IRREDUCIBILITY THEOREMS FOR THE PRINCIPAL SERIES

by

A.W. Knapp* and E.M. Stein+

Cornell University and Princeton University

1. Introduction

In earlier work [6] we have developed a class of intertwining integrals for semisimple Lie groups. These operators exhibit various members of the principal series of representations as unitarily equivalent in a way that mirrors the action of the Weyl group. Where members of the Weyl group act with fixed points, the operators give self-equivalences of representations of the principal series and thereby provide information about reducibility. One of the main results of the present announcement is that for (at least) some of these groups, the operators actually give complete information about reducibility of principal series representations.

To be more specific, let G be a connected semisimple Lie group of matrices and Let MAN be a minimal parabolic subgroup. Here M is compact, A is a vector group, and N is nilpotent. (For details of the notation, see §6 of [6].) The principal series consists of those representations $U(\sigma,\lambda)$ of G obtained by inducing from MAN the finite-dimensional representation $man \to \lambda(a)\sigma(m)$, where σ is an irreducible unitary representation of M and λ is a unitary character of A.

Let $W = M'/M$ be the Weyl group relative to A. The members w of M' act on representations of M and characters of A by $w\sigma(m) = \sigma(w^{-1}mw)$ and $w\lambda(m) = \lambda(w^{-1}mw)$. A central result of [6] is that, corresponding to each triple (w,σ,λ), there is a unitary

*Supported partly by a research associateship at Princeton University and partly by NSF Grant GP 28251 at Cornell University.
+National Science Foundation fellow.

operator $\mathcal{A}(w,\sigma,\lambda)$ with the property that

(1.1) $$U(w\sigma, w\lambda)\,\mathcal{A}(w,\sigma,\lambda) = \mathcal{A}(w,\sigma,\lambda)U(\sigma,\lambda).$$

The dependence of these operators on λ is holomorphic (in a neighborhood of λ unitary), and they satisfy a cocycle relation

(1.2) $$\mathcal{A}(w_1w_2,\sigma,\lambda) = \mathcal{A}(w_1, w_2\sigma, w_2\lambda)\,\mathcal{A}(w_2,\sigma,\lambda).$$

Fix (w,σ,λ) and suppose that $w\sigma$ is equivalent with σ and that $w\lambda = \lambda$. Then it is possible to extend σ to a representation of the subgroup of M' generated by M and w. (The enlarged σ operates on the same vector space as before.) With $\sigma(w)$ defined in this way, (1.1) yields

(1.3) $$U(\sigma,\lambda)[\sigma(w)\,\mathcal{A}(w,\sigma,\lambda)] = [\sigma(w)\,\mathcal{A}(w,\sigma,\lambda)]U(\sigma,\lambda).$$

If $\sigma(w)\,\mathcal{A}(w,\sigma,\lambda)$ is not scalar, then (1.3) exhibits $U(\sigma,\lambda)$ as reducible. With σ and λ fixed, we shall call the set of all such operators $\sigma(w)\,\mathcal{A}(w,\sigma,\lambda)$ the <u>set of intertwining operators for (σ,λ)</u>. This paper deals with the following two problems:

(1) Normally many of the operators $\sigma(w)\,\mathcal{A}(w,\sigma,\lambda)$ coincide. Give an explicit description of the distinct operators in the set.

(2) Decide whether the linear span of the set of intertwining operators for (σ,λ) is the entire set of bounded operators L such that $U(\sigma,\lambda)L = LU(\sigma,\lambda)$.

For G of real-rank one, problem 1 is solved by Theorem 5 of [6], and the question raised in problem 2 is answered affirmatively by the proof of Proposition 20. For G of higher real-rank, the two problems ostensibly are independent. However, progress by our methods on the second problem for a given G has occurred only after the first problem was solved for G. Our main conjectures are as follows:

<u>Conjecture 1</u>. <u>Let</u> $W_{\sigma,\lambda}$ <u>be the subgroup of elements</u> w <u>of</u> W

such that $w\sigma$ is equivalent with σ and $w\lambda = \lambda$. There exist subgroups W' and R of $W_{\sigma,\lambda}$ such that W' is abstractly isomorphic to a Weyl group, R is a direct sum of copies of $\mathbb{Z}_2$, W' is the subgroup on which $\sigma(w)\mathcal{A}(w,\sigma,\lambda)$ is scalar, $W_{\sigma,\lambda}$ is the semidirect product $W_{\sigma,\lambda} = W'R$ with W' normal, and the set of all operators $\sigma(r)\mathcal{A}(r,\sigma,\lambda)$ for r in R is linearly independent.

