September 1986

Duality Theorems in Relative Lie Algebra Cohomology

A. W. Knapp* and David A. Vogan, Jr.**

Although relative Lie algebra cohomology was introduced as early as 1956 (see [9]), the theory played only a minor role in the basic representation theory of Lie groups until the work of Zuckerman [17] in 1978.

In the representation theory of a semisimple Lie group G, the reasonable algebraic object corresponding to a representation of G is a module for both the complexified Lie algebra $\mathfrak{g}$ and a maximal compact subgroup K such that the actions are suitably compatible and the K action is the direct sum of finite-dimensional irreducible representations. Accordingly Zuckerman worked with the category $\mathcal{C}(\mathfrak{g},K)$ of all such modules. Already Bott [4], in the case of G compact, and Schmid [10,11], in the case of discrete series representations of noncompact G, had used sheaf and Dolbeault cohomology to give explicit realizations of representations. Zuckerman's idea was to set up an algebraic analog of these constructions that would be generalizable and would sidestep formidable analytic problems. He introduced the functor, now called $\Gamma : \mathcal{C}(\mathfrak{g},H) \to \mathcal{C}(\mathfrak{g},K)$ for $H \subseteq K$, that roughly speaking extracts the K-finite subspace of a $(\mathfrak{g},H)$ module.

His construction of representations was in two steps. In the first step, he applied an algebraic analog of holomorphic induction from a representation (often one-dimensional) of a certain kind of

---

* Supported by National Science Foundation Grant DMS 85-01793.

** Supported by National Science Foundation Grant DMS 85-04029.

subgroup. And in the second step, he wanted an algebraic analog of the extraction of global sections or global Dolbeault cohomology elements; for this purpose he used Γ or its derived functors Γ^i. The combined process is now known as cohomological induction and is explained in detail in [12, Chapter 6]. Zuckerman introduced his construction expressly as a device for constructing irreducible representations that had a good chance of being unitary.

For applications to unitary representations, a critical role is played by a "Duality Theorem" for Γ^i. Zuckerman [17] introduced the Duality Theorem as a conjecture and with P. Trauber gave several ideas toward proofs. Enright and Wallach found a complete proof, but their paper [8] suffers from a number of minor errors and one serious gap that we explain below in §8. (These matters are addressed also in Wallach's forthcoming book.)

Since a number of extensive investigations in the literature (e.g., [1], [7], [13], [14], [16]) rely on this Duality Theorem and since we were initially unaware of Enright's and Wallach's corrections, we found this situation to be rather unsatisfactory. Thus we began looking for a complete and correct proof of the Duality Theorem. The first such proof that we found was not too long and is sketched in §8 below. Its essential point is that it combines the approaches of Zuckerman and Trauber [17] and of Enright and Wallach [8], using both the explicit formulas sought by Zuckerman and Trauber and the abstract argument attempted by Enright and Wallach. But we investigated further because we wanted to show at the same time that the top derived functor of Γ has an interpretation that is geometrically dual to Γ, namely that it gives the closest thing possible to a largest K-finite quotient. We call this functor Π because it should be regarded as being related to periods or currents in the same way that Γ is related to sections.

We realized that identifying Π would be important for us for some applications that we have in mind, and the connection among the algebraic, geometric, and cohomological properties of Π is in fact our main reason for writing this paper. We knew that defining Π other than as a derived functor of Γ presents a problem: A $(\mathfrak{g}, H)$ module need not have a largest K-finite quotient. For example, if $\mathfrak{k}$ is the complexified Lie algebra of K, then every finite-dimensional K-invariant subspace of $L^2(K)$ is a quotient of the universal enveloping algebra $U(\mathfrak{k})$ of $\mathfrak{k}$, and every K-finite quotient of $U(\mathfrak{k})$ is finite-dimensional.

Thus we needed also a workable non-cohomological definition of Π. We were led to such a definition by a review of the work of Zuckerman and Trauber and of accompanying suggestions by A. Borel. Trauber and Borel had both speculated that it might be interesting to study the ring of all distributions on G that are K-finite and are supported on K, since members of $\mathcal{C}(\mathfrak{g}, K)$ are modules for this ring. It turned out that both Π and Γ could be defined in terms of this ring and that they amounted to change-of-ring functors. Moreover, the same thing was true of the functors pro and ind in [12, Chapter 6]. Thus there were really just two master functors in the theory, having to do with a change of rings by either $\otimes$ or Hom. (We call these functors P and I below.) In addition, many of the fundamental results in the theory were consequences of standard associativity formulas for $\otimes$ and Hom.

By this time it was clear to us that the foundations needed reworking in these terms. We carry out this effort in §§1-3 below and in part of §4. The remainder of §4 proves Poincaré duality for all of $\mathcal{C}(\mathfrak{g},K)$; previous written proofs (e.g., [3, p. 15]) used an additional hypothesis of admissibility. The proof of Zuckerman duality and the identification of Π is in §§5-7; we have given a proof that is longer than necessary because it gives a great deal of additional information.

We are grateful to K. S. Brown for helpful conversations at an early stage of this work.

Contents

1. The ring $R(\mathfrak{g},K)$

Let $\mathfrak{g}$ be a finite-dimensional complex Lie algebra, and let K be a compact Lie group (possibly disconnected). We assume that $\mathfrak{g}$ and K satisfy the compatibility conditions of [12, p. 299], namely that the complexified Lie algebra $\mathfrak{k}$ of K is a Lie subalgebra of $\mathfrak{g}$, that K acts on $\mathfrak{g}$ by automorphisms (called $\mathrm{Ad}(k)$), and that the differential at 1 of $\mathrm{Ad}(K)$ is $\mathrm{ad}\,\mathfrak{k} \subseteq \mathrm{ad}\,\mathfrak{g}$. We refer to $(\mathfrak{g},K)$ simply as a pair. Since K is compact, we can choose $\mathfrak{p} \subseteq \mathfrak{g}$ stable under $Ad(K)$ and $ad(\mathfrak{k})$ such that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

We shall associate a ring $R(\mathfrak{g},K)$ to the pair $(\mathfrak{g},K)$. To motivate abstract definitions, we shall make a special assumption about $(\mathfrak{g},K)$, introduce the ring concretely, and express everything with identities involving only $\mathfrak{g}$ and K. These identities can then be used as an abstract definition of $R(\mathfrak{g},K)$ without our special assumption. We leave to the reader all verifications that the abstract definitions make sense in general.

The special assumption is that there is a Lie group G with complexified Lie algebra $\mathfrak{g}$ such that K is a compact subgroup of G for which $\mathfrak{k}$ and Ad are compatible with the definitions imposed by G. This assumption is satisfied in applications to representation theory. But it is not always satisfied. For example, one can take $\mathfrak{g}$ to be semisimple and K to be a nontrivial covering of a compact torus in a simply connected group corresponding to a real form of $\mathfrak{g}$.

In any event, suppose G exists. Following Zuckerman [17], we take $R(\mathfrak{g},K)$ to be the ring of bi-K-finite distributions on G (i.e., on $C^\infty_{com}(G)$) that are supported on K, with convolution as multiplication. Here are some examples. We start from the convolution algebra C_K of K-finite functions on K and from the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$. (It may help the reader to regard members of C_K as measures by adjoining normalized Haar measure dk on K to them; convolution is then convolution of measures.) We identify $U(\mathfrak{g})$ with the left-invariant differential operators on G. Associate

$$c \otimes u \in C_K \otimes_{\mathbb{C}} U(\mathfrak{g}) \to Df = \int_K (uf)(k)c(k)\,dk. \tag{1.1}$$

Let ℓ and r denote the left and right regular representations of K on C_K and of $U(\mathfrak{g})$ on $U(\mathfrak{g})$, namely

$$\begin{aligned} \ell(k_0)c(k) &= c(k_0^{-1}k) \quad \text{and} \quad r(k_0)c(k) = c(kk_0) \\ \ell(v)u &= vu \quad \text{and} \quad r(v)u = uv^{tr}. \end{aligned} \tag{1.2}$$

Here v^{tr} refers to the antiautomorphism of $U(\mathfrak{g})$ that extends the map $X \to -X$ of $\mathfrak{g}$ into $U(\mathfrak{g})$. Also let L and R be the regular representations on distributions:

$$\begin{aligned} (L(k_0)D)(f) &= D(\ell(k_0)^{-1}f) \quad \text{and} \quad (R(k_0)D)(f) = D(r(k_0)^{-1}f) \\ (L(v)D)(f) &= D(\ell(v^{tr})f) \quad \text{and} \quad (R(v)D)(f) = D(r(v^{tr})f). \end{aligned} \tag{1.3}$$

In (1.1) we readily check that

$$\ell(k_0)c \otimes u \to L(k_0)D \quad \text{for } k_0 \in K \tag{1.4a}$$

$$c \otimes r(v)u \to R(v)D \quad \text{for } v \in U(\mathfrak{g}) \tag{1.4b}$$

$$r(v^{tr})c \otimes u - c \otimes \ell(v)u \to 0 \quad \text{for } v \in U(\mathfrak{k}). \tag{1.4c}$$

Thus (1.1) gives us a mapping of $C_K \otimes_{U(\mathfrak{k})} U(\mathfrak{g})$ into $R(\mathfrak{g},K)$ that respects the left action by K and the right action by $U(\mathfrak{g})$.

<u>Proposition 1.1</u>. The mapping of $C_K \otimes_{U(\mathfrak{k})} U(\mathfrak{g})$ into $R(\mathfrak{g},K)$ given in (1.1) is one-one onto. The formula in $C_K \otimes_{U(\mathfrak{k})} U(\mathfrak{g})$ for what corresponds to convolution in $R(\mathfrak{g},K)$ is

$$(c \otimes u)(c' \otimes v) = c * c'(\cdot)(Ad(\cdot)^{-1}u)v. \tag{1.5}$$

Remark. The understanding in this convolution formula is that we expand $\mathrm{Ad}(k)^{-1}u$ in terms of a convenient basis, lump the coefficient functions with c' before convolution, and lump the members of $U(\mathfrak{g})$ with v to form the $U(\mathfrak{g})$ part.

Proof of "onto." Fix a bounded open subset E of G containing K, and let a distribution D be given. The support of D being compact, we have

$$\|Df\| \leq \sum_{n \in \mathfrak{h}} a_n \|u_n f\| , \quad f \in C^\infty(E),$$

for some finite set $\mathfrak{h}$ of left-invariant differential operators u_n and constants a_n. (The norms are supremum norms.) Consider the space of vector-valued functions $\{u_n f\}_{n \in \mathfrak{h}}$ as a subspace of the scalar-valued continuous functions F on $\overline{E} \times \mathfrak{h}$. D is a bounded linear functional on the subspace and extends to all such F without an increase in norm. Hence there exist signed measures μ_n, $n \in \mathfrak{h}$, on $\overline{E}$ such that

$$Df = \sum_{n \in \mathfrak{h}} \int_{\overline{E}} u_n f(k)\, d\mu_n(k)$$

for all f in $C^\infty_{com}(E)$. The support condition on D then forces

GAP IN PROOF HERE.

$$Df = \sum_{n \in \mathfrak{h}} \int_{K} u_n f(k)\, d\mu_n(k) .$$

Since D is left K-finite, there are finitely many translates $D^i = L(k_i)D$ such that $\Sigma\, \mathbb{C} D^i$ is stable under $L(K)$. We may assume the D^i are linearly independent. Then we can define scalar-valued functions $L_{ij}(k)$ such that $L(k)D^j = \Sigma_i L_{ij}(k) D^i$, and $k \to \{L_{ij}(k)\}$ will be a finite-dimensional representation. Choose a K-finite function χ on K such that $L_{ij} * \chi = L_{ij}$ for all i and j, and let c_n be the K-finite function $c_n = \overline{\chi} * d\mu_n$. Then it is straightforward to check that

$$Df = \sum_{n \in \mathfrak{n}} \int_K u_n f(k) c_n(k)\, dk .$$

Proof of "one-one." Let $X_1, \ldots, X_\ell$ be a basis of $\mathfrak{p}$, and suppose $\sum c_n \otimes u_n$ maps to 0:

$$\sum_{n \in \mathfrak{n}} \int_K u_n f(k) c_n(k)\, dk = 0 \quad \text{for all} \quad f \in C^\infty_{com}(G) . \qquad (1.6)$$

Using (1.4c) and the Birkhoff-Witt Theorem, we may assume that the u_n's are distinct monomials $X_1^{j_1} \cdots X_\ell^{j_\ell}$. Canonical coordinates of the second kind near the identity in G show that the map

$$(k, x_1, \ldots, x_\ell) \to k \exp x_1 X_1 \cdots \exp x_\ell X_\ell$$

is a local diffeomorphism about $(1, 0, \ldots, 0)$, and we can thus take f above to be of the form $f_1(k) f_2(x_1, \ldots, x_\ell)$, where f_1 is in $C^\infty(K)$ with support near the identity and f_2 is in $C^\infty(\mathbb{R}^\ell)$ with support near the origin. Our monomial $X_1^{j_1} \cdots X_\ell^{j_\ell}$ operates just on f_2 and gives

$$\frac{\partial^{j_1 + \cdots + j_\ell} f_2}{\partial x_1^{j_1} \cdots \partial x_\ell^{j_\ell}} (0, \ldots, 0) .$$

Choosing f_2 suitably, we can thus arrange that (1.6) reduces to

$$\int_K f_1(k) c_n(k)\, dk = 0$$

for whatever n we please. Since f_1 is arbitrary, $c_n(1) = 0$

for all n. Using (1.4a) and translating our zero distribution, we conclude that $c_n(k) = 0$ for all n and all k. Thus the map is one-one.

Proof of convolution formula. In obvious notation

$$D(c\otimes u) * D(c'\otimes v)(f) = \iint_{K\times K} f(ku\ell v)c(k)c'(\ell)\,dk\,d\ell$$

$$= \iint_{K\times K} f(k\ell(Ad(\ell)^{-1}u)v)c(k)c'(\ell)\,dk\,d\ell$$

$$= \iint_{K\times K} f(k(Ad(\ell)^{-1}u)v)c(k\ell^{-1})c'(\ell)\,d\ell\,dk\,,$$

and (1.5) follows.

