Status of Classification of Irreducible Unitary Representations

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One of the first questions that one would like to answer for Fourier analysis with a particular group is: "What are all the irreducible unitary representations of the group?" For semisimple groups this problem remains unsolved——in fact, very far from solved. Our intention here is to give a survey of some aspects of what is known about the problem for semisimple Lie groups. For an earlier survey of this kind, see [23].

Most of the survey will be of old results, but we shall include some new facts as well:

1) a useful reformulation of the known criterion [23] for unitarity of an irreducible admissible representation. This is given as Theorem 1.2. Progress to date in applying this or some equivalent criterion to settle concrete unitarity questions is summarized in §2.

2) a description, given in a diagram in §3, of some representations of $SU(N,2)$ that we can prove are unitary. The diagram is complicated enough to illustrate the difficulty of the general problem yet simple enough to suggest a number of inductive approaches to a solution. In §4 we summarize briefly some techniques, including those needed for our result about $SU(N,2)$, for applying the unitarity criterion to determine whether a particular irreducible admissible representation is unitary.

3) an extension in §5 of one of the techniques listed in §4,

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namely use of explicit scalar formulas obtained from intertwining operators. We have already applied this extended technique to our own classification [19] of the irreducible unitary representations of $SU(2,2)$.

§1. Unitarity criterion

Let G denote a connected semisimple Lie group with a faithful matrix representation, let K be a maximal compact subgroup, and let θ be the corresponding Cartan involution. Fix a minimal parabolic subgroup P_{min}, and let $P_{min} = M_{min}A_{min}N_{min}$ be its Langlands decomposition. Here M_{min} is compact, A_{min} is a vector group, N_{min} is simply-connected nilpotent, and $G = KA_{min}N_{min}$ is an Iwasawa decomposition of G. A <u>standard parabolic subgroup</u> P of G is any closed subgroup containing P_{min}. There are finitely many such subgroups P, and each has a Langlands decomposition $P = MAN$. Here M is noncompact unless $P = P_{min}$, and also $A \subseteq A_{min}$ and $N \subseteq N_{min}$. The group P is called <u>cuspidal</u> if $\operatorname{rank} M = \operatorname{rank}(K \cap M)$.

Let ω be an irreducible unitary representation of G. A vector v is <u>K-finite</u> if the span of $\omega(K)v$ is finite-dimensional. Then ω defines an irreducible admissible representation of the Lie algebra of G on the space of K-finite vectors, by [11]. (We shall abuse notation and speak of an irreducible admissible representation of G.) We recall the statement of the Langlands classification [27] of irreducible admissible representations; the statement below has been sharpened by the incorporation of a result of Miličić [28].

Langlands classification [27]. The (equivalence classes of) irreducible admissible representations of G stand in one-one correspondence with all triples (P,π,ν), where

$P = MAN$ is a standard parabolic subgroup

π is an irreducible "tempered" unitary representation (equivalence class) of M

ν is a complex-valued linear functional on the Lie algebra of A with $\operatorname{Re}\nu$ in the open positive Weyl chamber.

The Langlands representation $J(P,\pi,\nu)$ is the unique irreducible quotient of the induced representation

$$U(P,\pi,\nu) = \operatorname{ind}_{MAN}^{G}(\pi \otimes e^{\nu} \otimes 1) \tag{1.1}$$

and is given as the image of an explicit intertwining operator $A(\theta P:P:\pi:\nu)$ applied to $U(P,\pi,\nu)$.

In (1.1) we have arranged parameters so that unitary representations induce to unitary representations, and we adopt the convention that G acts on the left. The intertwining operator $A(\theta P:P:\pi:\nu)$ is given by a convergent integral in the context of the theorem; its general definition and properties may be found in §§6-7 of [22]. The representation π is assumed "tempered" in the sense that $(\pi(m)\varphi,\psi)$ is in $L^{2+\epsilon}(M)$ for every $\epsilon > 0$ and for all $(K\cap M)$-finite vectors φ and ψ. The irreducible tempered representations were classified in 1976, with details appearing in [24]; their classification will be combined with the Langlands classification in Theorem 1.1 below.

An irreducible admissible representation comes from the space of K-finite vectors of a unitary representation if and only if it is infinitesimally unitary (in the sense of admitting a Hermitian inner product such that the Lie algebra of G acts in skew-Hermitian fashion), and in this case the unitary representation is unique (up to unitary equivalence) and irreducible.

Corollary [23]. $J(P,\pi,\nu)$ is infinitesimally unitary if and only if

(i) the formal symmetry conditions hold: there exists w in K normalizing A with $wPw^{-1} = \theta P$, $w\pi \cong \pi$, and $w\nu = -\bar{\nu}$, and

(ii) the Hermitian intertwining operator $\pi(w)R(w)A(\theta P: P:\pi:\nu)$, where $R(w)$ denotes right translation of functions by w, is positive or negative semidefinite.

For connected linear semisimple groups, it is proved in [24] that the irreducible tempered representations are all induced from cuspidal parabolic subgroups $M_1A_1N_1$ with a discrete series or limit of discrete series representation on M_1 and a unitary character on A_1; moreover, the limit of discrete series representation may be assumed to be given with nondegenerate data. Conversely such an induced representation is always tempered, and it is irreducible if and only if a certain finite group, known as the R group, is trivial.

Most of the steps needed to extend this result to handle an irreducible tempered representation π of the (possibly disconnected) group M obtained from a standard parabolic subgroup of G are already present in [24], and it is easy to complete the argument. Then we can substitute for π in the Langlands classification, and we arrive at Theorem 1.1 below (Theorem 5 of [23]). The information from the R group ensuring that π is irreducible needs to be built into the statement, and we accordingly recall some definitions from [23]. Let MAN be a cuspidal parabolic subgroup of G, let $W(A:G)$ be the Weyl group of A, let $\mathfrak{a}$ be the Lie algebra of A, and let σ be a discrete series or limit of discrete series of M with nondegenerate data. For each $\mathfrak{a}$ root α, let $\mu_{\sigma,\alpha}(\nu)$ be the Plancherel factor of §7 of [24]. Define

$$\Delta' = \{\text{useful } \mathfrak{a} \text{ roots } \alpha \mid s_\alpha \nu = \nu \text{ and } \mu_{\sigma,\alpha}(\nu) = 0\} \tag{1.2}$$

and

$$W'_{\sigma,\nu} = \text{Weyl group of root system } \Delta' . \tag{1.3}$$

The group $W'_{\sigma,\nu}$ is a subgroup of

$$W_{\sigma,\nu} = \{w \in W(A:G) \mid w\sigma \cong \sigma \text{ and } w\nu = \nu\} . \tag{1.4}$$

We can then reformulate the completeness of the Langlands classification as Theorem 1.1. The idea is that the R group of the concealed tempered representation π is isomorphic to $W_{\sigma,\nu}/W'_{\sigma,\nu}$.

<u>Theorem 1.1</u> [23]. Let $P = MAN$ be a cuspidal standard parabolic subgroup of G, let σ be a discrete series or limit of discrete series representation of M with nondegenerate data, and let ν be a complex-valued linear functional on $\mathfrak{a}$ with $\operatorname{Re}\nu$ in the closed positive Weyl chamber. Suppose that $W_{\sigma,\nu} = W'_{\sigma,\nu}$. Then the induced representation $U(P,\sigma,\nu)$ has a unique irreducible quotient $J'(P,\sigma,\nu)$, and every irreducible admissible representation of G is of the form $J'(P,\sigma,\nu)$ for some such triple (P,σ,ν).

The effect of Theorem 1.1 is to rewrite the completeness of the Langlands classification in terms of more manageable representations.

What is lost is the simple criterion for equivalences, but equivalences can always be sorted out by going back to the earlier statement. If we take these matters into account, then we can translate into the present language the unitarity criterion given in the corollary stated earlier.