Conjecture 2. In every case the intertwining operators for (σ,λ) do span the space of bounded operators L such that $U(\sigma,\lambda)L = LU(\sigma,\lambda)$.

The first conjecture if true is a sufficiently precise answer to problem (1) provided the subgroups W' and R are defined explicitly enough.

As evidence for these conjectures we have the following new results:

(1a) a proof of Conjecture 1, together with an explicit description of W' and R, for the case that G is split over $\mathbb{R}$ and σ and λ are arbitrary. See §2.

(1b) a proof of part of Conjecture 1 for general G. In this case the subgroups W' and R are not defined explicitly enough to provide a useful solution to problem (1). See the end of §2.

(2) a proof of Conjecture 2 when G has real-rank 2 and when $G = SL(n,\mathbb{R})$. The method in these cases applies to other groups as well; no group is known for which it fails. However, we as yet do not have an argument that works simultaneously for all G. See §3 and §4.

We should mention that the solution (1b) shows that the basic intertwining operators $\sigma(r)\mathcal{A}(r,\sigma,\lambda)$ are those whose normalizing factors (see §18 of [6]) are regular at λ. If Conjecture 2 is true, it would appear that the order of R (and hence the decision between reducibility and irreducibility) could be expressed in terms of the Plancherel measure and similar quantities (cf. Theorem 5 of [6]).

Other authors have worked on the problem of deciding irreducibility of principal series. In addition to [6], one should consult Gelfand-Graev [4], Bruhat [3], Kostant [7], Helgason [5], Zelobenko [12], and Wallach [11].

2. Operators when G is split over $\mathbb{R}$

In this section we assume that the group G, satisfying the conditions of §1, has the further property of being split over $\mathbb{R}$. Then the Lie algebra $\mathfrak{A}$ of A is a Cartan subalgebra, $\mathfrak{m}$ is 0, and M is a finite abelian group. To simplify the exposition, we shall assume that G is simple, so that $\mathfrak{G}$ is completely determined by one of the standard Dynkin diagrams.

From work of Satake [8, p. 93], for example, M is completely understood. For each root α let H_α be the member of $\mathfrak{A}$ corresponding to α and let $H'_\alpha = 2\langle\alpha,\alpha\rangle^{-1}H_\alpha$. Set $\gamma_\alpha = \exp \pi i H'_\alpha$. The element γ_α is in M and has order 1 or 2. Let $\varepsilon_1,\ldots,\varepsilon_n$ be the simple roots. Then the elements γ_{ε_i}, $1 \leq i \leq n$, generate M.

Let σ be an irreducible unitary representation of M. Then σ is one-dimensional and $\sigma(\gamma_\alpha) = \pm 1$ for each root α. In view of the remarks above, σ is completely determined by specifying which simple roots ε_i satisfy $\sigma(\gamma_{\varepsilon_i}) = -1$.

As in §1, the group M' acts on the representations σ and characters λ. Since σ is one-dimensional, equivalence becomes identity, and the action of M' reduces to an action of W. Then our concern is with the subgroup $W_{\sigma,\lambda}$ of elements of W that leave σ and λ fixed. Again since σ is one-dimensional, we can disregard $\sigma(w)$ in (1.3). By (1.3) the operators $\mathcal{A}(w,\sigma,\lambda)$ for w in $W_{\sigma,\lambda}$ commute with the principal series representation $U(\sigma,\lambda,x)$. Normally many of these operators coincide. Our problem in this section is to give an explicit description of the distinct operators in the set. The case that σ is trivial is of no interest for the problem since $\mathcal{Q}(w,1,\lambda)$ is easily seen to be scalar if w is in $W_{1,\lambda}$.

For simplicity we shall assume until after Theorem 1 that the character λ of A is trivial. Let $W_\sigma = W_{\sigma,1}$.

We say that the representation σ of M is fundamental if there is exactly one simple root ϵ_k such that $\sigma(\gamma_{\epsilon_k}) = -1$. In this case we write $\sigma = \sigma_k$. The condition that σ be fundamental for G is a mod 2 analog of the condition for G^C that an integral form be dominant.

Proposition 2.1. Each nontrivial representation σ of M is equivalent under W with a fundamental representation.