We can express members of $R(\mathfrak{g},K)$ also in terms of right-invariant differential operators. For u in $U(\mathfrak{g})$, let $u_R f = (uf^{\sim})^{\sim}$, where $f^{\sim}(x) = f(x^{-1})$. This definition makes u_R a right-invariant differential operator, and we have $(uv)_R = u_R v_R$. Then we can associate

$$u\otimes c \in U(\mathfrak{g})\otimes_{\mathbb{C}} C_K \rightarrow Df = \int_K ((u^{tr})_R f)(k)c(k)\,dk\,. \qquad (1.7)$$

This realization has properties complementary to (1.4):

$$u\otimes r(k_0)c \rightarrow R(k_0)D \qquad \text{for } k_0\in K \qquad (1.8a)$$

$$\ell(v)u\otimes c \rightarrow L(v)D \qquad \text{for } v\in U(\mathfrak{g}) \qquad (1.8b)$$

$$R(v^{tr})u\otimes c - u\otimes \ell(v)c \rightarrow 0 \qquad \text{for } v\in U(\mathfrak{k})\,. \qquad (1.8c)$$

The map $U(\mathfrak{g})\otimes_{U(\mathfrak{k})} C_K \rightarrow R(\mathfrak{g},K)$ is an isomorphism onto, and the convolution formula is

$$(u\otimes c)(v\otimes c') = u(\mathrm{Ad}(\cdot)v)c(\cdot) * c' . \qquad (1.9)$$

The two copies of C_K, given in (1.1) as $C_K\otimes 1$ and in (1.7) as $1\otimes C_K$, are the same, namely those distributions involving no derivatives transverse to K. On C_K, our multiplication reduces to convolution.

More generally we can ask for the relationship between (1.1) and (1.7). Abusing notation slightly, let us write $u\otimes c\otimes v$ for the distribution

$$Df = \int_K (u_R^{tr} vf)(k)c(k)\,dk .$$

Then we find

$$u\otimes c\otimes v = 1\otimes c\otimes (\mathrm{Ad}(\cdot)^{-1}u)v = u(\mathrm{Ad}(\cdot)v)\otimes c\otimes 1 , \qquad (1.10)$$

from which it follows that

$$u_1u_2\otimes c\otimes v = u_1\otimes c\otimes (\mathrm{Ad}(\cdot)^{-1}u_2)v \qquad (1.11a)$$

$$u\otimes c\otimes v_1v_2 = u(\mathrm{Ad}(\cdot)v_1)\otimes c\otimes v_2 . \qquad (1.11b)$$

The ring $R(\mathfrak{g},K)$ does not have an identity unless K is a finite group, but it does have an approximate identity. Namely for each irreducible representation τ of K, let $\chi_\tau \in C_K$ be the product of the degree of τ by the character of τ. For a finite set F of irreducible representations of K, put $\chi_F = \Sigma_{\tau\in F}\,\chi_\tau$. As F varies through such finite sets, the members $e_F = \chi_F\otimes 1 = 1\otimes\chi_F$ comprise the approximate identity. The use of the approximate identity in connection with $R(\mathfrak{g},K)$ modules will be discussed in the next section. For now let us notice that for any $c\otimes u$ in $R(\mathfrak{g},K)$, $e_F(c\otimes u) = c\otimes u$ if F is sufficiently large. Also for any $u\otimes c$

in $R(\mathfrak{g},K)$, $(u\otimes c)e_F = u\otimes c$ if F is sufficiently large. These facts follow immediately from (1.5) and (1.9), respectively.

The ring $R(\mathfrak{g},K)$ has a special antiautomorphism $(c\otimes u)^{tr} = u^{tr}\otimes c^{\sim}$. Thus every left $R(\mathfrak{g},K)$ module can be converted canonically into a right $R(\mathfrak{g},K)$ module, and vice versa. The antiautomorphism stablizes the approximate identity, sending cofinal sets to cofinal sets.

Traditional homological algebra for rings and modules assumes that each ring has an identity and that the identity acts as 1 in any module. This theory applies to $R(\mathfrak{g},K)$ if we simply adjoin an identity. The place where care is needed is where $R(\mathfrak{g},K)$ is treated as a module, especially when the ring in question is $R(\mathfrak{g},K)$ itself.

Now suppose that $(\mathfrak{g},K)$ and $(\mathfrak{q},H)$ are two pairs and that $\mathfrak{g}\supseteq\mathfrak{q}$ and $K\supseteq H$. (In this situation we shall always tacitly assume that the Lie algebra imbeddings and Ad actions are consistent.) Except in special circumstances, $R(\mathfrak{q},H)$ is not naturally a subring of $R(\mathfrak{g},K)$. Nevertheless, $R(\mathfrak{g},K)$ is a right and left $R(\mathfrak{q},H)$ module, and any element in $R(\mathfrak{g},K)$ is fixed by elements far out in the approximate identity of $R(\mathfrak{q},H)$. The module action informally is by convolution on G of the associated distributions. If $c\otimes u$ and $u'\otimes c'$ are in $R(\mathfrak{g},K)$ and $a\otimes q$ and $q'\otimes a'$ are in $R(\mathfrak{q},H)$, the actions work out to be

$$(c\otimes u)(a\otimes q) = c *_H a(\cdot)(\mathrm{Ad}(\cdot)^{-1}u)q\,, \tag{1.12a}$$

with

$$(c *_H b)(k) = \int_H c(kh^{-1})b(h)\,dh\,,$$

and

$$(q'\otimes a')(u'\otimes c') = q'(\mathrm{Ad}(\cdot)u')a'(\cdot) *_H c'\,, \tag{1.12b}$$

with

$$(b' *_H c')(k) = \int_H b'(h)c'(h^{-1}k)\,dh\,.$$

2. The category $\mathcal{C}(\mathfrak{g},K)$

For a pair $(\mathfrak{g},K)$ we let $\mathcal{C}(\mathfrak{g},K)$ be the usual category of all $(\mathfrak{g},K)$ modules, as in [12, p. 299]: A member of $\mathcal{C}(\mathfrak{g},K)$ is a complex vector space carrying representations of $\mathfrak{g}$ and of K such that

(i) the K representation splits as the (possibly infinite) direct sum of finite-dimensional irreducible representations

(ii) the differential of the K action is the restriction to $\mathfrak{k}$ of the $\mathfrak{g}$ action

(iii) $(\mathrm{Ad}(k)u)x = k(u(k^{-1}x))$ for $k \in K$, $u \in U(\mathfrak{g})$, and $x \in V$.

The morphisms $\mathrm{Hom}_{\mathfrak{g},K}(V,W)$ are the linear maps respecting the $\mathfrak{g}$ actions and the K actions.

A left $R(\mathfrak{g},K)$ module V will be called unital if, for each x in V, $e_F x = x$ for all e_F sufficiently far out in the approximate identity. In this case $C_K x$ is finite-dimensional for all x in V (since $C_K * e_F$ is finite-dimensional).

Proposition 2.1. A $(\mathfrak{g},K)$ module V in a natural way is a unital $R(\mathfrak{g},K)$ module. Conversely any unital $R(\mathfrak{g},K)$ module is a $(\mathfrak{g},K)$ module in a natural way. For any two such V and W, $\mathrm{Hom}_{\mathfrak{g},K}(V,W)$ coincides with $\mathrm{Hom}_{R(\mathfrak{g},K)}(V,W)$.

Remark. The correspondence between the $R(\mathfrak{g},K)$ action and the $(\mathfrak{g},K)$ action will be given by (2.1) and (2.2) in the course of the proof.

Proof. Let (π,V) be a $(\mathfrak{g},K)$ module, and define

$$(c \otimes u)x = \pi(c)\pi(u)x\,. \tag{2.1}$$

To see that we have an $R(\mathfrak{g},K)$ module, we need only check multiplicative associativity, the other properties being obvious. Let $(\cdot,\cdot)$ be an $\mathrm{Ad}(K)$-invariant inner product on the subset $U^n(\mathfrak{g})$ of members of $U(\mathfrak{g})$ of degree $\leq n$, for a suitably chosen n, and let u_i be an orthonormal basis of $U^n(\mathfrak{g})$. We have

$$\begin{aligned}
((c\otimes u)(c'\otimes v))x &= (c(\cdot) * c'(\cdot)(\mathrm{Ad}(\cdot)^{-1}u)v)x && \text{by (1.5)}\\
&= \sum_i \pi(c(\cdot) * c'(\cdot)(\mathrm{Ad}(\cdot)^{-1}u,u_i))\pi(u_i v)x && \text{by (2.1)}\\
&= \sum_i \pi(c)\pi(c'(\cdot)(\mathrm{Ad}(\cdot)^{-1}u,u_i))\pi(u_i)\pi(v)x\\
&= \sum_i \pi(c)\int_K c'(k)(\mathrm{Ad}(k)^{-1}u,u_i)\pi(k)\pi(u_i)\pi(v)x\,dk\\
&= \pi(c)\int_K c'(k)\pi(k)\pi(\mathrm{Ad}(k)^{-1}u)\pi(v)x\,dk\\
&= \pi(c)\int_K c'(k)\pi(u)\pi(k)\pi(v)x\,dk && \text{by (iii)}\\
&= \pi(c)\pi(u)\pi(c')\pi(v)x\\
&= (c\otimes u)((c'\otimes v)x) && \text{by (2.1).}
\end{aligned}$$

To see V is unital, we write $e_F x = \pi(\chi_F\otimes 1)x = \pi(\chi_F)x$, and this is x for F large since x is K-finite.

Parenthetically let us record the formula that expresses how an element $u\otimes c$ of $R(\mathfrak{g},K)$ acts:

$$(u\otimes c)x = \pi(u)\pi(c)x. \tag{2.2}$$

In fact, (1.10) gives

$$\begin{aligned}
(u\otimes c)x &= (c\otimes \mathrm{Ad}(\cdot)^{-1}u)x\\
&= \sum_i \pi(c(\cdot)(\mathrm{Ad}(\cdot)^{-1}u,u_i))\pi(u_i)x\\
&= \sum_i \int_K c(k)(\mathrm{Ad}(k)^{-1}u,u_i)\pi(k)\pi(u_i)x\,dk
\end{aligned}$$

$$= \int_K c(k)\pi(k)\pi(\mathrm{Ad}(k)^{-1}u)x\,dk$$

$$= \int_K c(k)\pi(u)\pi(k)x\,dk \qquad \text{by (iii)}$$

$$= \pi(u)\pi(c)x .$$

Conversely let V be a unital $R(\mathfrak{g},K)$ module, and let $u \in U(\mathfrak{g})$ and $x \in V$ be given. Choose F so that $e_F x = x$, and let

$$\pi(u)x = (u \otimes \chi_F)x . \tag{2.3}$$

To see this is well defined, let $F' \supseteq F$, so that $\chi_{F'} * \chi_F = \chi_F$. Then

$$(u\otimes \chi_{F'})x = (u\otimes\chi_{F'})e_F x = (u\otimes\chi_{F'})(1\otimes\chi_F)x$$
$$= (u\otimes(\chi_{F'} * \chi_F))x = (u\otimes\chi_F)x .$$

So $\pi(u)x$ is well defined.

Clearly $\pi(1) = 1$. Before checking that $\pi(uv) = \pi(u)\pi(v)$, let us define π on K. The action of $R(\mathfrak{g},K)$ gives us an action of the subring C_K by restriction, and $C_K x$ is finite-dimensional for all x since V is unital. Then it follows that we have a $(\mathfrak{k},K)$ action of K defined compatibly with the action of C_K:

$$(c\otimes 1)x = (1\otimes c)x = \int_K c(k)\pi(k)x\,dk .$$

The claim is that $e_F x = x$ implies

$$(u\otimes c)x = \pi(u)\pi(c)x \tag{2.4}$$

for u in $U(\mathfrak{g})$ and c in C_K. (In view of (2.2), our definitions are then inverse to those in the first part of the proof.) In fact, we have

$$e_F\pi(c)x = \pi(\chi_F)\pi(c)x = \pi(\chi_F * c)x$$

$$= \pi(c * \chi_F)x \qquad \text{since } \chi_F \text{ is central in } C_K$$

$$= \pi(c)\pi(\chi_F)x = \pi(c)e_F x = \pi(c)x. \qquad (2.5)$$

Thus (2.3) says

$$\pi(u)\pi(c)x = (u\otimes\chi_F)\pi(c)x = (u\otimes\chi_F)(1\otimes c)x = (u\otimes(\chi_F * c))x$$

$$= (u\otimes(c*\chi_F))x = (u\otimes c)(1\otimes\chi_F)x$$

$$= (u\otimes c)e_F x = (u\otimes c)x\,,$$

and (2.4) follows.