Theorem 1.2. Let (P,σ,ν) be such that the irreducible admissible representation $J'(P,\sigma,\nu)$ is defined. Then $J'(P,\sigma,\nu)$ is infinitesimally unitary if and only if

(i) there exists w in $W(A:G)$ such that $w^2 = 1$, $w\sigma \cong \sigma$, and $w\nu = -\bar{\nu}$, and

(ii) the standard intertwining operator $\sigma(w)A_P(w,\sigma,\nu)$ of §§7-8 of [22], when normalized to be pole-free and not identically zero as

$$\sigma(w)\,\mathcal{A}_P(w,\sigma,\nu) \ , \tag{1.5}$$

is positive or negative semidefinite.

If $J'(P,\sigma,\nu)$ is infinitesimally unitary, then every w satisfying (i) is such that the operator (1.5) is positive or negative semidefinite.

Proof. By way of preliminaries let us introduce notation that makes clear how to regard $J'(P,\sigma,\nu)$ as a Langlands quotient. With $P = MAN$, let $\mathfrak{m}$, $\mathfrak{a}$, and $\mathfrak{n}$ be the Lie algebras of M, A, and N. Define $\mathfrak{a}_*$ to be the span in $\mathfrak{a}$ of the vectors H_α such that the $\mathfrak{a}$ root α is orthogonal to $\operatorname{Re}\nu$. Let $\mathfrak{a}_1$ be the orthocomplement of $\mathfrak{a}_*$ in $\mathfrak{a}$. Define $\mathfrak{n}_*$ to be the centralizer of $\mathfrak{a}_1$ in $\mathfrak{n}$, $\mathfrak{n}_1$ to be the natural complement of $\mathfrak{n}_*$ in $\mathfrak{n}$, and $\mathfrak{m}_1$ to be

$$\mathfrak{m}_1 = \mathfrak{m} \oplus \mathfrak{a}_* \oplus \mathfrak{n}_* \oplus \theta\mathfrak{n}_* \, .$$

Then we can form a corresponding standard parabolic subgroup $P_1 = M_1A_1N_1$ of G with

$$M_1A_1N_1 \supseteq MAN$$

and with MA_*N_* a parabolic subgroup of M_1. These definitions are arranged so that $\nu|_{\mathfrak{a}_*}$ is imaginary and so that $\mathrm{Re}(\nu|_{\mathfrak{a}_1})$ is in the open positive Weyl chamber of $\mathfrak{a}_1'$. The representation

$$\pi = \mathrm{ind}_{MA_*N_*}^{M_1}(\sigma \otimes \exp(\nu|_{\mathfrak{a}_*}) \otimes 1)$$

is tempered and, by assumption, irreducible. Then we have

$$J'(P,\sigma,\nu) \cong J(P_1,\pi,\nu|_{\mathfrak{a}_1}).$$

Now we come to the proof of the theorem. The main step will be to prove that $J'(P,\sigma,\nu)$ infinitesimally unitary implies that (i) holds. Once this is done, we can argue as follows: If (i) holds for some w, then (1.5) is defined (by Lemma 7.9 of [22]), and Corollary 8.7 of [22] shows that the sesquilinear form

$$\langle u,v\rangle = \int_K (\sigma(w)\mathcal{A}_P(w,\sigma,\nu)u(k),v(k))\,dk \tag{1.6}$$

is invariant (in the sense that the Lie algebra of G acts by skew-Hermitian operators) and Hermitian. Since $w\nu = -\bar{\nu}$, we have

$$w(\mathrm{Re}\,\nu) = -\,\mathrm{Re}\,\nu\ . \tag{1.7}$$

From (1.7) it follows that $w\mathfrak{a}_* = \mathfrak{a}_*$ and therefore that $w\mathfrak{a}_1 = \mathfrak{a}_1$. Another application of (1.7) then shows that

$$w(\mathrm{Re}\,\nu|_{\mathfrak{a}_1}) = -\,\mathrm{Re}\,\nu|_{\mathfrak{a}_1}\ .$$

Since $\operatorname{Re} \nu|_{\mathfrak{a}_1}$ is in the open positive Weyl chamber of $\mathfrak{a}_1'$, $w\mathfrak{n}_1 = \theta\mathfrak{n}_1$. Thus $wP_1w^{-1} = \theta P_1$. From this equality and Corollary 7.7 of [22], we see that $\mathcal{Q}_P(w,\sigma,\nu)$ can be regarded as a composition of the Langlands operator $A(\theta P_1 : P_1 : \pi : \nu|_{\mathfrak{a}_1})$ followed by another operator. Since the image of the Langlands operator is irreducible, the image of (1.5) must be equivalent with $J'(P,\sigma,\nu)$. Consequently (1.6) descends to a nonzero invariant Hermitian form on $J'(P,\sigma,\nu)$. By irreducibility of $J'(P,\sigma,\nu)$, such a form is unique up to a scalar, and $J'(P,\sigma,\nu)$ is infinitesimally unitary if and only if a nonzero such form is semidefinite.

Thus the theorem will be proved if we show that $J'(P,\sigma,\nu)$ infinitesimally unitary implies that (i) holds. Thus suppose $J'(P,\sigma,\nu)$ is infinitesimally unitary. Then so is the equivalent representation $J(P_1,\pi,\nu|_{\mathfrak{a}_1})$. By the corollary above, there exists w_1 in the normalizer $N_K(\mathfrak{a}_1)$ such that

$$w_1P_1w_1^{-1} = \theta P_1, \quad w_1\pi \cong \pi, \quad \text{and} \quad \operatorname{Ad}(w_1)\nu|_{\mathfrak{a}_1} = -\bar{\nu}|_{\mathfrak{a}_1}. \tag{1.8}$$

We shall apply the equivalence criterion for irreducible tempered representations to the formula $w_1\pi \cong \pi$. (See Theorem 4 of [23] or Theorem 14.2 of [24]. These theorems are stated in the connected case, but they extend to groups like M without difficulty.) The criterion says that the equivalence of

$$\pi = \operatorname{ind}_{MA_*N_*}^{M_1}(\sigma \otimes \exp \nu|_{\mathfrak{a}_*} \otimes 1)$$

and

$$w_1\pi \cong \operatorname{ind}_{w_1(MA_*N_*)w_1^{-1}}^{M_1} (w_1\sigma \otimes \exp(\mathrm{Ad}(w_1)\nu)|_{\mathrm{Ad}(w_1)\mathfrak{a}_*} \otimes 1)$$

implies there is an element w_2 in $K \cap M_1$ with

$$w_1 M w_1^{-1} = w_2 M w_2^{-1} \tag{1.9a}$$

$$w_1 A_* w_1^{-1} = w_2 A_* w_2^{-1} \tag{1.9b}$$

$$w_1\sigma \cong w_2\sigma \tag{1.9c}$$

$$(\mathrm{Ad}(w_1)\nu)|_{\mathrm{Ad}(w_1)\mathfrak{a}_*} = (\mathrm{Ad}(w_2)\nu)|_{\mathrm{Ad}(w_2)\mathfrak{a}_*} . \tag{1.9d}$$