That is, there is a p in W and there is a k such that $p\sigma = \sigma_k$. Now it is shown in [6] that $\mathcal{Q}(p,\sigma,1)$ is a unitary equivalence of $U(\sigma,1)$ and $U(\sigma_k,1)$, and it is easy to see from Theorem 7 of [6] that $\mathcal{Q}(p,\sigma,1)$ conjugates the intertwining operators $\{\mathcal{Q}(w,\sigma,1) \mid w \in W_\sigma\}$ into

$$\{\mathcal{Q}(w',\sigma_k,1) \mid w' \in W_{\sigma_k} = p\, W_\sigma\, p^{-1}\}.$$

Thus if we characterize the intertwining operators for σ_k, we have characterized them for σ. So for the rest of the section we assume σ is fundamental. Say $\sigma = \sigma_k$.

Let Δ and Π be, respectively, the roots and simple roots for G. Consider the following conditions on a positive non-simple root α:

(i) $\sigma(\gamma_\alpha) = 1$.

(ii) $\sigma(\gamma_\beta) = -1$ for every $\beta > 0$ different from α such that $p_\alpha\beta < 0$.

(ii') $\langle\alpha,\epsilon_i\rangle \leq 0$ for $i \neq k$.

Here (ii) implies (ii'). [In fact, if $\langle\alpha,\epsilon_i\rangle > 0$, then $p_\alpha \epsilon_i < 0$ and (ii) gives $\sigma(\gamma_{\epsilon_i}) = -1$. But $\sigma = \sigma_k$ and $i \neq k$ imply $\sigma(\gamma_{\epsilon_i}) = 1$.]

Lemma 2.2. _There is at most one positive non-simple root_ α _satisfying both (i) and (ii)._

We shall define a new root system Δ' in terms of α. If there is no α in Lemma 1, let Π' consist of the ϵ_i for $i \neq k$. If α does exist, let Π' consist of α and the ϵ_i for $i \neq k$. Let Δ' be the subset of Δ generated by Π' and the Weyl group reflections corresponding to members of Π'. By (ii') Δ' is a root system in which Π' can be taken as the set of simple roots. The Dynkin diagram of Π' we shall call the _α-diagram_ of G and σ_k. Ordinarily the α-diagram is not connected.

Computation of α in examples is simplified by the following lemma.

Lemma 2.3. _The least positive_ α _in_ Δ _satisfying (i) and (ii') satisfies (ii)._

Let $W(\Delta')$ be the Weyl group for the root system Δ'. One can show that $W(\Delta') \subseteq W_\sigma$. Let R_σ be the subgroup of members w of W_σ such that $w(\Pi') \subseteq \Pi'$. Each element of R_σ defines an automorphism of the α-diagram of G and σ. If α exists, distinct members of R_σ lead to distinct automorphisms. In this case, in particular, if the α-diagram admits no nontrivial automorphism, then $R_\sigma = \{1\}$.

Theorem 1. W_σ _is the semidirect product_ $W_\sigma = W(\Delta')R_\sigma$ _with_ $W(\Delta')$ _normal._ $W(\Delta')$ _is the subgroup of_ W_σ _on which_ $\mathcal{A}(w,\sigma,1)$ _is scalar. Consequently if_ $w = w_1 r$ _is the decomposition of a member of_ W_σ _according to the semidirect product, then_ $\mathcal{A}(w,\sigma,1) = c\,\mathcal{A}(r,\sigma,1)$ _for a scalar_ c _of modulus one. Moreover, the set of all operators_ $\mathcal{A}(r,\sigma,1)$ _for_ r _in_ R_σ _is linearly independent._

With a case-by-case argument, one can check that $|R_\sigma| = 1, 2$, or 4. The case $|R_\sigma| = 4$ occurs only for G of type D_n with n even, and when $|R_\sigma| = 4$, R_σ is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. If G is of type A_n, $|R_\sigma| = 1$ or 2; this case is discussed in §4. For G of type B_n or C_n, $|R_\sigma|$ can be 1 or 2. But in F_4 and G_2, $|R_\sigma| = 1$ for all σ. In E_7 there is a fundamental σ for which $|R_\sigma| = 2$.

For an explicit example, take G of type C_n. In standard notation the simple roots are

$$\epsilon_1 = e_1 - e_2,\ \epsilon_2 = e_2 - e_3,\ \ldots,\ \epsilon_{n-1} = e_{n-1} - e_n,\ \epsilon_n = 2e_n.$$

Choose $\sigma = \sigma_n$. Then $\alpha = e_{n-1} + e_n$, and

$$\Pi' = \{\epsilon_1, \epsilon_2, \ldots, \epsilon_{n-1}, \alpha\}.$$

The α-diagram of σ is of type D_n, and $R_\sigma = \{1, p_n\}$.

