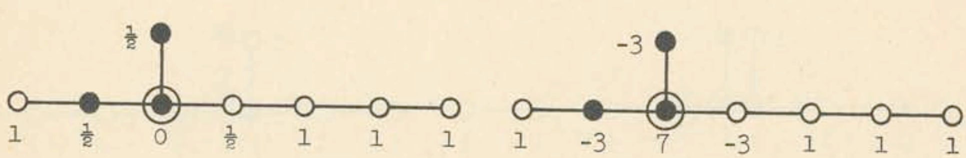
Reflections: $s_2 | s_1 s_3 s_6 s_5$. Singular set: $\{0,3,7\}$.

(k) E_8 with $\nu = \frac{5}{2}\alpha$.



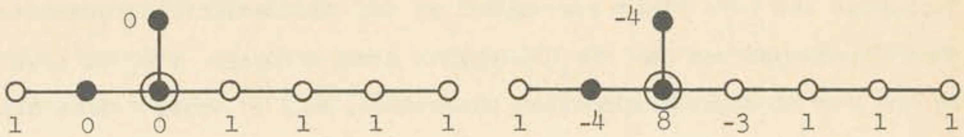
Reflections: $s_3 s_2 | s_0 s_1 s_3 s_6 s_5$. Singular set: $\{0,1,4,7\}$.

(ℓ) E_8 with $\nu = \frac{7}{2}\alpha$.



Reflections: $s_4 s_1 s_3 s_2 | s_0 s_1 s_6 s_5 s_4$. Singular set: $\{3,5,7\}$.

(m) E_8 with $\nu = 4\alpha$.



Reflections: $s_3 s_5 s_4 s_1 s_3 s_2 | s_0 s_1 s_6 s_5 s_4$. Singular set: $\{2,4,6,7\}$.

As we said, $U(P,\sigma,\nu)$ is irreducible in all 13 cases. We turn to the proof. Let Δ^+ be the positive system indicated above, let w be the product of the reflections to the right of the vertical line, let n be the number of reflections to the left of the vertical line, and put $\Delta_n^+ = w\Delta^+$. Then define $\Delta_{n-1}^+, \ldots, \Delta_0^+$ by applying one at a time the reflections that are to the left of the vertical line. Then we can define $\gamma_0 = \lambda + \nu + \mu$ as in Section 1, and we can reflect it to obtain γ_j dominant for Δ_j^+, $0 \leq j \leq n$. Theorem 2.1 enables us to decompose $\pi(\gamma_n, a \leftrightarrow \alpha)$ as the sum of four irreducible characters.

In every case, the first reflection to the left of the vertical line is s_2. Moreover, the root 2 is not in the τ-invariant of any of the four irreducible characters, nor is it in the τ-invariant of any constituent of any of the corresponding π's for these four characters. Applying s_2, we then obtain a decomposition of $\pi(\gamma_{n-1}, a \leftrightarrow \alpha)$ as the sum of eight irreducible characters. (This is all very similar to what happened in Section 5.)

When $n = 1$, we have only to check that the singular set meets the τ-invariant of 7 of these 8 characters, and then we have irreducibility for $U(P,\sigma,\nu)$. Thus we are done in cases (a), (c), (d), (h), and (j).

When $n = 2$, the next (and last) reflection is s_3, and we prepare to apply s_3 to our expansion of $\pi(\gamma_1, a \leftrightarrow \alpha)$ into the sum of eight irreducible characters. The four characters that occurred also in $\pi(\gamma_2, a \leftrightarrow \alpha)$ have 3 in their τ-invariants and may be disregarded. In order to handle the Θ terms, it is necessary to decompose the π's that correspond to the four additional characters that first appeared in $\pi(\gamma_1, a \leftrightarrow \alpha)$. Each of these π's is seen to be the sum of eight irreducible characters, all of whose τ-invariants meet the singular set. Therefore all of the Θ terms may be disregarded. We thus need to consider only our four new characters and their main new terms under U_3. Of these 8 characters, 7 have τ-invariants that meet the singular set. Thus $U(P,\sigma,\nu)$ is irreducible in cases (b), (e), (i), and (k).

In cases (g) and (ℓ), we have $n = 4$. The reflections to the left of the vertical line are $s_4 s_1 s_3 s_2$, and it is necessary to calculate the decomposition of $\pi(\gamma_2, a \leftrightarrow \alpha) = s_3 s_2 \pi(\gamma_4, a \leftrightarrow \alpha)$ exactly. This calculation is more complicated in case (ℓ) than in case (g) but can be done without tools more advanced than Theorem 1.7. One handles the last two wall crossings in the spirit of the previous paragraph, discarding as early as possible any terms that will not affect the final irreducibility. The details are fairly long and will be omitted, but the result is that $U(P,\sigma,\nu)$ is irreducible in cases (g) and (ℓ).

Case (f) has $n = 3$. The reflections to the left of the vertical line are $s_6 s_3 s_2$, and we calculate $\pi(\gamma_1, a \leftrightarrow \alpha) = s_3 s_2 \pi(\gamma_3, a \leftrightarrow \alpha)$ as in case (g) above. The resulting character identity involves ten irreducible characters. The root 6 is not in the τ-invariant of any of these ten characters, nor is it in the τ-invariant of any constituent of the corresponding π's for these ten characters. Applying s_6, we obtain a decomposition of $\pi(\gamma_0, a \leftrightarrow \alpha)$ as the sum of 20 irreducible characters. The singular set meets the τ-invariant of 19 of these 20 characters, and thus $U(P,\sigma,\nu)$ is irreducible in case (f).

Finally we consider case (m), in which $n = 6$ and the reflections are $s_3 s_5 s_4 s_1 s_3 s_2$. Here we calculate $\pi(\gamma_3, a \leftrightarrow \alpha) = s_1 s_3 s_2 \pi(\gamma_6, a \leftrightarrow \alpha)$ exactly, using Theorem 1.7, and we calculate $\pi(\gamma_2, a \leftrightarrow \alpha)$

$= s_4\pi(\gamma_3, \mathfrak{a} \leftrightarrow \alpha)$ except for two $\otimes$ terms. One handles the final two wall crossings in the spirit of the cases $n = 2$ and $n = 4$. The details, however, are much more complicated in this situation. But at any rate the result is that $U(P, \sigma, \nu)$ is irreducible in case (m).

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