Now we show that $\pi(uv) = \pi(u)\pi(v)$. If $e_F x = x$ and $e_{F'}\pi(v)x = \pi(v)x$, then

$$\pi(u)\pi(v)x = (u\otimes\chi_{F'})(v\otimes\chi_F)x$$

$$= ((u\,\mathrm{Ad}(\cdot)v)\chi_{F'}(\cdot) * \chi_F)x$$

$$= \Big(\sum_j uv_j \otimes \Big\{\int_K (\mathrm{Ad}(\cdot\ell^{-1})v, v_j)\chi_{F'}(\cdot\ell^{-1})\chi_F(\ell)\,d\ell\Big\}\Big)x$$

$$= \sum_j \pi(uv_j)\pi\Big(\Big\{\int_K (\mathrm{Ad}(\cdot\ell^{-1})v, v_j)\chi_{F'}(\cdot\ell^{-1})\chi_F(\ell)\,d\ell\Big\}\Big)x \quad \text{by (2.4)}$$

$$= \sum_j \pi(uv_j)\iint_{K\times K} (\mathrm{Ad}(k\ell^{-1})v, v_j)\chi_{F'}(k\ell^{-1})\chi_F(\ell)\pi(k)x\,d\ell\,dk$$

$$= \sum_j \pi(uv_j)\iint_{K\times K} (\mathrm{Ad}(\ell^{-1})v, \mathrm{Ad}(k^{-1})v_j)\chi_{F'}(k\ell^{-1})\chi_F(\ell)\pi(k)x\,d\ell\,dk$$

$$= \sum_j \pi(uv_j)\int_K (\mathrm{Ad}(k^{-1})v, \mathrm{Ad}(k^{-1})v_j)\chi_F(k)\pi(k)x\,dk$$

if F' is large enough

$$= \pi(uv)\pi(\chi_F)x$$

$$= \pi(uv)x \qquad \text{since } e_F x = x\,.$$

We have to verify (ii) and (iii). For (ii), let $X\in\mathfrak{k}$ be given. We are to show $\pi(X)x = \frac{d}{dt}\pi(\exp tX)x\Big|_{t=0}$. Choose F so that $e_F x = x$. If $F' \supseteq F$, then

$$\pi(\chi_{F'})\frac{d}{dt}\pi(\exp tX)x\Big|_{t=0} = \frac{d}{dt}\int_K \chi_{F'}(k)\pi(k \exp tX)x\,dk\Big|_{t=0}$$

$$= \frac{d}{dt}\int_K \chi_{F'}(k(\exp tX)^{-1})\pi(k)x\,dk\Big|_{t=0}$$

$$= \frac{d}{dt}\int_K \chi_{F'}((\exp tX)^{-1}k)\pi(k)x\,dk\Big|_{t=0}$$

since $\chi_{F'}$ is central in C_K

$$= \int_K (\ell(X)\chi_{F'})(k)\pi(k)x\,dk$$

$$= \pi(\ell(X)\chi_{F'})x$$

$$= (1\otimes \ell(X)\chi_{F'})x \qquad \text{by (2.4)}$$

$$= (X\otimes \chi_{F'})x \qquad \text{by (1.4c)}$$

$$= \pi(X)x \qquad \text{by (2.3).}$$

Since F' can be arbitrarily large, (ii) follows.

For (iii) we are to show that

$$\pi(\mathrm{Ad}(k_0)u)\pi(k_0)x = \pi(k_0)\pi(u)x$$

for u in $U(\mathfrak{g})$ and k_0 in K. Choose F with $e_F x = x$. Then

$$\pi(k_0)x = \pi(k_0)\pi(\chi_F)x = \int_K \chi_F(k)\pi(k_0)\pi(k)x\,dk = \pi(\ell(k_0)\chi_F)x. \qquad (2.6)$$

Applying (2.4), we obtain

$$\pi(\mathrm{Ad}(k_0)u)\pi(k_0)x = \pi(\mathrm{Ad}(k_0)u)\pi(\ell(k_0)\chi_F)x$$

$$= (\mathrm{Ad}(k_0)u \otimes \ell(k_0)\chi_F)x. \qquad (2.7)$$

If $F' \supset F$ is large enough so that $e_{F'}\pi(u)x = \pi(u)x$, then

$$\pi(k_0)\pi(u)x = \pi(\ell(k_0)\chi_{F'})\pi(u)x = (\ell(k_0)\chi_{F'}\otimes 1)(u\otimes\chi_F)x$$
$$= (\ell(k_0)\chi_{F'}\otimes 1)(\chi_F\otimes \mathrm{Ad}(\cdot)^{-1}u)x$$
$$= (\ell(k_0)\chi_{F'} * \chi_F(\cdot)(\mathrm{Ad}(\cdot)^{-1}u))x$$
$$= (\int_K \chi_{F'}(k_0^{-1}\cdot\ell^{-1})\chi_F(\ell)\mathrm{Ad}(\ell)^{-1}u\,d\ell)x$$
$$= (\chi_F(k_0^{-1}\cdot)\mathrm{Ad}(k_0^{-1}\cdot)^{-1}u)x \quad \text{if } F' \text{ is increased sufficiently}$$
$$= (\mathrm{Ad}(k_0)u\otimes \ell(k_0)\chi_F)x \qquad \text{by (1.10).}$$

Comparison with (2.7) completes the proof of (iii).

Finally let us show that $\mathrm{Hom}_{R(\mathfrak{g},K)}(V,W) = \mathrm{Hom}_{\mathfrak{g},K}(V,W)$. Let π act on V and π' act on W. If φ is in $\mathrm{Hom}_{R(\mathfrak{g},K)}(V,W)$ and F is chosen with $e_F x = x$, then (2.6) applied twice gives

$$\varphi(\pi(k_0)x) = \varphi((\ell(k_0)\chi_F\otimes 1)x) = (\ell(k_0)\chi_F\otimes 1)\varphi(x) = \pi'(k_0)\pi'(\chi_F)\varphi(x).$$

For large F the right side is $\pi'(k_0)\varphi(x)$, as required. Also u in $U(\mathfrak{g})$ implies

$$\varphi(\pi(u)x) = \varphi((u\otimes\chi_F)x) = (u\otimes\chi_F)\varphi(x) = \pi'(u)\varphi(x)$$

if F is large enough. So φ is in $\mathrm{Hom}_{\mathfrak{g},K}(V,W)$.

Conversely let φ be in $\mathrm{Hom}_{\mathfrak{g},K}(V,W)$. Then $\varphi(\pi(u)x) = \pi'(u)\varphi(x)$ for u in $U(\mathfrak{g})$ and $\varphi(\pi(k_0)x) = \pi'(k_0)\varphi(x)$ for k_0 in K. Hence also $\varphi(\pi(c)x) = \pi'(c)\varphi(x)$ for c in C_K. Consequently (2.1) gives

$$\varphi((c\otimes u)x) = \varphi(\pi(c)\pi(u)x) = \pi'(c)\pi'(u)\varphi(x) = (c\otimes u)\varphi(x),$$

and φ is in $\mathrm{Hom}_{R(\mathfrak{g},K)}(V,W)$. This completes the proof.

Proposition 2.1 is so fundamental that we shall often use it without specific reference. As a consequence of the proposition, we can exploit the interplay between representation theory and ring theory in studying $(\mathfrak{g},K)$ modules. Of particular importance will be the standard associativity formulas for $\otimes$ and Hom that are given on pp. 27-28 of [6]. In reproducing these formulas, we use subscripts to denote ring actions, left subscripts for left actions and right subscripts for right actions. Left and right actions on the same module are assumed to commute. Let R and S be rings. Then

$$(A\otimes_R B)\otimes_S C \cong A\otimes_R (B\otimes_S C) \qquad \text{for } (A_R, {}_RB_S, {}_SC) \tag{2.8a}$$

$$\mathrm{Hom}_R(A,\mathrm{Hom}_S(B,C)) \cong \mathrm{Hom}_S(B\otimes_R A, C) \quad \text{for } ({}_RA, {}_SB_R, {}_SC) \tag{2.8b}$$

$$\mathrm{Hom}_R(A,\mathrm{Hom}_S(B,C)) \cong \mathrm{Hom}_S(A\otimes_R B, C) \quad \text{for } (A_R, {}_RB_S, C_S). \tag{2.8c}$$

We shall use these identities to derive some simple properties of $C(\mathfrak{g},K)$. It is important to remember that not every left $R(\mathfrak{g},K)$ module is in $C(\mathfrak{g},K)$, only those meeting the conditions of Proposition 2.1. The decisive properties of the subcategory $C(\mathfrak{g},K)$ of all left $R(\mathfrak{g},K)$ modules are the following:

(i) $C(\mathfrak{g},K)$ is closed under passage to submodules, quotients, and finite direct sums

(ii) Every member of $C(\mathfrak{g},K)$ is the image of a projective in $C(\mathfrak{g},K)$

(iii) Every member of $C(\mathfrak{g},K)$ imbeds in an injective in $C(\mathfrak{g},K)$

(iv) All $R(\mathfrak{g},K)$ maps between two members of $C(\mathfrak{g},K)$ are $C(\mathfrak{g},K)$ morphisms.

Property (i) is clear, (iv) is in Proposition 2.1, and (ii) and (iii) will be proved in the next section.

Proposition 2.2. If V is in $\mathcal{C}(\mathfrak{g},K)$, then

$$R(\mathfrak{g},K) \otimes_{R(\mathfrak{g},K)} V \cong V \tag{2.9}$$

as a natural isomorphism of left $R(\mathfrak{g},K)$ modules.

Remarks. Here $R(\mathfrak{g},K)$ is a two-sided module under $R(\mathfrak{g},K)$. The right action is tensored out, and the left action survives.

Proof. The map $r \otimes v \to rv$ is an $R(\mathfrak{g},K)$ map of the left side of (2.9) into the right side. It is onto because of the presence of approximate identities. Let us prove it is one-one. If $\Sigma\,(r_i \otimes v_i)$ maps to 0, then $\Sigma\, r_i v_i = 0$. Choose F so that $e_F r_i = r_i$ for all i. Then

$$\Sigma\,(r_i \otimes v_i) = \Sigma\,(e_F r_i \otimes v_i) = \Sigma\,(e_F r_i \otimes v_i) - \Sigma\, e_F \otimes r_i v_i$$
$$= \Sigma\,(e_F r_i \otimes v_i - e_F \otimes r_i v_i)$$

shows $\Sigma\,(r_i \otimes v_i) = 0$ on the left side of (2.9). So the map is one-one. Clearly the isomorphism is natural in V.

The expected dual situation involves $\mathrm{Hom}_{R(\mathfrak{g},K)}(R(\mathfrak{g},K),V)$. Here the left action on $R(\mathfrak{g},K)$ disappears with the Hom, and the right action survives to define a left $R(\mathfrak{g},K)$ module: $(r\varphi)(s) = \varphi(sr)$. Although $\mathrm{Hom}_{R(\mathfrak{g},K)}(R(\mathfrak{g},K),V)$ is a left $R(\mathfrak{g},K)$ module, it is not necessarily in $\mathcal{C}(\mathfrak{g},K)$. As we shall see below, we can obtain a member of $\mathcal{C}(\mathfrak{g},K)$ by using the subset $\mathrm{Hom}_{R(\mathfrak{g},K)}(R(\mathfrak{g},K),V)_K$ of K-finite members (members φ with $C_K\varphi$ finite-dimensional).

Lemma 2.3. Let V and W be in $\mathcal{C}(\mathfrak{k},K)$ with trivial action on W. Make $\mathrm{Hom}_{\mathbb{C}}(V,W)$ into an $R(\mathfrak{k},K)$ module by defining

$$(r\varphi)(v) = \varphi(r^{tr}v),$$

where r^{tr} refers to the natural antiautomorphism of $R(\mathfrak{k},K)$. Let $\mathrm{Hom}_{\mathbb{C}}(V,W)_K$ be the subset of K-finite members of $\mathrm{Hom}_{\mathbb{C}}(V,W)$. Then $\mathrm{Hom}_{\mathbb{C}}(V,W)_K$ consists of those members of $\mathrm{Hom}_{\mathbb{C}}(V,W)$ that vanish except on finitely many K-isotypic subspaces of V.

Proof. Let φ be K-finite, and let V_0 be an irreducible K-stable subspace of V on which φ is not identically 0. Let V_0^* be the contragredient representation, and let $W_0 = \varphi(V_0)$. Then $\varphi|_{V_0}$ is in $\mathrm{Hom}_{\mathbb{C}}(V_0,W_0) \cong V_0^* \otimes_{\mathbb{C}} W_0$, which is the sum of a finite number of copies of V_0^*. Hence $\varphi|_{V_0}$ transforms under K according to V_0^*. Since φ lies in a finite-dimensional K-stable subspace, V_0^* is one of the finitely many K types occurring in $C_K\varphi$. So φ vanishes on the K-isotypic subspaces for all K types whose contragredients do not occur in $C_K\varphi$.

In the converse direction it is enough to show that if just one K type occurs in V, then every φ in $\mathrm{Hom}_{\mathbb{C}}(V,W)$ is K-finite. Thus write $V = \sum \oplus V_\alpha$ with all V_α K-isomorphic and irreducible. Single out some V_{α_0}, and let $\psi_\alpha : V_{\alpha_0} \to V_\alpha$ be a K isomorphism. By irreducibility, choose a basis of $\mathrm{End}_{\mathbb{C}} V_{\alpha_0}$ of the form $k_j^{-1}|_{V_{\alpha_0}}$, $1 \le j \le n$. Here $\{k_j\}$ refers to a subset of the actions by K. If k is in K, then there exist $c_j(k)$, $1 \le j \le n$, such that

$$k^{-1}x = \sum c_j(k)k_j^{-1}x \quad \text{for all } x \in V_{\alpha_0}.$$

Now $x \in V_{\alpha_0}$ implies

$$\begin{aligned}(k\varphi)(\psi_\alpha(x)) &= \varphi(k^{-1}\psi_\alpha(x)) = \varphi(\psi_\alpha(k^{-1}x)) \\ &= \varphi(\psi_\alpha(\textstyle\sum c_j(k)k_j^{-1}x)) = \sum c_j(k)(k_j\varphi)(\psi_\alpha(x)).\end{aligned}$$

Hence $k\varphi = \sum c_j(k)(k_j\varphi)$, and φ is K-finite.