We shall list some properties of $w_2^{-1}w_1$. Since w_2 is in M_1, (1.8) gives

$$(w_2^{-1}w_1)P_1(w_2^{-1}w_1)^{-1} = \theta P_1 . \tag{1.10a}$$

Also w_1 in $N_K(\mathfrak{a}_1)$ and w_2 in the centralizer $Z_K(\mathfrak{a}_1)$ imply $w_2^{-1}w_1$ is in $N_K(\mathfrak{a}_1)$, and (1.9b) shows $w_2^{-1}w_1$ is in $N_K(\mathfrak{a}_*)$. Thus

$$w_2^{-1}w_1 \in N_K(\mathfrak{a}) \cap N_K(\mathfrak{a}_*) . \tag{1.10b}$$

From (1.9c) we have

$$w_2^{-1}w_1\sigma \cong \sigma . \tag{1.10c}$$

By (1.9d) and (1.10b), we have $\mathrm{Ad}(w_2^{-1}w_1)(\nu|_{\mathfrak{a}_*}) = (\nu|_{\mathfrak{a}_*})$, which is imaginary. Hence

$$\mathrm{Ad}(w_2^{-1}w_1)(\nu|_{\mathfrak{a}_*}) = -(\bar{\nu}|_{\mathfrak{a}_*}) ,$$

and (1.8) gives

$$\mathrm{Ad}(w_2^{-1}w_1)\nu = -\bar{\nu} . \tag{1.10d}$$

Dropping "Ad" for simplicity, let us observe that $w_2^{-1}w_1$ normalizes the system Δ' of (1.2). [In fact, α in Δ' implies $w_2^{-1}w_1\alpha$ useful, and we have

$$s_{w_2^{-1}w_1\alpha}\nu = (w_2^{-1}w_1)s_\alpha(w_2^{-1}w_1)^{-1}\nu = -\,w_2^{-1}w_1s_\alpha\bar{\nu}$$

$$= -\,w_2^{-1}w_1\bar{\nu} = \nu$$

by two applications of (1.10d). Also

$$\mu_{\sigma,w_2^{-1}w_1\alpha}(\nu) = \mu_{(w_2^{-1}w_1)^{-1}\sigma,\alpha}((w_2^{-1}w_1)^{-1}\nu)$$

$$= \mu_{\sigma,\alpha}((w_2^{-1}w_1)^{-1}\nu) \qquad \text{by (1.10c)}$$

$$= \mu_{\sigma,\alpha}(\nu)$$

since $\mu_{\sigma,\alpha}$ depends only on the $\mathfrak{a}_*$ component and since $w_2^{-1}w_1$ fixes $\nu|_{\mathfrak{a}_*}$.] Then we can choose w_3 in $K \cap M_1$ representing a member of $W'_{\sigma,\nu}$ such that

$$w_3^{-1}w_2^{-1}w_1\Delta'^{+} = \Delta'^{+} . \tag{1.11}$$

Then it is clear that

$$(w_3^{-1}w_2^{-1}w_1)P_1(w_3^{-1}w_2^{-1}w_1)^{-1} = \theta P_1 \tag{1.12a}$$

and

$$w_3^{-1}w_2^{-1}w_1 \in N_K(\mathfrak{a}) \cap N_K(\mathfrak{a}_*) . \tag{1.12b}$$

Since $W'_{\sigma,\nu} \subseteq W_{\sigma,\nu}$, (1.10c) and (1.10d) give

$$w_3^{-1}w_2^{-1}w_1\sigma \cong \sigma \qquad (1.12c)$$

and

$$\mathrm{Ad}(w_3^{-1}w_2^{-1}w_1)\nu = -\bar{\nu} \ . \qquad (1.12d)$$

Let w be the class of $w_3^{-1}w_2^{-1}w_1$ in $W(A:G)$. Then w^2 fixes σ and ν, by (1.12c) and (1.12d), and so is in $W_{\sigma,\nu}$. Since π is irreducible (in order to have $J'(P,\sigma,\nu)$ defined), we have $W_{\sigma,\nu} = W'_{\sigma,\nu}$. Thus w^2 is in $W'_{\sigma,\nu}$. From (1.11), $w^2\Delta'^+ = \Delta'^+$, and thus $w^2 = 1$. This identity and formulas (1.12c) and (1.12d) together prove (i) and complete the proof of the theorem.

§2. Progress

The problem of classifying irreducible unitary representations comes down to deciding which parameters (P,σ,ν) in Theorem 1.2 satisfy (i) and (ii) of the theorem. Here (i) is easy to decide, but (ii) is often hard. There are several sufficient conditions for deciding one way or the other, and we shall list a number of them in §4. It is unlikely that the final answer will be a group-by-group investigation, but it does give some idea of the nature of the problem to tell what simple noncompact matrix groups have been completely settled.

The groups handled so far are the following.

Before the Langlands classification historically:

$SL(2,\mathbb{R})$	Bargmann [2], 1947
$SL(2,\mathbb{C})$	Bargmann [2], 1947
	Gelfand-Naimark [9], 1947
$SL(3,\mathbb{R})$	Vahutinskii [36], 1968
$SL(3,\mathbb{C})$	Tsuchikawa [35], 1968
$SO_e(n,1)$, double cover	Hirai [13], 1962
$SU(n,1)$	Ottoson [29], 1968
	Kraljević [26], 1973

After the Langlands classification historically:

$Sp(2,\mathbb{C})$ and $G_2^{\mathbb{C}}$	Duflo [4], 1976
$SL(4,\mathbb{R})$ (actually $GL(4,\mathbb{R})$)	Speh [33,34], 1977
$Sp(n,1)$	Baldoni Silva [1], 1980
$SL(4,\mathbb{C})$ and $SL(5,\mathbb{C})$	Duflo [5], 1980
$SU(2,2)$	Knapp-Speh [19], 1981
$SO_e(3,2)$	Several people independently, including the authors.

In retrospect, the groups done before 1973 can be settled quickly with the aid of the Langlands classification. The groups settled since 1973 are qualitatively different in that the pattern of unitarity is substantially more subtle.

Let us list some approaches to the general problem that cut across broad classes of groups:

1) Unusual constructions of unitary representations.

a) Representations with a highest weight vector. These were introduced by Harish-Chandra [12], and a systematic study of which ones are unitary was begun by Wallach [38] and Rossi-Vergne [31] and has now been completed by Enright-Howe-Wallach [7] and Jakobsen [43].

b) Dual reductive pairs. Two groups form a dual reductive pair if they are each other's centralizer in a real symplectic group. Howe earlier conjectured and recently proved [15] that the reduction of the Weil representation of the symplectic group should lead to a natural one-one correspondence between some unitary representations of one of the groups and some unitary representations of the other.

c) Occurrence in $L^2(G/H)$. See Flensted-Jensen [8].

2) Zuckerman's method. Zuckerman [41] has given a construction, by means of an analog of the realization of discrete series, of some irreducible admissible representations that are conjectured to be unitary, and this conjecture has been verified in some special cases.

3) Results for complex groups. Enright [6] has classified the infinitesimally unitary representations (of complex groups) with regular integral infinitesimal character. More recently, Enright has results on infinitesimally unitary representations (of complex groups) with a K-fixed vector and with regular infinitesimal character.

§3. Some results for SU(N,1) and SU(N,2)

In this section we announce the unitarity of some degenerate series representations of $SU(N,2)$ that one might not have expected to be unitary. We shall give the proof on another occasion, being content now with the statement of the result, a diagram illustrating the result, and the necessary preliminaries reformulating the length of the complementary series in $SU(n,1)$.

We begin with the preliminaries about $SU(n,1)$. These matters were brought to our attention by G. Zuckerman in 1973-74. Our concern will be with induced representations from P_{min}. Here A_{min} is one-dimensional, and we can therefore write $\nu = z\rho$, where z is in $\mathbb{C}$ and ρ is the half-sum of the positive $\mathfrak{a}_{\min}$ roots with multiplicities counted. If σ is an irreducible representation of $M_{\min}$, then the results of §§14-15 of [21] show that $J'(P_{\min},\sigma,z\rho)$ is infinitesimally unitary for $\operatorname{Re} z > 0$ if and only if z is real with $0 < z \leq z_c$, where z_c is the critical abscissa, defined as the least $z \geq 0$ that is a multiple of $1/n$ such that

(i) $z^{-1}\mu_{\sigma,\alpha}(z\rho) = \infty$ (with α the short positive $\mathfrak{a}_{\min}$ root), or

(ii) $\mu_{\sigma,\alpha}(z\rho) \neq 0$ and $A_{P_{\min}}(w,\sigma,-z\rho)$ has a pole (with w representing the nontrivial element of $W(A_{\min}:G)$).

It is always the case that $0 \leq z_c \leq 1$.