We pass to the case of general σ and λ. If (σ,λ) is given, we first apply to (σ,λ) a member of W that makes λ dominant. After this change it is quite easy to check that $W_{1,\lambda}$ is the Weyl group generated by the simple reflections that fix λ. These simple reflections correspond to a Dynkin diagram and fall into components corresponding to the components of the Dynkin diagram. We then operate with a second member of W, this one in $W_{1,\lambda}$, to make σ fundamental on each component. Let Δ' be the roots corresponding to the union of the α-diagrams for the components, and let Π' be the corresponding simple system. Let

$$R_{\sigma,\lambda} = \{w \in W_{\sigma,\lambda} \mid w(\Pi') \subseteq \Pi'\}.$$

<u>Theorem 2.</u> $W_{\sigma,\lambda}$ <u>is the semidirect product</u> $W_{\sigma,\lambda} = W(\Delta')R_{\sigma,\lambda}$ <u>with</u> $W(\Delta')$ <u>normal.</u> $W(\Delta')$ <u>is the subgroup of</u> $W_{\sigma,\lambda}$ <u>on which</u> $\mathcal{A}(w,\sigma,\lambda)$ <u>is scalar. Consequently if</u> $w = w_1 r$ <u>is the decomposition of a member of</u> W_σ <u>according to the semidirect product, then</u> $\mathcal{A}(w,\sigma,\lambda) = c\ \mathcal{A}(r,\sigma,1)$ <u>for a scalar</u> c <u>of modulus one. Moreover,</u>

<u>the set of all operators</u> $\mathcal{A}(r,\sigma,\lambda)$ <u>for</u> r <u>in</u> $R_{\sigma,\lambda}$ <u>is linearly independent.</u>

An example of this situation is in §4. In any event, one can use the form of $R_\sigma = R_{\sigma,1}$ to show that $R_{\sigma,\lambda}$ is a subgroup of a (perhaps large) direct sum of copies of $\mathbb{Z}_2$ and therefore itself is a direct sum of copies of $\mathbb{Z}_2$. This completely settles Conjecture 1 for G split over $\mathbb{R}$.

For the case of a general G not necessarily split over $\mathbb{R}$, one can prove an analog of Theorem 2 but without a satisfactory description of Δ' and $R_{\sigma,\lambda}$. To do so, let

$$\Delta' = \{\beta \mid p_\beta \in W_{\sigma,\lambda} \text{ and } \sigma(p_\beta)\mathcal{A}(p_\beta,\sigma,\lambda) \text{ is scalar}\},$$

where p_β is the reflection relative to β, and let

$$R_{\sigma,\lambda} = \{p \in W_{\sigma,\lambda} \mid p\beta > 0 \text{ for every } \beta > 0 \text{ in } \Delta'\}.$$

It is easy to see that Δ' is a root system and thus has a Weyl group $W(\Delta')$. Once again we have the semidirect product decomposition $W_{\sigma,\lambda} = W(\Delta')R_{\sigma,\lambda}$ with $W(\Delta')$ the normal subgroup corresponding to trivial operators and with the operators $\sigma(r)\mathcal{A}(r,\sigma,\lambda)$ for r in $R_{\sigma,\lambda}$ independent. This result is considerably easier to prove than Theorems 1 and 2. However, it has the shortcoming that Δ' is defined in such a way that one can prove rather little about $R_{\sigma,\lambda}$. These facts make clearer the thrust of Theorems 1 and 2, namely the possibility of conjugating σ and λ suitably so that the simple roots of Δ' can be expressed readily in terms of Π and special roots α.

3. <u>Completeness theorem for G of real-rank two</u>

Return to the notation of §1. Fix an irreducible unitary representation σ of M and a unitary character λ of A. If w is a member of M' for which σ is equivalent with $w\sigma$, then the operator $\sigma(w)\mathcal{A}(w,\sigma,\lambda)$ is defined. Here $\sigma(w)$ is determined up to a scalar factor; once a choice is made for the scalar, the operator depends only on the coset of w in $W = M'/M$. We make such a choice of scalars for each member of $W_{\sigma,\lambda}$ without imposing any consistency conditions on the different choices. Then we can speak unambiguously of the linear span of the operators $\sigma(w)\mathcal{A}(w,\sigma,\lambda)$ for w in $W_{\sigma,\lambda}$.

Theorem 3. Let G be of real-rank two. For any (σ,λ), the linear span of the operators $\sigma(w)\mathcal{A}(w,\sigma,\lambda)$ for w in $W_{\sigma,\lambda}$ is the set of all bounded linear operators L such that $U(\sigma,\lambda)L = LU(\sigma,\lambda)$.