Let $(\mathfrak{g},K)$ and $(\mathfrak{q},H)$ be pairs with $\mathfrak{g} \supseteq \mathfrak{q}$ and $K \supseteq H$ compatibly. If V is in $\mathcal{C}(\mathfrak{q},H)$, we define

$$P(V) = P^{\mathfrak{g},K}_{\mathfrak{q},H}(V) = R(\mathfrak{g},K) \otimes_{R(\mathfrak{q},H)} V \qquad (2.10)$$

as a left $R(\mathfrak{g},K)$ module under $r(s \otimes x) = rs \otimes x$; the convention for $\otimes_{R(\mathfrak{q},H)}$ is that $rq \otimes x = r \otimes qx$. Also we make

$$\mathrm{Hom}_{R(\mathfrak{q},H)}(R(\mathfrak{g},K),V) \qquad (2.11)$$

into a left $R(\mathfrak{g},K)$ module by letting $r\varphi(s) = \varphi(sr)$; the convention for $\mathrm{Hom}_{R(\mathfrak{q},H)}$ is that $\varphi(qr) = q(\varphi(r))$. We define a subset $I(V)$ to be the subset of K-finite members:

$$I(V) = I^{\mathfrak{g},K}_{\mathfrak{q},H}(V) = \mathrm{Hom}_{R(\mathfrak{q},H)}(R(\mathfrak{g},K),V)_K . \qquad (2.12)$$

There is a second way of defining $P(V)$, namely

$$P'(V) = V \otimes_{R(\mathfrak{q},H)} R(\mathfrak{g},K)$$

as a left $R(\mathfrak{g},K)$ module under $r(x \otimes s) = x \otimes sr^{tr}$; the convention for $\otimes_{R(\mathfrak{q},H)}$ is that $q^{tr}x \otimes r = x \otimes qr$. These definitions give naturally isomorphic $R(\mathfrak{g},K)$ modules under the map $r \otimes x \to x \otimes r^{tr}$. Thus we are free to use either definition. Normally we shall use the first definition, but it will be more convenient in §6 to use the second definition.

Similarly there is a second way of defining $I(V)$, namely with $r\varphi(s) = \varphi(r^{tr}s)$; the convention for $\mathrm{Hom}_{R(\mathfrak{q},H)}$ is that $\varphi(rq) = q^{tr}(\varphi(r))$. The two definitions give naturally isomorphic $R(\mathfrak{g},K)$ modules, and we are free to use either definition. However, we shall invariably use the first definition.

Proposition 2.4. For V in $\mathcal{C}(\mathfrak{q},H)$ and notation as above, $P(V)$ and $I(V)$ are in $\mathcal{C}(\mathfrak{g},K)$. Both P and I are covariant functors, P is right exact, and I is left exact.

Remark. The choice of notation P and I is to be a reminder of facts proved in the next section about preservation of projectives and injectives.

Proof. We know that the left $R(\mathfrak{g},K)$ module $R(\mathfrak{g},K)$ is unital and that every member is K-finite. Hence the same is true of $P(V)$, and $P(V)$ is in $\mathcal{C}(\mathfrak{g},K)$ by Proposition 2.1. If $\varphi : V \to W$ is a morphism, then $P(\varphi)$ is the map $1 \otimes \varphi$. Hence P is a covariant right exact functor.

For $I(V)$, we regard $R(\mathfrak{g},K)$ as a right $R(\mathfrak{g},K)$ module and V as a trivial $R(\mathfrak{g},K)$ module, and we apply Lemma 2.3 to $\mathrm{Hom}_{\mathbb{C}}(R(\mathfrak{g},K),V)_K$. Members of this set are characterized by vanishing except on finitely many K types of $R(\mathfrak{g},K)$, and hence members of the subset $I(V)$ of (2.11) are characterized by this same property. This property is preserved under the left $R(\mathfrak{g},K)$ action on (2.11), and thus $I(V)$ is a left $R(\mathfrak{g},K)$ module. Using an element e_F that fixes finitely many K types of $R(\mathfrak{g},K)$, we see that $I(V)$ is a unital left $R(\mathfrak{g},K)$ module. The members of $I(V)$ are K-finite by definition, and Proposition 2.1 thus shows that $I(V)$ is in $\mathcal{C}(\mathfrak{g},K)$.

To see that I is a functor, let $\varphi : V \to W$ be a morphism in $\mathcal{C}(\mathfrak{q},H)$. Then $I(\varphi) = \varphi \circ (\cdot)$ carries $I(V)$ into $\mathrm{Hom}_{R(\mathfrak{q},H)}(R(\mathfrak{g},K),W)$, and the problem is to show that the image is in the subset $I(W)$. If ψ is in $I(V)$, then ψ vanishes on all but finitely many K types of $R(\mathfrak{g},K)$, by Lemma 2.3. Hence so does $\varphi \circ \psi$. Then $\varphi \circ \psi$ is in

$I(W)$ by Lemma 2.3. So I is a functor. Clearly I is covariant, and it is easy to see it is left exact.

Proposition 2.5. If W is in $\mathcal{C}(\mathfrak{g},K)$, then

$$\mathrm{Hom}_{R(\mathfrak{g},K)}(R(\mathfrak{g},K),W)_K \cong W \tag{2.13}$$

as a natural isomorphism of left $R(\mathfrak{g},K)$ modules.

Proof. Let $F(W)$ be the left side of (2.13), and let V be in $\mathcal{C}(\mathfrak{g},K)$. Then we have natural isomorphisms

$$\begin{aligned}\mathrm{Hom}_{R(\mathfrak{g},K)}(V,F(W)) &= \mathrm{Hom}_{R(\mathfrak{g},K)}(V,\mathrm{Hom}_{R(\mathfrak{g},K)}(R(\mathfrak{g},K),W)) \\ &\cong \mathrm{Hom}_{R(\mathfrak{g},K)}(R(\mathfrak{g},K)\otimes_{R(\mathfrak{g},K)} V, W) \quad \text{by (2.8b)} \\ &\cong \mathrm{Hom}_{R(\mathfrak{g},K)}(V,W) \quad \text{by Proposition 2.2,}\end{aligned}$$

and it follows by a standard argument in homological algebra that F is naturally equivalent with the identity functor.

Let $S(\mathfrak{g})$ be the symmetric algebra of $\mathfrak{g}$. This is a $(\mathfrak{g},K)$ module under $(\mathrm{ad},\mathrm{Ad})$, and the symmetrization map $\mathcal{S}: S(\mathfrak{g}) \to U(\mathfrak{g})$ commutes with ad and Ad. With $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{p}$ as usual, let $S(\mathfrak{p}) \subseteq S(\mathfrak{g})$ be the symmetric algebra of $\mathfrak{p}$. This is a $(\mathfrak{k},K)$ module under $(\mathrm{ad},\mathrm{Ad})$.

We denote left and right regular representations on $R(\mathfrak{g},K)$ by L and R.

Proposition 2.6. Let $(\mathfrak{g},K)$ and $(\mathfrak{g},H)$ be two pairs with $K \supseteq H$ compatibly, and write $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{p}$ as usual. Then

$$R(\mathfrak{k},H)\otimes_{\mathbb{C}} S(\mathfrak{p}) \cong R(\mathfrak{g},H) \tag{2.14}$$

under $(c \otimes u) \otimes p \to c \otimes u\mathfrak{S}(p)$. The isomorphism carries the left $R(\mathfrak{k},H)$ module structure $L \otimes 1$ on the left side to L on the right side, and it carries the left $R(\mathfrak{k},H)$ structure $R \otimes Ad$ on the left side to R on the right side.

Proof. The chain of vector space isomorphisms

$$R(\mathfrak{k},H) \otimes_{\mathbb{C}} S(\mathfrak{p}) \cong (C_H \otimes_{U(\mathfrak{h})} U(\mathfrak{k})) \otimes_{\mathbb{C}} S(\mathfrak{p}) \cong C_H \otimes_{U(\mathfrak{h})} (U(\mathfrak{k}) \otimes_{\mathbb{C}} S(\mathfrak{p}))$$

$$\cong C_H \otimes_{U(\mathfrak{h})} U(\mathfrak{g}) \cong R(\mathfrak{g},H)$$

shows that our map is one-one onto. Clearly our map respects the action of H under $L \otimes 1$ and the action of $\mathfrak{k}$ under $R \otimes ad$. It respects the action of $\mathfrak{k}$ under $L \otimes 1$ because Y in $\mathfrak{k}$ implies

$$(L \otimes 1)(Y)((c \otimes u) \otimes p) = (c \otimes (Ad(\cdot)^{-1}Y)u) \otimes p$$

$$\to c \otimes (Ad(\cdot)^{-1}Y)u\mathfrak{S}(p) = (L \otimes 1)(Y)(c \otimes u\mathfrak{S}(p)).$$

It respects the action of H under $R \otimes Ad$ because h_0 in H implies

$$(R \otimes Ad)(h_0)((c \otimes u) \otimes p) = R(h_0)(Ad(\cdot)u \otimes c) \otimes Ad(h_0)p$$

$$= (R(h_0)c \otimes Ad(h_0)u) \otimes Ad(h_0)p$$

$$\to R(h_0)c \otimes (Ad(h_0)u)\mathfrak{S}(Ad(h_0)p)$$

$$= R(h_0)c \otimes (Ad(h_0)u)Ad(h_0)\mathfrak{S}(p)$$

$$= (R \otimes Ad)(h_0)(c \otimes u\mathfrak{S}(p)).$$

<u>Proposition 2.7</u>. Let $(\mathfrak{g},K)$ and $(\mathfrak{g},H)$ be two pairs with $K \supseteq H$ compatibly. Then

$$C_K \otimes_{R(\mathfrak{k},H)} R(\mathfrak{g},H) \cong R(\mathfrak{g},K) \tag{2.15}$$

under convolution: $c_1 \otimes (c_2 \otimes u) \to (c_1 *_H c_2) \otimes u$. The isomorphism carries the left K action to the left regular representation of K on $R(\mathfrak{g},K)$, and it carries the $R(\mathfrak{g},H)$ module structure $1\otimes R$ to the right regular representation on $R(\mathfrak{g},K)$.

Proof. Proposition 2.6 gives

$$C_K \otimes_{R(\mathfrak{k},H)} R(\mathfrak{g},H) \cong C_K \otimes_{R(\mathfrak{k},H)} (R(\mathfrak{k},H) \otimes_{\mathbb{C}} S(\mathfrak{p}))$$

with the left $\mathfrak{k}$ action on $R(\mathfrak{g},H)$ going into $L\otimes 1$. The right side here by (2.8a) is

$$\begin{aligned} &\cong (C_K \otimes_{R(\mathfrak{k},H)} R(\mathfrak{k},H)) \otimes_{\mathbb{C}} S(\mathfrak{p}) \\ &\cong C_K \otimes_{\mathbb{C}} S(\mathfrak{p}) && \text{by Proposition 2.2} \\ &\cong R(\mathfrak{k},K) \otimes_{\mathbb{C}} S(\mathfrak{p}) \\ &\cong R(\mathfrak{g},K) && \text{by Proposition 2.6.} \end{aligned}$$

We readily check that the actions correspond as indicated.

3. Projectives and injectives

Let $(\mathfrak{g},K)$ and $(\mathfrak{q},H)$ denote two pairs with $\mathfrak{g} \supseteq \mathfrak{q}$ and $K \supseteq H$ compatibly. If V is a $(\mathfrak{g},K)$ module, then we can regard V as a $(\mathfrak{q},H)$ module by restricting the action. We denote the resulting forgetful functor $\mathcal{F}$ by

$$\mathcal{F}_{\mathfrak{g},K}^{\mathfrak{q},H} : \mathcal{C}(\mathfrak{g},K) \to \mathcal{C}(\mathfrak{q},H)\,.$$

Clearly $\mathcal{F}$ is covariant and exact.

In the context of $R(\mathfrak{g},K)$ and $R(\mathfrak{q},H)$, the definition of $\mathcal{F}$ is less transparent, since $R(\mathfrak{q},H)$ need not be a subring of $R(\mathfrak{g},K)$. But we can define the action this way: Let $q \in \mathfrak{q}$ and $h \in H$. If

$x \in V$ is given, choose e_F in $R(\mathfrak{g},K)$ with $e_F x = x$. Then qx and hx are given as $(qe_F)x$ and $(he_F)x$, where qe_F and he_F are computed with $R(\mathfrak{g},K)$ considered as a left $R(\mathfrak{q},H)$ module. (See the end of §1.)

Let W be a $(\mathfrak{g},K)$ module. We define a pseudo-forgetful functor $\mathfrak{F}^\vee$ with

$$(\mathfrak{F}^\vee)^{\mathfrak{q},H}_{\mathfrak{g},K} : \mathcal{C}(\mathfrak{g},K) \to \mathcal{C}(\mathfrak{q},H)$$

by

$$\mathfrak{F}^\vee(W) = \operatorname{Hom}_{R(\mathfrak{g},K)}(R(\mathfrak{g},K),W)_H .$$

The fact that $\mathfrak{F}^\vee(W)$ is a $(\mathfrak{q},H)$ module and $\mathfrak{F}^\vee$ is a functor is proved by the same arguments as in the second half of Proposition 2.4.

<u>Lemma 3.1</u>. The covariant functor $(\mathfrak{F}^\vee)^{\mathfrak{q},H}_{\mathfrak{g},K}$ is exact.

Proof. Let

$$0 \longrightarrow W' \xrightarrow{\psi} W \xrightarrow{\varphi} W'' \longrightarrow 0$$

be an exact sequence in $\mathcal{C}(\mathfrak{g},K)$. If η is in $\mathfrak{F}^\vee(W')$ and $\psi \circ \eta = 0$, then $\psi(\eta(r)) = 0$ for all r, and $\eta(r) = 0$ for all r. Hence $\eta = 0$. Thus $\mathfrak{F}^\vee(\psi)$ is one-one.

If η is in $\mathfrak{F}^\vee(W)$ and $\varphi \circ \eta = 0$, then $\varphi(\eta(r)) = 0$ for all r, $\eta(r)$ is in $\ker \varphi = \text{image}\ \psi$, and $\eta(r) = \psi(w')$. Define $\xi(r) =$ this w', necessarily unique. Then ξ is in $\operatorname{Hom}_{R(\mathfrak{g},K)}(R(\mathfrak{g},K),W')$. Since η vanishes off finitely many H types of $R(\mathfrak{g},K)$, so does ξ. Thus ξ is in $\mathfrak{F}^\vee(W')$. Then $\psi(\xi(r)) = \psi(w') = \eta(r)$ shows $\ker \mathfrak{F}^\vee(\varphi) = \text{image}\ \mathfrak{F}^\vee(\psi)$.