The Plancherel factor $\mu_{\sigma,\alpha}$ in the definition of "critical abscissa" is given as follows: If λ denotes the highest weight of σ and ρ^- denotes the half-sum of the positive roots of $M_{\min}$, then apart from a multiplicative constant we have

$$\mu_{\sigma,\alpha}(z\rho) = z\Big(\prod_{\beta|_{\mathfrak{a}_{\min}}=\alpha} \langle \lambda + \rho^- + z\rho\, ,\, \beta\rangle\Big)\, f_\alpha(n\pi z/2)\ ,$$

where f_α is tan or cot according as whether $\sigma(\gamma_\alpha)$ is $(-1)^{n+1}I$ or $(-1)^n I$, for a certain element γ_α of order 2 in $M_{\min}$ defined in (3.1) below. (See §12 of [21].) The values of z for which f_α has a pole are exactly the values of z for which the infinitesimal character of $U(P_{\min},\sigma,z\rho)$ is integral. The result we need is as follows, and we give a proof after Lemma 3.2.

Proposition 3.1 (Zuckerman). In $SU(n,1)$, if σ is an irreducible representation of M_{min}, then the critical abscissa z_c for σ is the least $z \geq 0$ such that the infinitesimal character of $U(P_{min},\sigma,z\rho)$ is integral and is not orthogonal to two linearly independent complex roots. Equivalently, z_c is the least nonnegative

(i) odd multiple of $1/n$ in the tangent case such that

$$\langle \lambda + \rho^- + z\rho \, , \beta \rangle = 0 \quad \text{for at most one } \beta \text{ with } \beta|_{\mathfrak{a}_{min}} = \alpha$$

(ii) even multiple of $1/n$ in the cotangent case such that

$$\langle \lambda + \rho^- + z\rho \, , \beta \rangle = 0 \quad \text{for at most one } \beta \text{ with } \beta|_{\mathfrak{a}_{min}} = \alpha .$$

Lemma 3.2 (Zuckerman). In the tangent case if $\mu_{\sigma,\alpha}(\frac{1}{n}\rho) \neq \infty$, then $A_{P_{min}}(w,\sigma,z\rho)$ has a pole at $z = -1/n$.

Before proving the lemma, we make the notation more concrete. We suppose that $SU(n,1)$ is given in its standard representation, with $K = S(U(n) \times U(1))$. Diagonal matrices in the Lie algebra form a compact Cartan subalgebra, and we let e_j denote evaluation of the j^{th} diagonal entry. The group A_{min} is one-dimensional, and we choose to form it by Cayley transform of the root $e_n - e_{n+1}$. Half the sum of the positive restricted roots, with multiplicities, is then

$$\rho = \frac{n}{2} \operatorname{Cayley}(e_n - e_{n+1}) \; .$$

The group M_{min} is given by

$$M_{\min} = \left\{ \begin{pmatrix} u & & \\ & e^{i\theta} & \\ & & e^{i\theta} \end{pmatrix} \begin{matrix} \}n-1 \\ \}\ 1 \\ \}\ 1 \end{matrix} \quad \begin{matrix} \text{with } u \text{ unitary and} \\ \text{total determinant} = 1 \end{matrix} \right\},$$

and γ_α is the element of $M_{\min}$ given by

$$\gamma_\alpha = \begin{pmatrix} I & & \\ & -1 & \\ & & -1 \end{pmatrix}. \tag{3.1}$$

We can write the highest weight of σ as

$$\lambda = \sum_{j=1}^{n-1} c_j e_j ,$$

where $\{c_j\}$ is a nonincreasing sequence of integers or half-integers. The function f_α is

$$\begin{cases} \tan & \text{if all } c_j \in \mathbb{Z} + \frac{1}{2}(n+1) \\ \cot & \text{if all } c_j \in \mathbb{Z} + \frac{1}{2}n . \end{cases} \tag{3.2}$$

A pair of complex conjugate roots β with $\beta|_{\mathfrak{a}_{\min}} = \alpha$ is of the form $\{e_i - e_n , e_i - e_{n+1}\}$ if we drop the notation "Cayley," and its contribution to $\mu_{\sigma,\alpha}(z\rho)$ is

$$[c_i + \tfrac{1}{2}(n-2i) + \tfrac{1}{2}zn]\,[c_i + \tfrac{1}{2}(n-2i) - \tfrac{1}{2}zn] . \tag{3.3}$$

<u>Proof of Lemma 3.2</u>. For $\mu_{\sigma,\alpha}(\frac{1}{n}\rho)$ not to have a pole, there must be an index i for which (3.3) vanishes at $z = 1/n$. Then

$$(c_i + \tfrac{1}{2}) + \tfrac{1}{2}(n-2i) + \tfrac{1}{2}\,0\,n = 0 \tag{3.4a}$$

or

$$(c_i - \tfrac{1}{2}) + \tfrac{1}{2}(n-2i) + \tfrac{1}{2}\,0\,n = 0\,. \tag{3.4b}$$

Let σ' and σ'' be irreducible representations of $M_{\min}$ with highest weights

$$\sum_{j=1}^{n-1}(c_j + \tfrac{1}{2})e_j \quad \text{and} \quad \sum_{j=1}^{n-1}(c_j - \tfrac{1}{2})e_j\,,$$

respectively. Then σ' and σ'' correspond to cotangent cases, by (3.2), and (3.4) says $\mu_{\sigma',\alpha}(0) = 0$ or $\mu_{\sigma'',\alpha}(0) = 0$, say the first.

We shall now use some detailed information about intertwining operators given in the results and proofs of [21]. By Proposition 27(vii) and Theorem 4 of [21], the kernel of the intertwining operator for σ' has mean value $\neq 0$. In the notation of page 558 of [21], σ equals $((n+1),\sigma_0)$, where σ_0 has highest weight $\Sigma[c_j + \tfrac{1}{2}(n+1)]e_j$; the kernel for σ at $z = 0$ is

$$(\|X\|^4 + |Y|^2)^{-n/2}\left(\frac{\|X\|^2 + Yi}{(\|X\|^4 + |Y|^2)^{1/2}}\right)^{-n-1}\sigma_0\left(I - \frac{2(\|X\|^2 + Yi)X\bar{X}^t}{\|X\|^4 + |Y|^2}\right). \tag{3.5}$$

Also σ' equals (n,σ_0), and the kernel for σ' at $z = 0$ is

$$(\|X\|^4 + |Y|^2)^{-n/2}\left(\frac{\|X\|^2 + Yi}{(\|X\|^4 + |Y|^2)^{1/2}}\right)^{-n}\sigma_0(\text{same})\ ;$$

since $\|X\|^4 + |Y|^2$ is radial, the mean value satisfies

$$\mathfrak{M}\,[(\|X\|^4 + |Y|^2)^{\text{power}}(\|X\|^2 + Yi)^{-n}\,\sigma_0(\text{same})] \neq 0\,. \tag{3.6}$$

Now we check the criterion for $A_{P_{\min}}(w,\sigma,-z\rho)$ to have a pole at $z = 1/n$. We refer to pp. 523-524 of [21]. If $y \leftrightarrow (X,Y)$, then we find that

$$e^{(1+z)\rho H(y)} = [1 + \|X\|^2 + \tfrac{1}{4}(\|X\|^4 + |Y|^2)]^{n(1+z)/2}. \qquad (3.7)$$

Choose a smooth f supported near the identity to be given near $y = 1$ by

$$f(\kappa(y)) = \|X\|^2 + Yi . \qquad (3.8)$$

The Taylor expansion of the product of (3.7) and (3.8), when rearranged so as to be homogeneous, is

$$(1 + \tfrac{n}{2}(1+z)\|X\|^2 + \text{higher order terms})(\|X\|^2 + Yi).$$

Thus, in the notation of pp. 523-524 of [21], we have

$$g_2(z,y) = \|X\|^2 + Yi \qquad (3.9)$$

and

$$\begin{aligned} h_2(z,y) &= (\|X\|^4 + |Y|^2)^{\text{power}}(\text{kernel for } \sigma) g_2(z,y) \\ &= (\|X\|^4 + |Y|^2)^{\text{power}}(\|X\|^2 + Yi)^{-n}\, \sigma_0(\text{same}) \end{aligned}$$

by (3.5) and (3.9). By (3.6), this has nonzero mean value, and then (9.13) of [21] shows the operator has a pole.