Consequently $U(\sigma,\lambda)$ is irreducible if and only if all the operators $\sigma(w)\mathcal{A}(w,\sigma,\lambda)$ are scalar. The detailed proof of the theorem shows exactly which operators $\sigma(w)\mathcal{A}(w,\sigma,\lambda)$ are scalar.

The exposition will be simpler if we sketch the proof for a particular case and then describe the extent to which the general case differs from the special case. Temporarily take G to be the real symplectic group $Sp(2,\mathbb{R})$. Let ϵ_1 be the shorter simple root and ϵ_2 be the longer one. In the notation of §2, let $\sigma = \sigma_2$ and $\lambda = 1$. Then $W_{\sigma,\lambda} = W_\sigma$ is all of W and has order 8. So

$$W_\sigma = \{1,\ p_1,\ p_2,\ p_1p_2,\ p_2p_1,\ p_1p_2p_1,\ p_2p_1p_2,\ p_1p_2p_1p_2\}.$$

Recall the Bruhat decomposition of G: The MAN double cosets are in one-to-one correspondence with the elements of W, and $G = \cup MANwMAN$ with the union over a system of representatives of the cosets of M' modulo M. Let $C(w)$ be the double coset corresponding to w.

Let σ operate in the space E^σ, let $C^{\sigma,1}$ be the subspace of functions in $C^\infty(G,E^\sigma)$ that lie in the representation space for $U(\sigma,1)$, and let $\pi_{\sigma,1}$ be the standard mapping of $C^\infty_{com}(G,E^\sigma)$ onto $C^{\sigma,1}$, given by the case $\lambda = 1$ of

$$(3.1)\quad (\pi_{\sigma,\lambda}f)(x) = \int_{MAN} e^{-\rho H(\xi)}\lambda(\exp H(\xi))^{-1}\sigma(m(\xi))^{-1}f(\xi x)d_r\xi.$$

If L is a bounded operator commuting with $U(\sigma,1)$, then the mapping δ_L given by

$$(3.2)\qquad f \to L(\pi_{\sigma,1}f)(1)$$

is an $\mathrm{End}(E_\sigma)$-valued distribution on G. (See [1].)

Bruhat [3] examined δ_L and found that it satisfies a functional equation under translation on the right and left by MAN. For w in W he showed essentially that if

(i) δ_L vanishes on $C(w')$ for each $w' \neq w$ such that $\overline{C(w')} \supseteq C(w)$ and

(ii) δ_L does not vanish on $C(w)$,

then

(i') w is in $W_{\sigma,1}$ (here this conclusion is empty)

(ii') the component of δ_L transverse to $C(w)$ vanishes

(iii') the restriction of δ_L to $C(w)$ is a multiple of a distribution δ_w that is independent of L.

The idea of the proof is to construct a "pseudo-operator" $T(w)$ for each w in W_σ such that the main part of the distribution corresponding to $T(w)$ is δ_w. Subtracting from L a suitable linear combination of the $T(w)$, we obtain an operator whose distribution vanishes on certain double cosets and satisfies a functional equation reflecting properties of both L and the $T(w)$'s. This functional equation will have no solutions unless each $T(w)$ that was subtracted from L is already one of the operators $\sigma(w)\mathcal{A}(w,\sigma,\lambda)$. Consequently L is in the span of the operators $\sigma(w)\mathcal{A}(w,\sigma,\lambda)$.

We shall describe parts of the proof in more detail, but first we must define the pseudo-operators $T(w)$. Let w be in W_σ, and let Δ_w be the set of all positive roots α such that $w\alpha < 0$ and

$\sigma(\gamma_\alpha) = +1$. If we regard λ as a variable on A or its Lie algebra, we can speak of differentiation D_α of a function of λ with respect to the vector α. Define the <u>pseudo-operator</u> $T(w)$ by

$$T(w) = \left(\prod_{\alpha\in\Delta_w} D_\alpha\right) \mathcal{A}(w,\sigma,\lambda)\Big|_{\lambda=1}.$$

This operator is well-defined if we use the compact picture for the induced representation. It has three main properties:

(1) It maps $C^{\sigma,1}$ into itself.

(2) It satisfies an obvious functional equation. This equation comes by applying the differential operator $\prod D_\alpha$ to both sides of (1.1), using the Liebritz rule for differentiating products and using a formula relating $U(\sigma,\lambda)$ to $U(\sigma,1)$ in the compact picture. The result is of the form

(3.3) $$U(\sigma,1,x)T(w) = T(w)U(\sigma,1,x) + \text{remainder terms}.$$

(3) Its distribution, defined in analogy with (3.2), coincides with a nonzero multiple of δ_w on the union of all double cosets of dimension $\geq$ dim $C(w)$.