Finally let η be in $\mathfrak{F}^\vee(W'')$. We seek ξ in $\mathfrak{F}^\vee(W)$ with $\varphi \circ \xi = \eta$. For each irreducible representation τ of K, the element

$\eta(e_{\{\tau\}})$ is in W'' and is thus of the form $\varphi(w_{\{\tau\}})$, $w_{\{\tau\}} \in W$. Now $R(\mathfrak{g},K)$ is a direct sum of left ideals $R(\mathfrak{g},K) = \sum_\tau R(\mathfrak{g},K)e_{\{\tau\}}$, as is apparent from (1.9). Thus we can define $\xi_0 \in \operatorname{Hom}_{R(\mathfrak{g},K)}(R(\mathfrak{g},K),W)$ consistently by

$$\xi_0(\sum r_\tau e_{\{\tau\}}) = \sum r_\tau w_{\{\tau\}} .$$

Then

$$\begin{aligned}\varphi(\xi_0(\sum r_\tau e_{\{\tau\}})) &= \varphi(\sum r_\tau w_{\{\tau\}}) = \sum r_\tau \varphi(w_{\{\tau\}}) \\ &= \sum r_\tau \eta(e_{\{\tau\}}) = \eta(\sum r_\tau e_{\{\tau\}})\end{aligned}$$

says $\varphi\circ\xi_0 = \eta$. Now $\eta = 0$ except on finitely many H types of $R(\mathfrak{g},K)$ under the right regular representation. Thus let us define ξ to be ξ_0 on the finitely many H types and to be 0 on the remaining H types. Then ξ respects the left action of $R(\mathfrak{g},K)$, is in $\mathcal{F}^\vee(W)$, and has $\varphi\circ\xi = \eta$. Hence ξ has the required properties.

Recall the functors P and I from $\mathcal{C}(\mathfrak{q},H)$ to $\mathcal{C}(\mathfrak{g},K)$ given in (2.10) and (2.12).

<u>Proposition 3.2</u>. Let $(\mathfrak{g},K)$ and $(\mathfrak{q},H)$ be two pairs with $\mathfrak{g} \supseteq \mathfrak{q}$ and $K \supseteq H$ compatibly. Then there are natural isomorphisms

$$\operatorname{Hom}_{R(\mathfrak{g},K)}(P^{\mathfrak{g},K}_{\mathfrak{q},H}(V),W) \cong \operatorname{Hom}_{R(\mathfrak{q},H)}(V,(\mathcal{F}^\vee)^{\mathfrak{q},H}_{\mathfrak{g},K}(W)) \tag{3.1}$$

$$\operatorname{Hom}_{R(\mathfrak{g},K)}(W,I^{\mathfrak{g},K}_{\mathfrak{q},H}(V)) \cong \operatorname{Hom}_{R(\mathfrak{q},H)}(\mathcal{F}^{\mathfrak{q},H}_{\mathfrak{g},K}(W),V) \tag{3.2}$$

for V in $\mathcal{C}(\mathfrak{q},H)$ and W in $\mathcal{C}(\mathfrak{g},K)$.

Remark. When $H = K$, $\mathcal{F}^\vee(W) \cong W$ by Proposition 2.5. The resulting identities are called "Frobenius reciprocity." It follows from these identities and (6.1.7) and (6.1.23) of [12] that

$$P^{\mathfrak{g},K}_{\mathfrak{q},K}(V) \cong \operatorname{ind}^{\mathfrak{g},K}_{\mathfrak{q},K}(V)$$

$$I^{\mathfrak{g},K}_{\mathfrak{q},K}(V) \cong \operatorname{pro}^{\mathfrak{g},K}_{\mathfrak{q},K}(V),$$

in the notation of [12]. These formulas show how pro and ind can be defined without a several-step process. Also they confirm that the problem of a good notation, satisfactory for everyone, is completely hopeless.

Proof of (3.1).

$$\begin{aligned}
\operatorname{Hom}_{R(\mathfrak{g},K)}(P^{\mathfrak{g},K}_{\mathfrak{q},H}(V),W) &= \operatorname{Hom}_{R(\mathfrak{g},K)}(R(\mathfrak{g},K)\otimes_{R(\mathfrak{q},H)} V, W)\\
&\cong \operatorname{Hom}_{R(\mathfrak{q},H)}(V, \operatorname{Hom}_{R(\mathfrak{g},K)}(R(\mathfrak{g},K),W)) \quad \text{by (2.8b)}\\
&= \operatorname{Hom}_{R(\mathfrak{q},H)}(V, \operatorname{Hom}_{R(\mathfrak{g},K)}(R(\mathfrak{g},K),W)_H)\\
&= \operatorname{Hom}_{R(\mathfrak{q},H)}(V, (\mathcal{F}^{\vee})^{\mathfrak{q},H}_{\mathfrak{g},K}(W)).
\end{aligned}$$

Proof of (3.2).

$$\begin{aligned}
\operatorname{Hom}_{R(\mathfrak{g},K)}(W,I^{\mathfrak{g},K}_{\mathfrak{q},H}(V)) &= \operatorname{Hom}_{R(\mathfrak{g},K)}(W, \operatorname{Hom}_{R(\mathfrak{q},H)}(R(\mathfrak{g},K),V)_K)\\
&= \operatorname{Hom}_{R(\mathfrak{g},K)}(W, \operatorname{Hom}_{R(\mathfrak{q},H)}(R(\mathfrak{g},K),V))\\
&\cong \operatorname{Hom}_{R(\mathfrak{q},H)}(R(\mathfrak{g},K)\otimes_{R(\mathfrak{g},K)} W, V) \quad \text{by (2.8b)}\\
&\cong \operatorname{Hom}_{R(\mathfrak{q},H)}(\mathcal{F}^{\mathfrak{q},H}_{\mathfrak{g},K}(W),V) \quad \text{by Proposition 2.2.}
\end{aligned}$$

It is worth tracking down the explicit formulas for (3.1) and (3.2). If we write (3.1) as $\Phi \longleftrightarrow \Phi'$, then the relationship is

$$\Phi(r\otimes v) = \Phi'(v)(r). \tag{3.3}$$

If we write (3.2) as $\Phi \longleftrightarrow \Phi'$, then the relationship is

$$\Phi(w)(r) = \Phi'(rw). \tag{3.4}$$

Corollary 3.3. If $(\mathfrak{g},K)$ and $(\mathfrak{q},H)$ are pairs with $\mathfrak{g}\supseteq\mathfrak{q}$ and $K\supseteq H$ compatibly, then $P_{\mathfrak{q},H}^{\mathfrak{g},K}$ carries projectives to projectives and $I_{\mathfrak{q},H}^{\mathfrak{g},K}$ carries injectives to injectives.

Remark. These results are analogous to those on pp. 30-31 of [6]. But here the rings $R(\mathfrak{q},H)$ and $R(\mathfrak{g},K)$ are not necessarily related by a homomorphism.

Proof. If P is projective in $\mathcal{C}(\mathfrak{q},H)$, then (3.1) says that

$$W \to \operatorname{Hom}_{R(\mathfrak{g},K)}(P_{\mathfrak{q},H}^{\mathfrak{g},K}(P)\,,\,W)$$

is equivalent with

$$W \to \operatorname{Hom}_{R(\mathfrak{q},H)}(P\,,\,(\mathcal{F}^{\vee})_{\mathfrak{g},K}^{\mathfrak{q},H}(W))\,,$$

which is exact, being the composition of the exact functors $(\mathcal{F}^{\vee})_{\mathfrak{g},K}^{\mathfrak{q},H}$ and $\operatorname{Hom}_{R(\mathfrak{q},H)}(P,\cdot)$. (See Lemma 3.1.) Hence $P_{\mathfrak{q},H}^{\mathfrak{g},K}(P)$ is projective.

If I is injective in $\mathcal{C}(\mathfrak{q},H)$, then (3.2) says that

$$W \to \operatorname{Hom}_{R(\mathfrak{g},K)}(W\,,\,I_{\mathfrak{q},H}^{\mathfrak{g},K}(I))$$

is equivalent with

$$W \to \operatorname{Hom}_{R(\mathfrak{q},H)}(\mathcal{F}_{\mathfrak{g},K}^{\mathfrak{q},H}(W)\,,\,I)\,,$$

which is exact, being the composition of the exact functors $\mathcal{F}_{\mathfrak{g},K}^{\mathfrak{q},H}$ and $\operatorname{Hom}_{R(\mathfrak{q},H)}(\cdot,I)$. Hence $I_{\mathfrak{q},H}^{\mathfrak{g},K}(I)$ is injective.

In $\mathcal{C}(\mathfrak{k},K)$ every module is both projective and injective, and thus Corollary 3.3 yields the known results that every $P_{\mathfrak{k},K}^{\mathfrak{g},K}(V)$ is projective (Corollary 6.1.8 of [12]) and every $I_{\mathfrak{k},K}^{\mathfrak{g},K}(V)$ is injective (Corollary 6.1.24 of [12]). In addition, if V is in $\mathcal{C}(\mathfrak{g},K)$, then V is a quotient of $P^{\mathfrak{g},K}_{\mathfrak{k},K}(\mathcal{F}^{\mathfrak{k},K}_{\mathfrak{g},K}V)$, as in Corollary 6.1.10 of [12], and V maps in one-one fashion into $I^{\mathfrak{g},K}_{\mathfrak{k},K}(\mathcal{F}^{\mathfrak{k},K}_{\mathfrak{g},K}V)$, as in Corollary 6.1.24 of [12]. One consequence is that axioms (ii) and (iii) before Proposition 2.2 are now seen to be verified for $\mathcal{C}(\mathfrak{g},K)$. Another consequence is that the projectives in $\mathcal{C}(\mathfrak{g},K)$ are the direct summands of all $P^{\mathfrak{g},K}_{\mathfrak{k},K}(V)$ for V in $\mathcal{C}(\mathfrak{k},K)$, and the injectives in $\mathcal{C}(\mathfrak{g},K)$ are the direct summands of all $I^{\mathfrak{g},K}_{\mathfrak{k},K}(V)$ for V in $\mathcal{C}(\mathfrak{k},K)$.

Fix a pair $(\mathfrak{g},K)$ and write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ as usual. Let $\mathcal{P}$ be the projection of $\mathfrak{g}$ on $\mathfrak{p}$ along $\mathfrak{k}$, and let $m = \dim \mathfrak{p}$. For $0 \leq n \leq m$, we can regard $\Lambda^n\mathfrak{p}$ as a $(\mathfrak{k},K)$ module under the adjoint representation. Corollary 3.3 shows that the $(\mathfrak{g},K)$ module

$$X_n = R(\mathfrak{g},K) \otimes_{R(\mathfrak{k},K)} \Lambda^n\mathfrak{p} \tag{3.5a}$$

is a projective in $\mathcal{C}(\mathfrak{g},K)$. Since

$$R(\mathfrak{k},K) = C_K \otimes_{U(\mathfrak{k})} U(\mathfrak{k}) \cong C_K,$$

(2.8a) and Proposition 2.2 give

$$R(\mathfrak{g},K) \otimes_{R(\mathfrak{k},K)} \Lambda^n\mathfrak{p} \cong (U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} R(\mathfrak{k},K)) \otimes_{R(\mathfrak{k},K)} \Lambda^n\mathfrak{p} \cong U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} \Lambda^n\mathfrak{p},$$

and thus X_n can be written in the more familiar form

$$X_n = U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} \Lambda^n\mathfrak{p}, \tag{3.5b}$$

with K acting by $\mathrm{Ad} \otimes \mathrm{Ad}$. The sequence of $\mathcal{C}(\mathfrak{g},K)$ modules and maps

$$0 \longleftarrow \mathbb{C} \xleftarrow{\ \epsilon\ } X_0 \xleftarrow{\ \partial_0\ } X_1 \xleftarrow{\ \partial_1\ } \cdots \xleftarrow{\ \partial_{m-1}\ } X_m \longleftarrow 0 \tag{3.6}$$

with

$$\partial_{n-1}(u\otimes Y_1\wedge\cdots\wedge Y_n) = \sum_{\ell=1}^{n}(-1)^{\ell+1}(uY_\ell\otimes Y_1\wedge\cdots\wedge\hat{Y}_\ell\wedge\cdots\wedge Y_n)$$

$$+\sum_{r<s}(-1)^{r+s}(u\otimes\rho[Y_r,Y_s]\wedge Y_1\wedge\cdots\wedge\hat{Y}_r\wedge\cdots\wedge\hat{Y}_s\wedge\cdots\wedge Y_n) \qquad (3.7)$$

and ϵ given by projection to the constant term is exact and is called the Koszul resolution in $\mathcal{C}(\mathfrak{g},K)$ of $\mathbb{C}$ with trivial action. (See [9] for the exactness.)

The axioms listed before Proposition 2.2, now known to be satisfied for $\mathcal{C}(\mathfrak{g},K)$, allow us to use derived functors in the usual way. Each module V in $\mathcal{C}(\mathfrak{g},K)$ has a standard projective resolution and a standard injective resolution that are sometimes helpful. We shall write them down after some preparation.

For V and W in $\mathcal{C}(\mathfrak{g},K)$, we make $V\otimes_{\mathbb{C}}W$ into a member of $\mathcal{C}(\mathfrak{g},K)$ in the usual way by defining

$$X(v\otimes w) = Xv\otimes w + v\otimes Xw \quad \text{for } X\in\mathfrak{g}$$

$$k(v\otimes w) = kv\otimes kw \quad \text{for } k\in K.$$

We attempt to make $\mathrm{Hom}_{\mathbb{C}}(V,W)$ into a member of $\mathcal{C}(\mathfrak{g},K)$ by defining

$$(X\varphi)(v) = X(\varphi(v)) - \varphi(Xv) \quad \text{for } X\in\mathfrak{g}$$

$$(k\varphi)(v) = k(\varphi(k^{-1}v)) \quad \text{for } k\in K.$$

However, not every member of $\mathrm{Hom}_{\mathbb{C}}(V,W)$ need be K-finite. Thus we extract the K-finite part $\mathrm{Hom}_{\mathbb{C}}(V,W)_K$, and the result is in $\mathcal{C}(\mathfrak{g},K)$. (See [12, p. 311].)