<u>Proof of Proposition 3.1</u>. We consider the tangent case first. When z is an even multiple of $1/n$, $f_\alpha(n\pi z/2) = 0$ and thus $\mu_{\sigma,\alpha}(z\rho) = 0$. Consequently the critical abscissa is an odd multiple of $1/n$. If $\mu_{\sigma,\alpha}(\frac{1}{n}\rho) = \infty$, then $z_c = 1/n$ and also

$$\langle \lambda + \rho^- + \tfrac{1}{n}\rho , \beta\rangle \neq 0 \quad \text{for all } \beta \text{ with } \beta|_{\mathfrak{a}_{\min}} = \alpha ;$$

hence z_c is indeed given as in (i) of the proposition. If $\mu_{\sigma,\alpha}(\frac{1}{n}\rho) \neq \infty$, then Lemma 3.2 says that the intertwining operator has a pole at $z = -1/n$. From the top of p. 551 of [21], the intertwining operator has a pole at every negative odd multiple of $1/n$. Hence z_c is the least positive odd multiple of $1/n$ such that $\mu_{\sigma,\alpha}(z\rho) \neq 0$, and thus z_c is given as in (i) of the proposition.

In the cotangent case, matters are easier. When z is an odd multiple of $1/n$, $f_\alpha(n\pi z/2) = 0$ and thus $\mu_{\sigma,\alpha}(z\rho) = 0$. Consequently the critical abscissa is an even multiple of $1/n$. If $\mu_{\sigma,\alpha}(0) \neq 0$, then $z_c = 0$ and also

$$\langle \lambda + \rho^- + 0\rho \, , \beta \rangle \neq 0 \quad \text{for all } \beta \text{ with } \beta|_{\mathfrak{a}_{min}} = \alpha \, ;$$

hence z_c is given as in (ii) of the proposition. If $\mu_{\sigma,\alpha}(0) = 0$, then Theorem 4 and Proposition 27 of [21] say the intertwining operator has a pole at $z = 0$. From the top of p. 551 of [21], the operator has a pole at every negative even multiple of $1/n$. Hence z_c is the least positive even multiple of $1/n$ such that $\mu_{\sigma,\alpha}(z\rho) \neq 0$, and z_c is given as in (ii) of the proposition.

<u>Corollary 3.3</u>. In $SU(n,1)$ let σ be an irreducible representation of M_{min}, and let z_c be the critical abscissa.

(i) In a tangent case, $z_c > 1/n$ if and only if $\mu_{\sigma,\alpha}(\frac{1}{n}\rho) = 0$, i.e., if and only if there exists i with $2 \leq i \leq n-1$ and

$$c_{i-1} = c_i = \tfrac{1}{2}(2i - n - 1).$$

(ii) In a tangent case, $z_c = \frac{n-k+1}{n}$ with $k < n$ if and only if k is the smallest integer $\equiv n \bmod 2$ such that the index i of conclusion (i) satisfies

$$1 \leq i - \tfrac{1}{2}(n-k) \quad \text{and} \quad i + \tfrac{1}{2}(n-k) \leq n$$

and

$$c_j = \tfrac{1}{2}(2i - n - 1) \quad \text{for all } j \text{ with } i - \tfrac{1}{2}(n-k) \leq j < i + \tfrac{1}{2}(n-k).$$

(iii) In a cotangent case, $z_c > 0$ if and only if $\mu_{\sigma,\alpha}(0) = 0$, i.e., if and only if there exists i with $1 \leq i \leq n-1$ and

$$c_i = \tfrac{1}{2}(2i - n) .$$

(iv) In a cotangent case, $z_c = \frac{n-k+1}{n}$ with $k < n+1$ if and only if k is the smallest integer $\not\equiv n \bmod 2$ such that the index i of conclusion (iii) satisfies

$$1 \leq i + 1 - \tfrac{1}{2}(n-k+1) \quad \text{and} \quad i + \tfrac{1}{2}(n-k+1) \leq n$$

and

$$c_j = \tfrac{1}{2}(2i - n) \quad \text{for all } j \text{ with}$$

$$i + 1 - \tfrac{1}{2}(n-k+1) \leq j < i + \tfrac{1}{2}(n-k+1) .$$

We omit the proof, which is a combinatorial exercise in the presence of Proposition 3.1.

Now we turn to $SU(N,2)$, and we suppose $N \geq 3$. The roots relative to $\mathfrak{a}_{\min}$ form a system of type $(BC)_2$, generated by $f_1 - f_2$, f_2, and $2f_2$. We can arrange that $M_{\min}$ consists of all matrices

$$\begin{pmatrix} e^{i\theta} & & & & \\ & e^{i\varphi} & & & \\ & & u & & \\ & & & e^{i\theta} & \\ & & & & e^{i\varphi} \end{pmatrix} \tag{3.10}$$

with u unitary of size $N-2$ and with the total determinant equal to one. An irreducible representation σ of M_{min}, when applied to (3.10), has the form

$$\sigma(\text{—}) = e^{i(n\theta+m\varphi)}\,\sigma_0(u) , \tag{3.11}$$

where σ_0 is an irreducible representation of the unitary group $U(N-2)$.

If we restrict σ to matrices (3.10) with $\varphi = 0$, we obtain an irreducible representation σ_1 of the M_{min} of an imbedded subgroup $SU(N-1, 1)$ of $SU(N,2)$. Let $z_c(\sigma_1)$ be the critical abscissa (see the previous discussion) for σ_1. Similarly if we restrict σ to matrices (3.10) with $\theta = 0$, we obtain an irreducible representation σ_2 of the M_{min} of a different imbedded subgroup $SU(N-1, 1)$. Let $z_c(\sigma_2)$ be the critical abscissa for σ_2.

We study the series of representations attached to P_{min} and to σ. The parameter ν on $\mathfrak{a}_{min}$ will satisfy the formal symmetry condition (i) of Theorem 1.2 if ν is real. Let us write $\nu = af_1 + bf_2$, with a and b real. To have ν in the closed positive Weyl chamber, we assume $a \geq b \geq 0$. We seek values of a and b for which $J'(P_{min}, \sigma, af_1+bf_2)$ is infinitesimally unitary. Under an assumption on σ, the nontrivial values of a and b for which we know the representation is infinitesimally unitary are given in the following theorem.

Theorem 3.4. In $SU(N,2)$ with $N \geq 3$, let σ be given as in (3.11). If $z_c(\sigma_1) > 0$ and $z_c(\sigma_2) > 0$, then $J'(P_{min}, \sigma, af_1+bf_2)$ is defined and infinitesimally unitary whenever

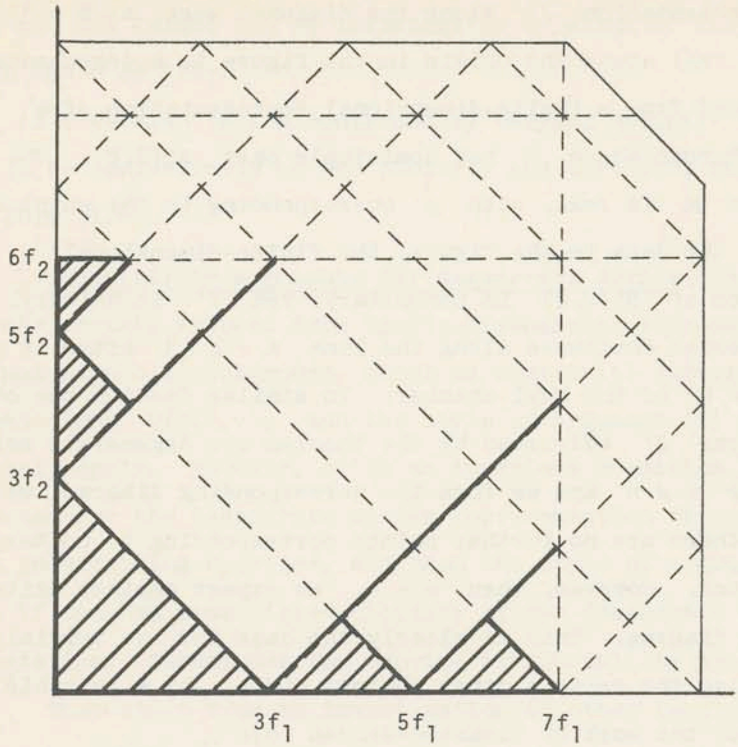
$$a \leq (N-1)z_c(\sigma_1),$$

$$0 \leq b \leq (N-1)z_c(\sigma_2),$$

and $$a = |n-m| + 2k + b \quad \text{for an integer } k \geq 1.$$

To understand the statement of the theorem, let us refer to Figure 1. There $N = 8$, $n = 0$, $m = 1$, and σ_0 is trivial. Two Weyl chambers of $\mathfrak{a}'_{min}$ are exhibited for clarity. The dotted rectangle marks off the influence of the end of the complementary series for $SU(N-1,1)$, and the assertion of the theorem is that the diagonal lines sloping up to the right correspond to unitary representations.