There is a helpful (though slightly inaccurate) notation for dealing with these operators. We write formally

$$T(p_1) = T_1 \qquad T(p_1p_2p_1) = T_1H_2T_1$$
$$T(p_2) = H_2 \qquad T(p_2p_1p_2) = H_2T_1H_2$$
$$T(p_1p_2) = T_1H_2 \qquad T(p_1p_2p_1p_2) = T_1H_2T_1H_2.$$
$$T(p_2p_1) = H_2T_1$$

Each $T(w)$ is written as a product of T's and H's, with the subscripts matching those in a minimal decomposition of w into the product of simple reflections. We use T or H in the kth factor according as the kth operator $\mathcal{A}$ in the expansion of $\mathcal{A}(w,\sigma,\lambda)$ by (1.2) is or is not, respectively, scalar for $\lambda = 1$. [The notation here is meant to suggest that $T(w)$ is formally the product of rank-one pseudo-operators, T being a rank-one pseudo-identity and H being a rank-one Hilbert transform. Actually the product of these operators is not exactly equal to $T(w)$, but it is equal in first approximation, in the sense that the main part of the distribution for the product operator is δ_w.]

Form δ_L as in equations (3.2) and (3.1), and consider the open double coset $C(p_1p_2p_1p_2)$. In view of property (3) of $T(w)$, we can find a constant c such that the distribution ν of $L - cT(p_1p_2p_1p_2)$ vanishes on $C(p_1p_2p_1p_2)$. Combining (3.3) and the commutativity of L with $U(\sigma,\lambda)$, we see that $L - cT(p_1p_2p_1p_2)$ satisfies an analog of (3.3). It follows that ν satisfies a functional equation under right translation by MAN. Also property (1) of $T(w)$ implies that ν satisfies another functional equation under left translation by MAN.

It turns out that the main contribution to ν is on $C(p_2p_1p_2)$, and one can show from the functional equation that ν has no transverse derivatives to this double coset. It follows readily that the restriction of ν to $C(p_2p_1p_2)$ makes sense and is a function. Evaluating this function at a representative w' of $p_2p_1p_2$ and using the functional equation for right translation by a in A and left translation by $wa^{-1}w^{-1}$, we are led to the equality of a bounded expression and $c\alpha(\log a)$ for a certain root α. Consequently $c = 0$ and δ_L vanishes on $C(p_1p_2p_1p_2)$.

Next we attempt to show that δ_L vanishes on $C(p_1p_2p_1)$ and $C(p_2p_1p_2)$. (The transverse derivatives to these double cosets must vanish, according to [3].) We form

$$L - c_1T(p_1p_2p_1) - c_2T(p_2p_1p_2)$$

and argue similarly. The conclusion $c_1 = 0$ will come immediately from considering $C(p_1p_2)$, but the conclusion $c_2 = 0$ will come only later by considering $C(1)$ after L has been handled on the double cosets that lie between $C(p_2p_1p_2)$ and $C(1)$.

The argument continues in this way, with L adjusted on double cosets of lower and lower dimension. The details are cumbersome to list, but we can say the following. For each w we push L off $C(w)$ by using $T(w)$. The contradiction that eliminates $T(w)$ comes from the double coset corresponding to the formal expansion of $T(w)$ in H's and T's, but with one T deleted. The root in the final equation that gives the contradiction is obtained as follows: If the factor p_{i_k} is deleted from $p_{i_1} \cdots p_{i_n}$, then the root is $p_{i_n} \cdots p_{i_{k+1}} \epsilon_{i_k}$. It is possible for a linear combination of as many as two roots to appear in the final equation, but these roots will be distinct and hence independent.

In the end, L will be expressed as a linear combination of $T(p_2)$ and $T(1)$, that is, of H_2 and I. These are the operators $\mathcal{A}(w,\sigma,1)$ for $w = p_2$ and $w = 1$, with no differentiations, and the theorem is proved for this special G and σ.