We shall use the identity

$$\mathrm{Hom}_{R(\mathfrak{g},K)}(U\otimes_{\mathbb{C}}V, W) \cong \mathrm{Hom}_{R(\mathfrak{g},K)}(U, \mathrm{Hom}_{\mathbb{C}}(V,W)_K) \qquad (3.8)$$

valid for U, V, and W in $C(\mathfrak{g},K)$ with actions as above. (See [12, p. 314].) To prove this identity, one begins with the vector space identity

$$\mathrm{Hom}_{\mathbb{C}}(U \otimes_{\mathbb{C}} V, W) \cong \mathrm{Hom}_{\mathbb{C}}(U, \mathrm{Hom}_{\mathbb{C}}(V,W))$$

given in (2.8c), then checks that the $\mathfrak{g}$ and K actions correspond, then restricts to $R(\mathfrak{g},K)$ maps, and finally puts the subscript K in place for free.

<u>Proposition 3.4</u>. Let W be in $C(\mathfrak{g},K)$. Then

(a) the functor $V \to V \otimes_{\mathbb{C}} W$ on $C(\mathfrak{g},K)$ is covariant and exact, and it sends projectives into projectives; if W is finite-dimensional, it sends injectives into injectives.[1]

(b) the functor $V \to \mathrm{Hom}_{\mathbb{C}}(V,W)_K$ on $C(\mathfrak{g},K)$ is contravariant and exact, and it sends projectives into injectives.

(c) the functor $V \to \mathrm{Hom}_{\mathbb{C}}(W,V)_K$ on $C(\mathfrak{g},K)$ is covariant and exact, and it sends injectives into injectives.

Proof. The results about covariance/contravariance are obvious, and the exactness follows from the corresponding fact in the category of vector spaces. For (a) let P be projective. We use (3.8). The functor

$$V \to \mathrm{Hom}_{R(\mathfrak{g},K)}(P \otimes_{\mathbb{C}} W, V) \cong \mathrm{Hom}_{R(\mathfrak{g},K)}(P, \mathrm{Hom}_{\mathbb{C}}(W,V)_K)$$

is the composition of $V \to \mathrm{Hom}_{\mathbb{C}}(W,V)_K$ and $U \to \mathrm{Hom}_{R(\mathfrak{g},K)}(P,U)$, both of which are exact. Hence $P \otimes_{\mathbb{C}} W$ is projective. Let I be injective,

---

[1] This last statement is a corrected version of a sentence in the proof of Lemma 3.3 of [8].

and suppose W is finite-dimensional. With $W^* = \mathrm{Hom}_{\mathbb{C}}(W,\mathbb{C})$ denoting the dual of W, we can check that $I \otimes_{\mathbb{C}} W \cong \mathrm{Hom}_{\mathbb{C}}(W^*, I)_K$ as $(\mathfrak{g}, K)$ modules. The functor

$$V \to \mathrm{Hom}_{R(\mathfrak{g},K)}(V, I \otimes_{\mathbb{C}} W) \cong \mathrm{Hom}_{R(\mathfrak{g},K)}(V, \mathrm{Hom}_{\mathbb{C}}(W^*, I)_K)$$

$$\cong \mathrm{Hom}_{R(\mathfrak{g},K)}(V \otimes_{\mathbb{C}} W^*, I)$$

is the composition of $V \to V \otimes_{\mathbb{C}} W^*$ and $U \to \mathrm{Hom}_{R(\mathfrak{g},K)}(U,I)$, both of which are exact. Hence $I \otimes_{\mathbb{C}} W$ is injective.

For (b) let P be projective. The functor

$$V \to \mathrm{Hom}_{R(\mathfrak{g},K)}(V, \mathrm{Hom}_{\mathbb{C}}(P,W)_K) \cong \mathrm{Hom}_{R(\mathfrak{g},K)}(V \otimes_{\mathbb{C}} P, W)$$

$$\cong \mathrm{Hom}_{R(\mathfrak{g},K)}(P, \mathrm{Hom}_{\mathbb{C}}(V,W)_K)$$

is the composition of $V \to \mathrm{Hom}_{\mathbb{C}}(V,W)_K$ and $U \to \mathrm{Hom}_{R(\mathfrak{g},K)}(P,U)$, both of which are exact. Hence $\mathrm{Hom}_{\mathbb{C}}(P,W)_K$ is injective.

For (c) let I be injective. The functor

$$V \to \mathrm{Hom}_{R(\mathfrak{g},K)}(V, \mathrm{Hom}_{\mathbb{C}}(W,I)_K) \cong \mathrm{Hom}_{R(\mathfrak{g},K)}(V \otimes_{\mathbb{C}} W, I)$$

is the composition of $V \to V \otimes_{\mathbb{C}} W$ and $U \to \mathrm{Hom}_{R(\mathfrak{g},K)}(U,I)$, both of which are exact. Hence $\mathrm{Hom}_{\mathbb{C}}(W,I)_K$ is injective. This completes the proof.

We write V^* for the dual of a vector space V. A special case of Proposition 3.4b is that the functor $V \to (V^*)_K$ on $\mathcal{C}(\mathfrak{g},K)$ is contravariant and exact, and it sends projectives into injectives. We denote $(V^*)_K$ by V^c.

Let V be in $\mathcal{C}(\mathfrak{g},K)$. Applying (a) and (b) of the proposition to the Koszul resolution, we obtain a projective resolution

$$0 \leftarrow V \leftarrow X_0 \otimes_{\mathbb{C}} V \leftarrow X_1 \otimes_{\mathbb{C}} V \leftarrow \cdots \leftarrow X_m \otimes_{\mathbb{C}} V \leftarrow 0 \quad (3.9)$$

and an injective resolution

$$0 \longrightarrow V \longrightarrow \mathrm{Hom}_{\mathbb{C}}(X_0, V)_K \longrightarrow \mathrm{Hom}_{\mathbb{C}}(X_1, V)_K \longrightarrow \cdots \longrightarrow \mathrm{Hom}_{\mathbb{C}}(X_m, V)_K \longrightarrow 0 . \quad (3.10)$$

These are called the <u>standard resolutions</u> of V.

We conclude this section with some auxiliary results. The first generalizes the induction-by-stages formula.

<u>Proposition 3.5</u>. Let $(\mathfrak{g}, K)$, $(\mathfrak{q}, H)$, and $(\mathfrak{q}_1, B)$ be three pairs with $\mathfrak{g} \supseteq \mathfrak{q} \supseteq \mathfrak{q}_1$ and $K \supseteq H \supseteq B$ compatibly. For V in $\mathcal{C}(\mathfrak{q}_1, B)$, there are natural isomorphisms

$$P^{\mathfrak{g},K}_{\mathfrak{q},H} \circ P^{\mathfrak{q},H}_{\mathfrak{q}_1,B}(V) \cong P^{\mathfrak{g},K}_{\mathfrak{q}_1,B}(V)$$

and

$$I^{\mathfrak{g},K}_{\mathfrak{q},H} \circ I^{\mathfrak{q},H}_{\mathfrak{q}_1,B}(V) \cong I^{\mathfrak{g},K}_{\mathfrak{q}_1,B}(V) .$$

Proof. Formula (2.8a) and Proposition 2.2 give

$$P^{\mathfrak{g},K}_{\mathfrak{q},H} \circ P^{\mathfrak{q},H}_{\mathfrak{q}_1,B}(V) = R(\mathfrak{g},K) \otimes_{R(\mathfrak{q},H)} R(\mathfrak{q},H) \otimes_{R(\mathfrak{q}_1,B)} V$$

$$\cong R(\mathfrak{g},K) \otimes_{R(\mathfrak{q}_1,B)} V = P^{\mathfrak{g},K}_{\mathfrak{q}_1,B}(V) ,$$

while (2.8c) and Proposition 2.2 give

$$I^{\mathfrak{g},K}_{\mathfrak{q},H} \circ I^{\mathfrak{q},H}_{\mathfrak{q}_1,B}(V) = \mathrm{Hom}_{R(\mathfrak{q},H)}(R(\mathfrak{g},K), \mathrm{Hom}_{R(\mathfrak{q}_1,B)}(R(\mathfrak{q},H),V)_H)_K$$

$$= \mathrm{Hom}_{R(\mathfrak{q},H)}(R(\mathfrak{g},K), \mathrm{Hom}_{R(\mathfrak{q}_1,B)}(R(\mathfrak{q},H),V))_K$$

$$\cong \mathrm{Hom}_{R(\mathfrak{q}_1,B)}(R(\mathfrak{g},K) \otimes_{R(\mathfrak{q},H)} R(\mathfrak{q},H), V)_K$$

$$\cong \mathrm{Hom}_{R(\mathfrak{q}_1,B)}(R(\mathfrak{g},K),V)_K = I^{\mathfrak{g},K}_{\mathfrak{q}_1,B}(V) .$$

Proposition 3.6. Let $(\mathfrak{g},K)$ and $(\mathfrak{q},H)$ be two pairs with $\mathfrak{g} \supseteq \mathfrak{q}$ and $K \supseteq H$ compatibly, and suppose H has finite index in K. Then the functors $P^{\mathfrak{g},K}_{\mathfrak{q},H}$ and $I^{\mathfrak{g},K}_{\mathfrak{q},H}$ are exact.

Proof. Proposition 3.5 gives $P^{\mathfrak{g},K}_{\mathfrak{q},H} = P^{\mathfrak{g},K}_{\mathfrak{g},H} \circ P^{\mathfrak{g},H}_{\mathfrak{q},H}$, and we consider the two factors separately. Propositions 2.7 and 2.2 give

$$P^{\mathfrak{g},K}_{\mathfrak{g},H}(V) = R(\mathfrak{g},K) \otimes_{R(\mathfrak{g},H)} V \cong C_K \otimes_{R(\mathfrak{k},H)} R(\mathfrak{g},H) \otimes_{R(\mathfrak{g},H)} V \cong C_K \otimes_{R(\mathfrak{k},H)} V .$$

Since H has finite index in K, $\mathfrak{k} = \mathfrak{h}$. Thus $R(\mathfrak{k},H) \cong C_H$. Hence

$$P^{\mathfrak{g},K}_{\mathfrak{g},H}(V) \cong C_K \otimes_{C_H} V = C_K \otimes_H V .$$

Now $V \to C_K \otimes_{\mathbb{C}} V$ is exact, and $C_K \otimes_{\mathbb{C}} V$ is fully reducible as an H module under the tensor product action. Thus $C_K \otimes_H V$ may be identified with the H-isotypic subspace for the trivial representation of H, and the passage to this subspace is exact. Hence $P^{\mathfrak{g},K}_{\mathfrak{g},H}$ is exact.

Let $\mathfrak{p}_1$ be a vector space complement to $\mathfrak{q}$ in $\mathfrak{g}$. Then $U(\mathfrak{g}) \cong S(\mathfrak{p}') \otimes_{\mathbb{C}} U(\mathfrak{q})$ as a right $\mathfrak{q}$ module (with trivial action on $S(\mathfrak{p}')$), and we have vector space isomorphisms

$$P^{\mathfrak{g},H}_{\mathfrak{q},H}(V) = R(\mathfrak{g},H) \otimes_{R(\mathfrak{q},H)} V \cong U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} C_H \otimes_{R(\mathfrak{q},H)} V$$

$$\cong S(\mathfrak{p}') \otimes_{\mathbb{C}} U(\mathfrak{q}) \otimes_{U(\mathfrak{h})} C_H \otimes_{R(\mathfrak{q},H)} V$$

$$\cong S(\mathfrak{p}') \otimes_{\mathbb{C}} R(\mathfrak{q},H) \otimes_{R(\mathfrak{q},H)} V \cong S(\mathfrak{p}') \otimes_{\mathbb{C}} V,$$

the last step holding by Proposition 2.2. Thus $P^{\mathfrak{g},H}_{\mathfrak{q},H}$ is exact.

We argue similarly for $I^{\mathfrak{g},K}_{\mathfrak{q},H}$. We have

$$I^{\mathfrak{g},K}_{\mathfrak{g},H}(V) = \mathrm{Hom}_{R(\mathfrak{g},H)}(R(\mathfrak{g},K),V)_K$$

$$\cong \mathrm{Hom}_{R(\mathfrak{g},H)}(C_K \otimes_{R(\mathfrak{k},H)} R(\mathfrak{g},H)\,,\, V)_K \quad \text{by Proposition 2.7}$$

$$\cong \mathrm{Hom}_{R(\mathfrak{k},H)}(C_K\,,\, \mathrm{Hom}_{R(\mathfrak{g},H)}(R(\mathfrak{g},H),V))_K \quad \text{by (2.8c)}$$

$$= \mathrm{Hom}_{R(\mathfrak{k},H)}(C_K\,,\, \mathrm{Hom}_{R(\mathfrak{g},H)}(R(\mathfrak{g},H),V)_H)_K$$

$$\cong \mathrm{Hom}_{R(\mathfrak{k},H)}(C_K, V)_K \quad \text{by Proposition 2.5.}$$

Since H has finite index in K, we can replace $R(\mathfrak{k},H)$ by C_H and then by H. Then $V \to \mathrm{Hom}_{\mathbb{C}}(C_K, V)_K$ is exact, and so is the passage to the H invariants. Hence $I^{\mathfrak{g},K}_{\mathfrak{g},H}$ is exact.

Finally the same isomorphism $R(\mathfrak{g},H) \cong S(\mathfrak{p}') \otimes_{\mathbb{C}} R(\mathfrak{q},H)$ as above gives a vector space isomorphism

$$I^{\mathfrak{g},H}_{\mathfrak{q},H}(V) = \mathrm{Hom}_{R(\mathfrak{q},H)}(R(\mathfrak{g},H),V)_H \cong \mathrm{Hom}_{R(\mathfrak{q},H)}(S(\mathfrak{p}') \otimes_{\mathbb{C}} R(\mathfrak{q},H),V)_H$$

$$\cong \mathrm{Hom}_{\mathbb{C}}(S(\mathfrak{p}')\,,\, \mathrm{Hom}_{R(\mathfrak{q},H)}(R(\mathfrak{q},H),V)_H)_H \quad \text{by (2.8c)}$$

$$\cong \mathrm{Hom}_{\mathbb{C}}(S(\mathfrak{p}'),V)_H \quad \text{by Proposition 2.5.}$$

Thus $I^{\mathfrak{g},H}_{\mathfrak{q},H}$ is exact.