There are other points in the figure that are asserted to correspond to unitary representations that are not addressed by the theorem, and they are easy to check: The triangle that abuts the origin gives unitary representations because the operator (1.5) at $\nu = 0$ for $w = -1$ is scalar and has to remain semidefinite within the triangle by continuity. Along the axes out to the horizontal or vertical dotted lines, J' can be regarded as unitarily induced from a parabolic subgroup of $SU(N,2)$ whose M has semisimple part $SU(N-1,1)$; the unitary representation of M is in the complementary series. The other triangles correspond to unitary representations by a continuity argument. (Analogous triangles were



$\nu = af_1 + bf_2$, two chambers showing

Positive chamber: $0 \leq b \leq a$

Key: and unitary points

reducible points

edge of region of bounded matrix coefficients

Figure 1. Unitary $J'(P_{min}, \sigma, \nu)$ in $SU(8,2)$ with ν real, $\sigma_0 = 1$, $n = 0$, and $m = 1$.

discovered earlier by Guillemonat [10] for another group.)

The representation J' along the diagonal edge $a+b=3$ of the triangle that abuts the origin in the figure is a degenerate series, induced from a finite-dimensional representation of a parabolic subgroup whose M has semisimple part $SL(2,\mathbb{C})$. The linear functional on $\mathfrak{a}$ is real, with 0 corresponding to the point where $a=b$. For the data in the figure, the finite-dimensional representation of $SL(2,\mathbb{C})$ is nonunitary, yet J' is unitary. The degenerate series continues along the line $a=b+3$ after it meets the wall $b=0$ of the Weyl chamber. In similar fashion the other representations J' addressed by the theorem are degenerate series.

Whenever $m \neq n$ and we form the corresponding diagram, we expect that there are no further points corresponding to unitary representations. However, when $m=n$, we expect another unitary point in the diagram. This is clearly the case for σ trivial, and we know it also for certain other choices of σ, by a suitable application of the work of Flensted-Jensen [8].

§4. Techniques for deciding whether J' is unitary

We record here a number of known techniques for deciding whether a representation $J'(P,\sigma,\nu)$ satisfies condition (ii) of Theorem 1.2 and is therefore infinitesimally unitary.

1. Continuity arguments with Hermitian forms [20]. A continuous function from a connected set into nondegenerate Hermitian forms on a finite-dimensional complex vector space has values that are everywhere definite or nowhere definite. Formula (1.6) gives a Hermitian form on an infinite-dimensional space to which this fact can be applied, since the space is the orthogonal sum of its K-isotypic (finite-dimensional)

subspaces. Nondegeneracy will occur wherever $\mathcal{A}_P(w,\sigma,\nu)$ has zero kernel, and the kernel can be investigated by means of the identities given in §§6-8 of [22].

It is true also that a continuously varying definite Hermitian form will be semidefinite on the closure, and this fact can be applied to the form (1.6).

2. <u>Irreducibility arguments for degenerate series</u> [33]. Degenerate series, induced from finite-dimensional representations of nonminimal parabolic subgroups, occur as nontrivial quotients of representations $U(P,\sigma,\nu)$, and the style of argument (1) does not immediately apply. However, often an invariant Hermitian form can be constructed for the degenerate series representation directly, usually from an intertwining operator, and then the style of argument (1) applies if one can prove irreducibility of the degenerate series representation. Techniques for proving irreducibility are developed in [33]. They start from an investigation of other Langlands parameters with the same infinitesimal character. Theorem 3.4 above is proved in this way.

3. <u>Induction of representations from parabolic subgroups</u>. A representation induced from a unitary representation of $P = MAN$ is of course unitary. This observation produces interesting unitary representations when applied in the case of a complementary series representation of M and the trivial character of A, as was the case with $G = SU(N,2)$ in §3. A variant of this observation is useful [34]: Suppose π is an irreducible admissible representation of M that admits an invariant nondegenerate Hermitian nondefinite form and has π equivalent with $w\pi$, where $wPw^{-1} = \theta P$; if

$$\operatorname{ind}_{MAN}^{G}(\pi \otimes 1 \otimes 1)$$

is irreducible, then it is not infinitesimally unitary.

4. Sufficiency of studying cases with real infinitesimal character. In unpublished work Vogan has proved the following theorem: Any irreducible unitary representation with nonreal infinitesimal character is of the form

$$\mathrm{ind}_{MAN}^{G}(\pi \otimes e^{i\lambda} \otimes 1) ,$$

where MAN is a proper parabolic subgroup, π is an irreducible unitary representation of M with real infinitesimal character, and $e^{i\lambda}$ is a unitary character of A. The proof is constructive and therefore reduces the classification problem to the case of real infinitesimal character.

5. Dirac inequality. This inequality has its origins in work of Schmid [32] on realization of discrete series. The Dirac operator was used for it by Parthasarathy [30, p. 29] and Hotta-Parthasarathy [14, p. 163]. Later Enright [6] used a form of it in deciding whether certain representations were unitary. Baldoni Silva gave it in the form below and used it in dealing with $Sp(n,1)$. See Borel-Wallach [3] for further discussion.

Suppose that $\operatorname{rank} G = \operatorname{rank} K$ and that π is an irreducible unitary representation of G with real infinitesimal character Λ. Choose a positive system of roots relative to a compact Cartan subgroup, and let ρ_k and ρ_n be the half-sums of the positive compact roots and positive noncompact roots, respectively. Suppose that μ is the highest weight of a K-type appearing in $\pi(K)$. If w is a member of the Weyl group of K such that $w(\mu - \rho_n)$ is K-dominant, then

$$|\Lambda|^2 \leq |w(\mu - \rho_n) + \rho_k|^2 .$$

6. Minimal K-type arguments. Minimal K-types were introduced by Vogan [37], and Baldoni Silva [1] used them in proving that certain representations of $Sp(n,1)$ are not unitary. The key fact is that the Langlands intertwining operator is nonvanishing on a minimal K-type, under suitable hypotheses. Baldoni Silva has two ways of applying this fact.

For certain representations of M, she proves that the intertwining operator is nondefinite at $\nu = 0$ on the span of the minimal K-types, and she is able to conclude the same thing for other ν by the key fact and by (1) above.

For certain other representations of M, she applies the Dirac inequality above to $J'(P,\sigma,\nu)$, taking μ to be a minimal K-type of the induced representation $U(P,\sigma,\nu)$. It is the key fact that shows that a minimal K-type of $U(P,\sigma,\nu)$ occurs as a K-type in $J'(P,\sigma,\nu)$. The result is a computable bound on $|\nu|$ if $J'(P,\sigma,\nu)$ is unitary and ν is real.

7. Isolation of trivial representation (Kazhdan [17] and Wang [40]). If G is simple and $\dim A_{\min} \geq 2$, then the trivial representation is isolated in the Fell topology of the unitary dual of G. It follows that the trivial representation is isolated in the unitary points of $J'(P_{\min},1,\nu)$.