Now consider the case of general G of real-rank two. In view of the results in [6] concerning the real-rank one case, we may assume that G is simple. We are given σ and λ, but the same kind of argument as after Proposition 2.1 shows that there is no loss of generality in taking λ dominant. Observe that $W_{\sigma,\lambda} = W_{\sigma,1} \cap W_{1,\lambda}$. With λ dominant, $W_{1,\lambda}$ is generated by the simple reflections that it contains. If there are no simple reflections in $W_{1,\lambda}$, then $W_{1,\lambda} = \{1\}$, and the the theorem follows from Bruhat's results [3]. If there is one simple reflection, the problem is substantially a rank-one problem and is handled by a simpler version of the argument to follow. If both simple reflections are in $W_{1,\lambda}$, then $\lambda = 1$; this is the only hard case.

Thus suppose $\lambda = 1$. Call a restricted root $\alpha > 0$ <u>primitive</u> if $\alpha/2$ is not a restricted root. Fix w in $W_{\sigma,1}$ and consider the primitive restricted roots $\alpha > 0$ such that $w\alpha < 0$. Recall from §18 of [6] that the restriction $\sigma|_{M_\alpha}$ is equivalent with a multiple of a single irreducible representation σ_α of M_α. We shall say that a primitive α with $w\alpha < 0$ is in Δ_w if the rank-one intertwining operator $\sigma_\alpha(p_\alpha)\mathcal{A}_\alpha(p_\alpha,\sigma_\alpha,1)$ is scalar. (A necessary and sufficient condition for this is given in Theorem 5 of [6].) Define the <u>pseudo-operator</u> $T(w)$ by

$$T(w) = \Big(\prod_{\alpha\in\Delta_w} D_\alpha\Big)\sigma(w)\mathcal{A}(w,\sigma,\lambda)\Big|_{\lambda=1},$$

with notation as in the case of $Sp(2,\mathbb{R})$. Just as in $Sp(2,\mathbb{R})$ it is convenient to have symbolic notation for $T(w)$. If w decomposes minimally as $w = p_{i_1}\cdots p_{i_n}$, we write $T(w)$ formally as a product of T's and H's, using T_{i_j} at the jth stage if the associated root $p_{i_n}\cdots p_{i_{j+1}}\varepsilon_{i_j}$ is in Δ_w and using H_{i_j} otherwise. (Again we are thinking of $T(w)$ as a product of pseudo-identities T and nontrivial rank-one operators H, and again this is only an approximation. In this general case, the symbol H_i is standing for both an operator and its inverse, and each H in a string must be interpreted suitably.)

With this notation we describe $W_{\sigma,1}$ and the associated pseudo-operators. A case-by-case check using the results of [10] shows that we can conjugate σ by a member of M' in order to arrive at one of the following situations:

(1) A_2 as restricted root diagram. Here $|W| = 6$. W_σ can be any of $\{1\}$, $\{1,p_1\}$, $\{1,p_2\}$, W. In all cases each $T(w)$ is formally a product of T's.

(2) G_2 as restricted root diagram. Here $|W| = 12$. W_σ can be $\{1\}$, $\{1,p_1\}$, $\{1,p_2\}$, or W with each $T(w)$ a product of T's.

Alternatively W_σ can be $\{1, p_1, p_2p_1p_2p_1p_2, p_1p_2p_1p_2p_1p_2\}$ with formally $T(p_1) = H_1$, $T(p_2p_1p_2p_1p_2) = H_2H_1T_2H_1H_2$, and the other $T(w)$ given as the product.

(3) BC_2 as restricted root diagram. Here $|W| = 8$. Let ϵ_1 and ϵ_2 be the simple restricted roots, with ϵ_1 longer than ϵ_2. Then $2\epsilon_2$ is a restricted root. There are two possibilities for W_σ.

(a) $W_\sigma = W$. Then $T(p_1) = T_1$, $T(p_2) = H_2$ or T_2, and the other $T(w)$'s are given as products.

(b) $W_\sigma = \{1, p_2, p_1p_2p_1, p_2p_1p_2p_1\}$. Then $T(p_2) = H_2$ or T_2 and, independently, $T(p_1p_2p_1) = H_1T_2H_1$ or $H_1H_2H_1$. The other $T(w)$ is given as the product.

(4) B_2 as restricted root diagram. Here $|W| = 8$. Let ϵ_1 and ϵ_2 be the simple restricted roots, with ϵ_1 shorter then ϵ_2. Then W_σ and the pseudo-operators can be as in (3a) and (3b) above, or else W_σ can be $\{1\}$, $\{1,p_1\}$, or $\{1,p_2\}$ with only T's occurring.

We need one more fact. This is a result due to Steinberg [9, p. 127] for Chevalley groups, and to Borel and Tits [2] in the general case. Namely let $w = p_{i_1}\cdots p_{i_n}$ be a minimal decomposition into simple reflections. Then the closure of $C(w)$ is the union of the double cosets $C(w')$ as w' runs over all products (in order) of subsets of the p_{i_j}.