<u>Proposition 3.7</u>. Let $(\mathfrak{g},K)$ and $(\mathfrak{g},H)$ be pairs with $K \supseteq H$ compatibly. Then $\mathcal{F}^{\mathfrak{k},H}_{\mathfrak{g},H}$ carries projectives to projectives and injectives to injectives.

Proof. It is enough to handle standard projectives and injectives

$$P = R(\mathfrak{g},H) \otimes_{C_H} W \quad \text{and} \quad I = \mathrm{Hom}_{C_H}(R(\mathfrak{g},H),W)_H\,,$$

where W is an $(\mathfrak{h},H)$ module. Write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ as usual. Then we have

$$\mathcal{F}^{\mathfrak{k},H}_{\mathfrak{g},H}(P) \cong S(\mathfrak{p}) \otimes_{\mathbb{C}} (R(\mathfrak{k},H) \otimes_{C_H} W)$$

by Proposition 2.6. The factor in parentheses is projective by Corollary 3.3, and $\mathcal{F}^{\mathfrak{k},H}_{\mathfrak{g},H}(P)$ is thus projective by Proposition 3.4a. Also we have

$$\mathcal{F}^{\mathfrak{k},H}_{\mathfrak{g},H}(I) \cong \operatorname{Hom}_{C_H}(S(\mathfrak{p}) \otimes_{\mathbb{C}} R(\mathfrak{k},H),\, W)_H \cong \operatorname{Hom}_{\mathbb{C}}(S(\mathfrak{p}), \operatorname{Hom}_{C_H}(R(\mathfrak{k},H),W)_H)_H$$

by Proposition 2.6 and (2.8c). The inner Hom on the right is injective by Corollary 3.3, and $\mathcal{F}^{\mathfrak{k},H}_{\mathfrak{g},H}(I)$ is thus injective by Proposition 3.4c.

4. Poincaré duality

Fix a pair $(\mathfrak{g},K)$. The functor $V \to V \otimes_{R(\mathfrak{g},K)} \mathbb{C}$ is covariant and right exact from $\mathcal{C}(\mathfrak{g},K)$ into the category $\mathcal{C}(0,\{1\})$ of all vector spaces. Its left derived functors, defined by a projective resolution of V, are defined to be the relative homology functors $V \to H_n(\mathfrak{g},K;V)$.

Similarly, for V in $\mathcal{C}(\mathfrak{g},K)$, let $V^{R(\mathfrak{g},K)}$ be the subspace of $R(\mathfrak{g},K)$ invariants; these are the $\mathfrak{g}$ invariants that are fixed by K. The functor $V \to V^{R(\mathfrak{g},K)}$ is covariant and left exact from $\mathcal{C}(\mathfrak{g},K)$ into $\mathcal{C}(0,\{1\})$. Its right derived functors, defined by an injective resolution of V, are defined to be the relative cohomology functors $V \to H^n(\mathfrak{g},K;V)$.

Let us make matters explicit, using the standard resolutions (3.9) and (3.10). We need the following vector space identity, which is a companion result to (3.8). It is valid for U, V, and W in $\mathcal{C}(\mathfrak{g},K)$:

$$(U \otimes_{\mathbb{C}} V) \otimes_{R(\mathfrak{g},K)} W \cong U \otimes_{R(\mathfrak{g},K)} (V \otimes_{\mathbb{C}} W). \tag{4.1}$$

Here the tensor products over $\mathbb{C}$ carry the tensor product $(\mathfrak{g},K)$

action. For the tensor products over $R(\mathfrak{g},K)$, the factor on the left is converted from a left $R(\mathfrak{g},K)$ module into a right $R(\mathfrak{g},K)$ module in the usual way by the standard antiautomorphism of $R(\mathfrak{g},K)$.

Let us prove (4.1). The map is simply

$$(u \otimes v) \otimes w \to u \otimes (v \otimes w). \qquad (4.2)$$

We start with (2.8a) in the form

$$(U \otimes_{\mathbb{C}} V) \otimes_{\mathbb{C}} W \cong U \otimes_{\mathbb{C}} (V \otimes_{\mathbb{C}} W). \qquad (4.3)$$

We compose the map in (4.3) from left to right with the passage to the quotient on the right side of (4.1). Then we check that the composition descends to the quotient on the left side of (4.1). Thus (4.2) defines a map of the left side of (4.1) onto the right side of (4.1). The inverse of (4.2) defines the inverse map in (4.1).

Let us make homology more explicit. We consider the standard projective resolution (3.9) of V. A term is $X_n \otimes_{\mathbb{C}} V$, and we write $X_n = R(\mathfrak{g},K) \otimes_{R(\mathfrak{k},K)} \Lambda^n \mathfrak{p}$ as in (3.5a). Applying the functor $V \to V \otimes_{R(\mathfrak{g},K)} \mathbb{C}$ to the resolution and using (4.1), we have

$$\begin{aligned}
(X_n \otimes_{\mathbb{C}} V) \otimes_{R(\mathfrak{g},K)} \mathbb{C} &\cong X_n \otimes_{R(\mathfrak{g},K)} V \\
&= (\Lambda^n \mathfrak{p} \otimes_{R(\mathfrak{k},K)} R(\mathfrak{g},K)) \otimes_{R(\mathfrak{g},K)} V \\
&\cong \Lambda^n \mathfrak{p} \otimes_{R(\mathfrak{k},K)} (R(\mathfrak{g},K) \otimes_{R(\mathfrak{g},K)} V) && \text{by (2.8a)} \\
&\cong \Lambda^n \mathfrak{p} \otimes_{R(\mathfrak{k},K)} V && \text{by Proposition 2.2} \\
&= \Lambda^n \mathfrak{p} \otimes_K V.
\end{aligned}$$

Thus the complex from which we obtain homology is

$$0 \longleftarrow \Lambda^0\mathfrak{p} \otimes_K V \xleftarrow{\partial} \Lambda^1\mathfrak{p} \otimes_K V \xleftarrow{\partial} \cdots \xleftarrow{\partial} \Lambda^m\mathfrak{p} \otimes_K V \longleftarrow 0 , \qquad (4.4)$$

and we readily check that the operator ∂ is given on $\Lambda^n\mathfrak{p} \otimes_K V$ by

$$\partial(X_1 \wedge \dots \wedge X_n \otimes v) = \sum_{\ell=1}^{n} (-1)^{\ell} (X_1 \wedge \dots \wedge \hat{X}_\ell \wedge \dots \wedge X_n \otimes X_\ell v)$$
$$+ \sum_{r<s} (-1)^{r+s} (\wp[X_r, X_s] \wedge X_1 \wedge \dots \wedge \hat{X}_r \wedge \dots \hat{X}_s \wedge \dots \wedge X_n \otimes v) . \qquad (4.5)$$

Next let us make cohomology more explicit. We consider the standard injective resolution (3.10) of V. A term is $\mathrm{Hom}_{\mathbb{C}}(X_n, V)_K$, and we again write X_n as in (3.5a). Applying the functor $V \to V^{R(\mathfrak{g},K)}$ to the resolution, we have

$$\begin{aligned}
(\mathrm{Hom}_{\mathbb{C}}(X_n, V)_K)^{R(\mathfrak{g},K)} &= \mathrm{Hom}_{R(\mathfrak{g},K)}(X_n, V) \\
&= \mathrm{Hom}_{R(\mathfrak{g},K)}(R(\mathfrak{g},K) \otimes_{R(\mathfrak{k},K)} \Lambda^n\mathfrak{p}, V) \\
&\cong \mathrm{Hom}_{R(\mathfrak{k},K)}(\Lambda^n\mathfrak{p}, \mathrm{Hom}_{R(\mathfrak{g},K)}(R(\mathfrak{g},K), V)) \quad \text{by (2.8)} \\
&= \mathrm{Hom}_{R(\mathfrak{k},K)}(\Lambda^n\mathfrak{p}, \mathrm{Hom}_{R(\mathfrak{g},K)}(R(\mathfrak{g},K), V)_K) \\
&\cong \mathrm{Hom}_{R(\mathfrak{k},K)}(\Lambda^n\mathfrak{p}, V) \quad \text{by Proposition 2.5} \\
&= \mathrm{Hom}_K(\Lambda^n\mathfrak{p}, V) .
\end{aligned}$$

Thus the complex from which we obtain cohomology is

$$0 \longrightarrow \mathrm{Hom}_K(\Lambda^0\mathfrak{p}, V) \xrightarrow{d} \mathrm{Hom}_K(\Lambda^1\mathfrak{p}, V) \xrightarrow{d} \cdots \xrightarrow{d} \mathrm{Hom}_K(\Lambda^m\mathfrak{p}, V) \longrightarrow 0 , \qquad (4.6)$$

and we readily check that the operator d is given on $\mathrm{Hom}_K(\Lambda^{k-1}\mathfrak{p}, V)$ by

$$d\lambda(Y_1 \wedge \dots \wedge Y_k) = \sum_{\ell=1}^{k} (-1)^{\ell+1} Y_\ell(\lambda(Y_1 \wedge \dots \wedge \hat{Y}_\ell \wedge \dots \wedge Y_k))$$

$$+ \sum_{r<s} (-1)^{r+s} \lambda(\wp[Y_r, Y_s] \wedge Y_1 \wedge \dots \wedge \hat{Y}_r \wedge \dots \wedge \hat{Y}_s \wedge \dots \wedge Y_k) . \tag{4.7}$$

Since $V \to V \otimes_{R(\mathfrak{g},K)} \mathbb{C}$ is right exact, we have a natural isomorphism

$$H_0(\mathfrak{g},K;V) \cong V \otimes_{R(\mathfrak{g},K)} \mathbb{C} .$$

Since $V \to V^{R(\mathfrak{g},K)}$ is left exact, we have a natural isomorphism

$$H^0(\mathfrak{g},K;V) \cong V^{R(\mathfrak{g},K)} .$$

Recall that $\mathfrak{g}$ is unimodular if the adjoint action of $\mathfrak{g}$ on $\Lambda^{\dim \mathfrak{g}} \mathfrak{g}$ is the zero action. As a reductive Lie algebra, $\mathfrak{k}$ is unimodular. Consequently $\mathfrak{k}$ acts by the same character on $\Lambda^{\dim \mathfrak{g}} \mathfrak{g}$ and on $\Lambda^{\dim \mathfrak{p}} \mathfrak{p}$. We can therefore regard $\Lambda^{\dim \mathfrak{p}} \mathfrak{p}$ as a member of $C(\mathfrak{g},K)$ by insisting that $\mathfrak{g}$ act by the same character by which it acts on $\Lambda^{\dim \mathfrak{g}} \mathfrak{g}$ and that K act by the adjoint action; these actions are consistent.

<u>Theorem 4.1</u> (Poincaré duality). For any $\mathfrak{g}$, for $0 \leq i \leq m = \dim \mathfrak{p}$, and for V in $C(\mathfrak{g},K)$, there are vector space isomorphisms

$$H^i(\mathfrak{g},K;V^c) \cong H_i(\mathfrak{g},K;V)^* \tag{4.8}$$

and

$$H^i(\mathfrak{g},K;V) \cong H_{m-i}(\mathfrak{g},K;V \otimes_{\mathbb{C}} (\Lambda^m \mathfrak{p})^*) \tag{4.9}$$

natural in V. Consequently

$$H^i(\mathfrak{g},K,V^c) \cong H^{m-i}(\mathfrak{g},K, V \otimes_{\mathbb{C}} \Lambda^m \mathfrak{p})^* \tag{4.10}$$

naturally in V.

Remarks. The isomorphism (4.10) comes by substituting (4.9) into (4.8). If K acts trivially on $\Lambda^m \mathfrak{p}$ (as is the case if K is connected) and if $\mathfrak{g}$ is unimodular, then $(\Lambda^m \mathfrak{p})^*$ can pull out of the homology in (4.9) and $\Lambda^m \mathfrak{p}$ can pull out of the cohomology in (4.10).

Proof. Let $F(W) = W^{R(\mathfrak{g},K)}$ and $G(W) = W \otimes_{R(\mathfrak{g},K)} \mathbb{C}$. Then

$$F(V^c) = \operatorname{Hom}_{R(\mathfrak{g},K)}(\mathbb{C}, V^c) = \operatorname{Hom}_{R(\mathfrak{g},K)}(\mathbb{C}, \operatorname{Hom}_{\mathbb{C}}(V,\mathbb{C})_K)$$

$$\cong \operatorname{Hom}_{\mathbb{C}}(V \otimes_{R(\mathfrak{g},K)} \mathbb{C}, \mathbb{C}) = G(V)^*$$

by (2.8b) if we regard V as a left $\mathbb{C}$ module and a right $R(\mathfrak{g},K)$ module. Letting "dual" refer to $(\cdot)^c$ in $C(\mathfrak{g},K)$ and to $(\cdot)^*$ in $C(0,\{1\})$, we can rewrite this relation as a natural isomorphism

$$F \circ \text{dual}(V) \cong \text{dual} \circ G(V). \tag{4.11}$$

If $(\cdot)^i$ refers to a right derived functor and $(\cdot)_i$ refers to a left derived functor, we thus have

$(F^i \circ \text{dual})(V) \cong (F \circ \text{dual})^i(V)$		since dual is exact and carries projectives to injectives
	$\cong (\text{dual} \circ G)^i(V)$	by (4.11)
	$\cong (\text{dual} \circ G_i)(V)$	since dual is exact and contravariant.

This isomorphism is just (4.8).

Proof of (4.9). We break the argument into several steps.