8. Asymptotics of K-finite matrix coefficients (Howe-Moore [16]). The K-finite matrix coefficients of any nontrivial irreducible unitary representation vanish at infinity. Hence $J'(P,\sigma,\nu)$, for ν real, can be unitary only if ν lies in the interior of the convex hull of the half-sums of the positive roots in the various orderings.

9. Detailed analysis of intertwining operators. This is a technique introduced by Duflo [4] for complex groups and developed in

part from the work of Kostant [25]. Duflo uses explicit formulas for intertwining operators for $SL(2,\mathbb{C})$ and product formulas for intertwining operators for complex groups to show that certain representations are not unitary. This technique will be extended in the next section, whose results are used in [19] to handle $SU(2,2)$.

§5. Explicit formulas for intertwining operators

In [4] Duflo introduced a technique, developed in part from the work of Kostant [25], for analyzing intertwining operators for complex groups. In this section we shall generalize the technique to real groups by beginning with the analysis done by Wallach in §8.11 of [39].

We shall study only the series of representations associated to P_{min}, and we therefore systematically write P, M, A, and N in place of P_{min}, M_{min}, A_{min}, and N_{min}.

Let τ be an irreducible representation of K on a space V^τ with character χ_τ and degree d_τ, and let σ be an irreducible representation of M on a space V_σ with character χ_σ and degree d_σ. We recall that

$$L = \operatorname{ind}_M^K \sigma$$

acts in the space of V_σ-valued functions on K satisfying

$$f(km) = \sigma(m)^{-1} f(k)$$

by $L(k_0)f(k) = f(k_0^{-1}k)$, with norm given by the L^2 norm on K. Let $\mathbb{V}^\tau$ be the subspace of this space that transforms according to τ, i.e., is fixed by

$$d_\tau \int_K \overline{\chi_\tau(k)} L(k)\, dk \, .$$

The first result was noted by Wallach in §8.11 of [39].

<u>Proposition 5.1</u>. For v in V^τ and E in $\mathrm{Hom}_M(V^\tau, V_\sigma)$, define

$$\varphi_{v,E}(k) = E(\tau(k)^{-1}v) \, .$$

Then $\varphi_{v,E}$ is in $\mathbb{V}^\tau$ and the extension Φ of the resulting bilinear map

$$V^\tau \times \mathrm{Hom}_M(V^\tau, V_\sigma) \to \mathbb{V}^\tau$$

to the tensor product

$$V^\tau \otimes_{\mathbb{C}} \mathrm{Hom}_M(V^\tau, V_\sigma)$$

is an isomorphism onto $\mathbb{V}^\tau$.

Now we shall interpret the standard intertwining operators of §§4-8 of [22] in the context of Proposition 5.1. Let α be a simple $\mathfrak{a}$ root, V_α be the exponential of the sum of the root spaces for the negative multiples of α, and ρ_α be half the sum (counting multiplicities) of the positive multiples of α that are roots. We use the notation

$$g = \kappa(g)(\exp H(g))n$$

for the Iwasawa decomposition of g relative to $G = KAN$. One of the standard intertwining operators of [22] is given by the analytic continuation of

$$A(s_\alpha^{-1}Ps_\alpha : P:\sigma:\lambda)f(k) = \int_{V_\alpha} e^{-(\rho_\alpha+\lambda)H(\overline{n})} f(k\kappa(\overline{n}))\, d\overline{n} \; ; \tag{5.1}$$

this operator maps V^τ into itself. We define

$$a_{\tau,\alpha}(\lambda) = \int_{V_\alpha} e^{-(\rho_\alpha+\lambda)H(\overline{n})} \tau(\kappa(\overline{n}))^{-1}\, d\overline{n} \; . \tag{5.2}$$

The next result is stated and proved in §8.11 of Wallach [39].

Proposition 5.2. The operator $a_{\tau,\alpha}(\lambda)$ is in $\mathrm{Hom}_M(V^\tau, V^\tau)$, and the operator $A(s_\alpha^{-1}Ps_\alpha : P:\sigma:\lambda)$ is given on V^τ by the formula

$$\Phi^{-1} A(s_\alpha^{-1}Ps_\alpha : P:\sigma:\lambda)\, \Phi = I \otimes (\text{right by } a_{\tau,\alpha}(\lambda)) \; . \tag{5.3}$$

Remarks. The formula for the intertwining operator, in more concrete terms, is

$$A(s_\alpha^{-1}Ps_\alpha : P:\sigma:\lambda)\varphi_{v,E} = \varphi_{v,Ea_{\tau,\alpha}(\lambda)} \; . \tag{5.4}$$

There is a certain amount of flexibility in how we write (5.3) or (5.4); only the part of $a_{\tau,\alpha}(\lambda)$ that deals with the σ-subspace of V^τ is relevant. More precisely, define

$$P_\sigma^\tau = d_\sigma \int_M \overline{\chi_\sigma(m)}\, \tau(m)\, dm$$

and

$$P_\sigma = d_\sigma \int_M \overline{\chi_\sigma(m)}\, \sigma(m)\, dm \; .$$

(The latter operator is just the identity.) Then $E = P_\sigma E = EP_\sigma^\tau$. Since P_σ^τ is a projection and is in the span of the $\tau(m)$,

$$P_\sigma^\tau a_{\tau,\alpha}(\lambda) = (P_\sigma^\tau)^2 a_{\tau,\alpha}(\lambda) = P_\sigma^\tau a_{\tau,\alpha}(\lambda) P_\sigma^\tau \ .$$

Thus

$$Ea_{\tau,\alpha}(\lambda) = E(P_\sigma^\tau a_{\tau,\alpha}(\lambda) P_\sigma^\tau) \ . \tag{5.5}$$

Now suppose that w represents an element in the Weyl group $W(A:G)$, and suppose $w\sigma \cong \sigma$. We shall abuse notation, writing Weyl group elements and representatives in the same way. If we decompose w into a minimal product of simple reflections as $w = s_{\alpha_1} \cdots s_{\alpha_n}$, then the operator whose normalized version is (1.5) is the operator

$$\sigma(w) A_P(w,\sigma,\lambda) \ , \tag{5.6}$$

and it expands according to a cocycle relation as

$$= \sigma(w) A_P(s_{\alpha_1}, s_{\alpha_2} \cdots s_{\alpha_n} \sigma, s_{\alpha_2} \cdots s_{\alpha_n} \lambda) \cdot \ldots \cdot A_P(s_{\alpha_n}, \sigma, \lambda) \ .$$

Here

$$A_P(s_\alpha, \sigma', \lambda') = R(s_\alpha) A(s_\alpha^{-1} P s_\alpha : P : \sigma' : \lambda') \ ,$$

where R denotes right translation, and thus we can compute (5.6) from (5.3), as follows.

Proposition 5.3. If $w\sigma \cong \sigma$, then the operator $\sigma(w) A_P(w,\sigma,\lambda)$ is given on $\mathbb{V}^\tau$ by the formula

$$\Phi^{-1} \sigma(w) A_P(w,\sigma,\lambda) \Phi = I \otimes T \ ,$$

where T is the successive composition of the following endomorphisms of $\mathrm{Hom}_M(V^\tau, V_\sigma)$:

$$E \to \sigma(w)E\tau(w)^{-1}$$

right by $\tau(s_{\alpha_1}\cdots s_{\alpha_n})a_{\tau,\alpha_n}(\lambda)\tau(s_{\alpha_1}\cdots s_{\alpha_n})^{-1}$

right by $\tau(s_{\alpha_1}\cdots s_{\alpha_{n-1}})a_{\tau,\alpha_{n-1}}(s_{\alpha_n}\lambda)\tau(s_{\alpha_1}\cdots s_{\alpha_{n-1}})^{-1}$

$\vdots$

right by $\tau(s_{\alpha_1})a_{\tau,\alpha_1}(s_{\alpha_2}\cdots s_{\alpha_n}\lambda)\tau(s_{\alpha_1})^{-1}$.