Putting this fact and the detailed description of the pseudo-operators together, one sees that a simple modification of the completeness argument given for $Sp(2,\mathbb{R})$ and the special σ works for all G of real-rank two and all σ.

4. Completeness theorem for $SL(n,\mathbb{R})$

For $SL(n,\mathbb{R})$ the completeness theorem is as follows.

<u>Theorem 4</u>. <u>Let</u> $G = SL(n,\mathbb{R})$. <u>For any</u> (σ,λ) <u>the linear span</u>

<u>of the operators</u> $\mathcal{A}(w,\sigma,\lambda)$ <u>for</u> w <u>in</u> $W_{\sigma,\lambda}$ <u>is the set of all bounded linear operators</u> L <u>such that</u> $U(\sigma,\lambda)L = LU(\sigma,\lambda)$.

As we shall see, this linear span has dimension 1 or 2. Dimension 1 is necessary and sufficient for irreducibility. Partial results on irreducibility for $SL(n,\mathbb{R})$ were known already. Gelfand and Graev [4] settled n odd, and Wallach [11] proved the irreducibility of $U(\sigma,\lambda)$ when all $\mathcal{A}(w,\sigma,1)$ for w in $W_{\sigma,1}$ are scalar.

Before commenting on the proof, we introduce notation. In $SL(n,\mathbb{R})$, we shall take M to be diagonal matrices with ± 1 in the diagonal entries and A to be diagonal matrices with positive diagonal entries. The Weyl group W is the permutation group on n letters, and it operates by permuting the diagonal entries.

Turning to the proof, we may without loss of generality deal with a convenient image of (σ,λ) under the operation of W. First conjugate (σ,λ) so that λ is dominant. The effect of this is to decompose $\{1,\ldots,n\}$ into disjoint strings of consecutive integers in such a way that the members of $W_{1,\lambda}$ are exactly the permutations that leave each string stable. Next, σ is given as a product of certain signs of diagonal entries, and we make a further conjugation leaving each string stable so that the signs that are used within each string occur consecutively at the beginning of the string.

With (σ,λ) in this form, one can check that there are only two possibilities:

(1) $W_{\sigma,\lambda}$ is a direct product of smaller permutation groups, and each $T(w)$ for w in $W_{\sigma,\lambda}$ is formally the product of T operators only.

(2) n is even, and exactly the first half of the entries in each λ-string obtained above is used in computing σ. In this case $W_{\sigma,\lambda}$ has a subgroup W' of index 2 that is a direct product of

smaller permutation groups. Write $W_{\sigma,\lambda} = W' \cup w_0 W'$, where w_0 is the element of shortest length in the nontrivial coset. Then $T(w_0)$ is a product of H operators only, and therefore $\mathcal{A}(w_0,\sigma,\lambda)$ is not scalar by property (iii) of $T(w_0)$. Say, $\mathcal{A}(w_0,\sigma,\lambda) = H_0$. In addition, each $T(w)$ for w in W' is a product of T's, and each $T(w)$ for w in w_0W' is the product of H_0 by a product of T's.

The rest of the proof proceeds along the lines of §3. The only additional thing that is needed is an algebraic result to ensure that if Theorem 3 fails, then the functional equation satisfied by the difference of L and its first approximation is actually contradictory. This result is given as Proposition 4.2.

Let W be a Weyl group, and let $\ell(w)$ be the length of the element w of W. If p and q are members of W, we shall say that p is a parent of the child q if $\ell(p) = \ell(q)+1$ and if q can be obtained from some (or any, in view of Steinberg's result mentioned in §3) minimal decomposition of p by striking out one of the simple-reflection factors. In this case it is a simple matter to see that $p = qw_\alpha$ for some root-reflection w_α. We write $\alpha = \alpha_{p,q}$. For fixed q, let P_q be the set of parents of q.

Lemma 4.1. *Let W be any Weyl group, let p and q be members of W with $\ell(p) = \ell(q)+1$, and suppose $p = qw_\alpha$ for some α. Then p is a parent of the child q.*

Proposition 4.2. *Let W be the Weyl group of $SL(n,\mathbb{R})$, and fix a length ℓ. Suppose that to each element p in W of length ℓ is associated a complex number c_p in such a way that the set $\{c_p\}$ satisfies*

$$\sum_{p \in P_q} c_p \, \alpha_{p,q} = 0$$

for all q in W of length ℓ-1. Then all the c_p are equal to 0.

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