Proof. For E' in $\mathrm{Hom}_M(V^\tau, V_{\sigma'})$, we have

$$\begin{aligned} A_P(s_\alpha,\sigma',\lambda')\varphi_{v,E'}(k) &= (A(s_\alpha^{-1}Ps_\alpha:P:\sigma':\lambda')\varphi_{v,E'})(ks_\alpha) \\ &= Ea_{\tau,\alpha}(\lambda')\tau(s_\alpha)^{-1}\tau(k)^{-1}v \\ &= \varphi_{v,Ea_{\tau,\alpha}(\lambda')\tau(s_\alpha)^{-1}}(k)\ , \end{aligned}$$

with $Ea_{\tau,\alpha}(\lambda)\tau(s_\alpha)^{-1}$ in $\mathrm{Hom}_M(V^\tau, V_{s_\alpha\sigma})$. Thus we can calculate $A_P(w,\sigma,\lambda)\varphi_{v,E}$ inductively and obtain

$$\begin{aligned} \sigma(w)A_P(w,\sigma,\lambda)\varphi_{v,E} &= \sigma(w)Ea_{\tau,\alpha_n}(\lambda)\tau(s_{\alpha_n})^{-1}a_{\tau,\alpha_{n-1}}(s_{\alpha_n}\lambda)\tau(s_{\alpha_{n-1}})^{-1} \\ &\qquad \cdot\ldots\cdot a_{\tau,\alpha_1}(s_{\alpha_2}\cdots s_{\alpha_n}\lambda)\tau(s_{\alpha_1})^{-1} \\ &= \{\sigma(w)E\tau(w)^{-1}\}[\tau(s_{\alpha_1}\cdots s_{\alpha_n})a_{\tau,\alpha_n}(\lambda)\tau(s_{\alpha_1}\cdots s_{\alpha_n})^{-1}] \\ &\qquad \cdot[\tau(s_{\alpha_1}\cdots s_{\alpha_{n-1}})a_{\tau,\alpha_{n-1}}(s_{\alpha_n}\lambda)\tau(s_{\alpha_1}\cdots s_{\alpha_{n-1}})^{-1}] \\ &\qquad \cdot\ldots\cdot[\tau(s_{\alpha_1})a_{\tau,\alpha_1}(s_{\alpha_2}\cdots s_{\alpha_n}\lambda)\tau(s_{\alpha_1})^{-1}]\ . \end{aligned}$$

Each of the factors in brackets is easily seen to be in $\mathrm{Hom}_M(V^\tau, V^\tau)$, and the factor in braces is in $\mathrm{Hom}_M(V^\tau, V_\sigma)$. The proposition follows.

In practice one repeats for each "right by (——)" operator in Proposition 5.3 the argument that led to (5.5). The endomorphisms in the proposition may be thought of as endomorphisms of $\mathrm{Hom}_M(V_\sigma^\tau, V_\sigma)$, where V_σ^τ is the image of P_σ^τ on V^τ. They become

$$E \to \sigma(w) E \tau(w)^{-1}$$

and the various operators of right multiplication by

$$\tau(s_{\alpha_1} \cdots s_{\alpha_j}) P^\tau_{s_{\alpha_{j+1}} \cdots s_{\alpha_n} \sigma} a_{\tau,\alpha_j}(s_{\alpha_{j+1}} \cdots s_{\alpha_n} \lambda) P^\tau_{s_{\alpha_{j+1}} \cdots s_{\alpha_n} \sigma} \tau(s_{\alpha_1} \cdots s_{\alpha_j})^{-1} .$$

In some cases this operator will be scalar; in some others, only its determinant is needed. In such cases we can drop the conjugation by $\tau(s_{\alpha_1} \cdots s_{\alpha_j})$.

For these formulas to be useful, we have to know the value of $P_\sigma^\tau a_{\tau,\alpha}(\lambda) P_\sigma^\tau$. The idea is to reduce matters to computations of expressions

$$a_{\tau'}(\zeta) = \int_V e^{-(1+\zeta)\rho(\overline{n})} \tau'(\kappa(\overline{n}))^{-1} \, d\overline{n} \qquad (5.7)$$

in the real-rank-one connected simple group G_α built from α. (G_α is defined as the group generated by the exponentials of the root spaces for the nonzero multiples of α.) However, carrying out the reduction turns out to be a little trickier than is at first apparent.

Let $K_\alpha = K \cap G_\alpha$ and $M_\alpha = M \cap G_\alpha$. It is easy to see that M normalizes K_α. We now make the special assumption on G that every automorphism of K_α defined by conjugation by M is an inner automorphism of K_α. This assumption is valid if M is connected, as in a complex group G. It is valid also if $G = SU(2,2)$, but it is not valid if $G = SL(3,\mathbb{R})$ or $G = Sp(2,\mathbb{R})$.

Under this assumption we can proceed as follows. Decompose $\tau|_{K_\alpha}$ into its primary components

$$\tau|_{K_\alpha} = \Sigma\, n_j \tau_j$$

with $n_j\tau_j$ acting on V_j. Our special assumption implies that $\tau|_M$ leaves each V_j stable. Now $\sigma|_{M_\alpha}$ is primary by the corollary to Lemma 57 of [21]; thus find the subspace $(V_j)_{\sigma|M_\alpha}$ of each V_j that transforms according to $\sigma|_{M_\alpha}$. The operator

$$P^{\tau_j}_{\sigma|M_\alpha}\, a_{\tau_j,\alpha}(\lambda)\, P^{\tau_j}_{\sigma|M_\alpha}$$

is essentially just a real-rank-one expression to be formed from (5.7). Thus to complete the reduction, all we need to do is to reduce each $(V_j)_{\sigma|M_\alpha}$ under σ; this is possible since $M = M_\alpha Z_M(M_\alpha)$, by Lemma 57 of [21].

Finally we need to be able to compute (5.7) for G of real-rank one and τ an irreducible representation of K. Formulas for $\widetilde{SO}(n,1)$ and $SU(n,1)$ may be found in Klimyk-Gavrilik [18]. See also Theorem 8.11.9 of [39]. We reproduce here the most useful such formulas, which are the ones for $SL(2,\mathbb{R})$ and $SL(2,\mathbb{C})$.

For $SL(2,\mathbb{R})$, let

$$\tau_N\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = e^{iN\theta}.$$

If we put $\bar{n} = \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}$, we are led to the integral

$$a_{\tau_N}(\zeta) = \int_{-\infty}^{\infty} (1+y^2)^{-\frac{1}{2}(1+\zeta)} \left(\frac{1+iy}{(1+y^2)^{1/2}}\right)^N dy ,$$

which is easily evaluated as

$$a_{\tau_N}(\zeta) = \frac{\pi^{1/2}\Gamma\left(\frac{\zeta-|N|}{2}\right)}{\Gamma\left(\frac{\zeta+1-|N|}{2}\right)} \prod_{j=1}^{|N|}\left(\frac{\zeta-j}{\zeta+|N|+1-2j}\right) . \tag{5.8}$$

For $SL(2,\mathbb{C})$, let τ_N be the representation of $SU(2)$ in homogeneous polynomials of degree N in two complex variables. The monomials $\{z_1^k z_2^\ell,\ k+\ell=N\}$ form a basis, and $a_{\tau_N}(\zeta)$ is diagonal in this basis. If we put $\bar{n} = \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}$, we are led to the integral

$$a_{\tau_N}(\zeta)(z_1^k z_2^\ell) = \int_{y\in\mathbb{C}} (1+|y|^2)^{-(1+\zeta)-\frac{1}{2}(k+\ell)} (z_1-\bar{y}z_2)^k (yz_1+z_2)^\ell \, dy ,$$

which can be evaluated to give

$$a_{\tau_N}(\zeta)(z_1^k z_2^\ell) = \frac{\pi z_1 z_2 \{\prod_{j=1}^{k}(\zeta+\frac{1}{2}(k-\ell-2j))\}\{\prod_{j=1}^{\ell}(\zeta+\frac{1}{2}(\ell-k-2j))\}}{\prod_{j=1}^{k+\ell+1}(\zeta+\frac{1}{2}(k+\ell+2-2j))} . \tag{5.9}$$

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