(1) Preliminaries. We calculate homology and cohomology from the explicit complexes (4.4) and (4.6). In this setting we shall exhibit a natural isomorphism on the chain/cochain level and show that it

passes to the homology/cohomology. First we define ∂ on all of $\Lambda^n\mathfrak{p}\otimes_{\mathbb{C}} V\otimes_{\mathbb{C}} (\Lambda^m\mathfrak{p})^*$ by (4.5) and d on all of $\mathrm{Hom}_{\mathbb{C}}(\Lambda^{k-1}\mathfrak{p},V)$ by (4.7). It is trivial that ∂ commutes with the action of K on $\Lambda^n\mathfrak{p}\otimes_{\mathbb{C}} V\otimes_{\mathbb{C}} (\Lambda^m\mathfrak{p})^*$ and thus descends to the K invariants $\Lambda^n\mathfrak{p}\otimes_K (V\otimes_{\mathbb{C}} (\Lambda^m\mathfrak{p})^*)$. And, of course, our new d is an extension of the original one on $\mathrm{Hom}_K(\Lambda^{k-1}\mathfrak{p},V)$. We give the isomorphism on the level of a map

$$\Lambda^n\mathfrak{p}\otimes_{\mathbb{C}} V\otimes_{\mathbb{C}} (\Lambda^m\mathfrak{p})^* \longrightarrow \mathrm{Hom}_{\mathbb{C}}(\Lambda^{m-n}\mathfrak{p},V). \tag{4.12}$$

Namely for $\xi\in\Lambda^n\mathfrak{p}$, $\gamma\in\Lambda^{m-n}\mathfrak{p}$, and $\epsilon\in(\Lambda^m\mathfrak{p})^*$, we define $\xi\otimes v\otimes\epsilon\to\lambda_{\xi\otimes v\otimes\epsilon}$ by

$$\lambda_{\xi\otimes v\otimes\epsilon}(\gamma) = \epsilon(\xi\wedge\gamma)v. \tag{4.13}$$

Then $\xi\otimes v\otimes\epsilon\to\lambda_{\xi\otimes v\otimes\epsilon}$ extends to give the linear isomorphism (4.12). This map respects the K actions

$$k(\xi\otimes v\otimes\epsilon) = k\xi\otimes kv\otimes k\epsilon$$

$$(k\lambda)(\gamma) = k(\lambda(k^{-1}\gamma))$$

because

$$\lambda_{k(\xi\otimes v\otimes\epsilon)}(\gamma) = \lambda_{k\xi\otimes kv\otimes k\epsilon}(\gamma) = (k\epsilon)(k\xi\wedge\gamma)kv = \epsilon(\xi\wedge k^{-1}\gamma)kv$$
$$= k(\epsilon(\xi\wedge k^{-1}\gamma)v) = k(\lambda_{\xi\otimes v\otimes\epsilon}(k^{-1}\gamma)) = (k\lambda_{\xi\otimes v\otimes\epsilon})(\gamma).$$

Restricted to the K invariants, our map thus gives a linear isomorphism

$$\Lambda^n\mathfrak{p}\otimes_K (V\otimes_{\mathbb{C}} (\Lambda^m\mathfrak{p})^*) \longrightarrow \mathrm{Hom}_K(\Lambda^{m-n}\mathfrak{p},V).$$

Equation (4.9) will thus follow if we show that, under our mapping $\xi\otimes v\otimes\epsilon\to\lambda_{\xi\otimes v\otimes\epsilon}$, ∂ corresponds to $(-1)^{|\xi|}d$, where $|\xi|$ is the degree of ξ. That is, we are to show that

$$d\lambda_{\xi\otimes v\otimes\epsilon} = (-1)^{|\xi|}\lambda_{\partial(\xi\otimes v\otimes\epsilon)} . \tag{4.14}$$

(2) More explicit formula for action of $\mathfrak{p}$ on $\Lambda^m\mathfrak{p}$. Put $t = \dim \mathfrak{k}$. Until (4.17) below, fix nonzero elements

$$\eta = X_1 \wedge \ldots \wedge X_m \quad \text{in } \Lambda^m\mathfrak{p}$$

$$\tau = T_1 \wedge \ldots \wedge T_t \quad \text{in } \Lambda^t\mathfrak{k} .$$

If Z belongs to $\mathfrak{g}$, then Z acts on $\Lambda^m\mathfrak{p}$ by multiplication by the scalar $c(Z)$ defined by

$$c(Z)(\eta\wedge\tau) = Z\cdot(\eta\wedge\tau) .$$

More explicitly,

$$c(Z)(\eta\wedge\tau) = \sum_{i=1}^{m} (-1)^{i+1}([Z,X_i]\wedge X_1\wedge\ldots\wedge\widehat{X}_i\wedge\ldots\wedge X_m\wedge\tau)$$

$$+ \sum_{j=1}^{t} (-1)^{j+1}(\eta\wedge[Z,T_j]\wedge T_1\wedge\ldots\wedge\widehat{T}_j\wedge\ldots\wedge T_t) . \tag{4.15}$$

Suppose now that Z belongs to $\mathfrak{p}$. Then $[Z,T_j]$ belongs to $\mathfrak{p}$, and its exterior product with η is 0. Thus the second sum on the right side of (4.15) is 0. For the same reason we can replace $[Z,X_i]$ by its projection on $\mathfrak{p}$ in the first sum. Finally we can drop τ on both sides, obtaining

$$c(Z)\eta = \sum_{i=1}^{m} (-1)^{i+1}(\wp[Z,X_i]\wedge X_1\wedge\ldots\wedge\widehat{X}_i\wedge\ldots\wedge X_m) \tag{4.16}$$

for all Z in $\mathfrak{p}$.

For the action on the element ϵ of $(\Lambda^m\mathfrak{p})^*$, (4.16) gives

$$(Z\cdot\epsilon)(\eta) = \sum_{i=1}^{m} (-1)^{i}\epsilon(\wp[Z,X_i]\wedge X_1\wedge\ldots\wedge\widehat{X}_i\wedge\ldots\wedge X_m) \tag{4.17}$$

for all Z in $\mathfrak{p}$.

(3) Verification of (4.14). Let $\xi = X_1 \wedge \ldots \wedge X_n$ and $\gamma = Y_0 \wedge \ldots \wedge Y_{m-n}$ be in $\Lambda^n \mathfrak{p}$ and $\Lambda^{m-n+1}\mathfrak{p}$, respectively. To prove (4.14), we want to prove that

$$d\lambda_{\xi\otimes v\otimes\epsilon}(\gamma) + (-1)^{n+1}\lambda_{\partial(\xi\otimes v\otimes\epsilon)}(\gamma) = 0. \tag{4.18}$$

Actually we shall prove that

$$(-1)^n d\lambda_{\xi\otimes v\otimes\epsilon}(\gamma) - \lambda_{\partial(\xi\otimes v\otimes\epsilon)}(\gamma) = d(\epsilon\otimes v)(\xi\wedge\gamma). \tag{4.19}$$

On the right side, $\epsilon\otimes v$ is to be regarded as a member of $\mathrm{Hom}(\Lambda^m\mathfrak{p},\gamma)$. Since $\xi\wedge\gamma$ belongs to $\Lambda^{m+1}\mathfrak{p}$, which is zero, the right side of (4.19) is 0. Thus (4.18) follows from (4.19).

To prove (4.19), we introduce

$$\begin{aligned}
\xi_i &= X_1\wedge\ldots\wedge\hat{X}_i\wedge\ldots\wedge X_n && \text{in } \Lambda^{n-1}\mathfrak{p}\\
\xi_{ij} &= X_1\wedge\ldots\wedge\hat{X}_i\wedge\ldots\wedge\hat{X}_j\wedge\ldots\wedge X_n && \text{in } \Lambda^{n-2}\mathfrak{p}\\
\gamma_i &= Y_0\wedge\ldots\wedge\hat{Y}_i\wedge\ldots\wedge Y_{m-n} && \text{in } \Lambda^{m-n}\mathfrak{p}\\
\gamma_{ij} &= Y_0\wedge\ldots\wedge\hat{Y}_i\wedge\ldots\wedge\hat{Y}_j\wedge\ldots\wedge Y_{m-n} && \text{in } \Lambda^{m-n-1}\mathfrak{p}.
\end{aligned} \tag{4.20}$$

We compute the left side of (4.19), using (4.5) and (4.7) to obtain

$$(-1)^n\sum_{i=0}^{m-n}(-1)^i\epsilon(\xi\wedge\gamma_i)Y_i v + (-1)^n\sum_{i<j}(-1)^{i+j}\epsilon(\xi\wedge\rho[Y_i,Y_j]\wedge\gamma_{ij})v$$

$$-\sum_{r=1}^{n}(-1)^r\epsilon(\xi_r\wedge\gamma)X_r v - \sum_{r=1}^{n}(-1)^r(X_r\epsilon)(\xi_r\wedge\gamma)v$$

$$-\sum_{r<s}(-1)^{r+s}\epsilon(\rho[X_r,X_s]\wedge\xi_{rs}\wedge\gamma)v.$$

We use (4.17) to compute $X_r\epsilon$, rearrange terms a little, and find that this expression is

$$= \sum_{r=1}^{n} (-1)^{r+1}\epsilon(\xi_r \wedge \gamma) X_r v + \sum_{i=0}^{m-n} (-1)^{i+n}\epsilon(\xi \wedge \gamma_i) Y_i v$$

$$- \sum_{r<s} (-1)^{r+s}\epsilon(\rho[X_r, X_s] \wedge \xi_{rs} \wedge \gamma) v + \sum_{i<j} (-1)^{i+j}\epsilon(\rho[Y_i, Y_j] \wedge \xi \wedge \gamma_{ij}) v$$

$$- \sum_{r=1} \sum_{s<r} (-1)^{r+s}\epsilon(\rho[X_r, X_s] \wedge \xi_{rs} \wedge \gamma) v + \sum_{r=1} \sum_{s>r} (-1)^{r+s}\epsilon(\rho[X_r, X_s] \wedge \xi_{rs} \wedge \gamma) v$$

$$- \sum_{r=1}^{n} \sum_{i=0}^{m-n} (-1)^{r+i+n-1}\epsilon(\rho[X_r, Y_i] \wedge \xi_r \wedge \gamma_i) v .$$

(Here the last three terms represent the contribution from the sum involving $X_r\epsilon$.) The fifth and sixth terms are each the negative of the third. Thus the expression simplifies to

$$= \sum_{r=1}^{n} (-1)^{r+1}\epsilon(\xi_r \wedge \gamma) X_r v + \sum_{i=0}^{m-n} (-1)^{i+n}\epsilon(\xi \wedge \gamma_i) Y_i v$$

$$+ \sum_{r<s} (-1)^{r+s}\epsilon(\rho[X_r, X_s] \wedge \xi_{rs} \wedge \gamma) v + \sum_{i<j} (-1)^{i+j}\epsilon(\rho[Y_i, Y_j] \wedge \xi \wedge \gamma_{ij}) v$$

$$+ \sum_{r,i} (-1)^{r+i+n}\epsilon(\rho[X_r, Y_i] \wedge \xi_r \wedge \gamma_i) v ,$$

which is precisely the right side of (4.19), by (4.7). Thus (4.19) is proved, and Theorem 4.1 follows.

5. Mackey isomorphisms

We shall need two associativity formulas that lie deeper than the ones in (2.8), (3.8), and (4.1). These formulas generalize the well known fact about induced representations that induction commutes with tensor product under certain circumstances. Accordingly we call these formulas Mackey isomorphisms.

Theorem 5.1. Let $(\mathfrak{g},K)$ and $(\mathfrak{q},H)$ be pairs with $\mathfrak{g} \supseteq \mathfrak{q}$ and $K \supseteq H$ compatibly, let U be in $\mathcal{C}(\mathfrak{g},K)$ with action π, and let V be in $\mathcal{C}(\mathfrak{q},H)$. Then there exists a unique isomorphism of $\mathcal{C}(\mathfrak{g},K)$ modules

$$\Phi : P^{\mathfrak{g},K}_{\mathfrak{q},H}(U \otimes_{\mathbb{C}} V) \to U \otimes_{\mathbb{C}} P^{\mathfrak{g},K}_{\mathfrak{q},H}(V) \tag{5.1a}$$

such that

$$\Phi(c \otimes (f \otimes v)) = \sum_i f_i \otimes ((\pi(\cdot)f, f_i)c(\cdot) \otimes v) \tag{5.1b}$$

for c in $C_K \subseteq R(\mathfrak{g},K)$, f in U, and v in V; here f lies in some finite-dimensional K-invariant subspace of U, $(\cdot,\cdot)$ is a K-invariant inner product for this space, and $\{f_i\}$ is an orthonormal basis of this space. The isomorphism is natural with respect to U and V.

Remark. For intuition, one can regard the right side of (5.1b) as the image of $f \otimes (\delta \otimes v)$ acted on by c; here δ is a point mass at 1 in K.

Proof. Uniqueness is clear since $U(\mathfrak{g})$ carries C_K within $R(\mathfrak{g},K)$ onto $R(\mathfrak{g},K)$ and since Φ is a $\mathfrak{g}$ map. To prove existence with (5.1a), we first reduce matters to the case of $P^{\mathfrak{g},K}_{\mathfrak{g},H}$. Let U and V be as in the theorem, let X be in $\mathcal{C}(\mathfrak{g},K)$, and let Y be in $\mathcal{C}(\mathfrak{g},H)$. Then we have natural isomorphisms

$$\mathrm{Hom}_{R(\mathfrak{g},H)}(Y, \mathrm{Hom}_{R(\mathfrak{g},K)}(R(\mathfrak{g},K), \mathrm{Hom}_{\mathbb{C}}(U,X)_K)_H)$$

$$\cong \mathrm{Hom}_{R(\mathfrak{g},K)}(R(\mathfrak{g},K) \otimes_{R(\mathfrak{g},H)} Y, \mathrm{Hom}_{\mathbb{C}}(U,X)_K) \qquad \text{by (2.8)}$$

$$\cong \mathrm{Hom}_{R(\mathfrak{g},K)}(U\otimes_{\mathbb{C}}(R(\mathfrak{g},K)\otimes_{R(\mathfrak{g},H)} Y), X) \qquad \text{by (3.8)}$$

$$\cong \mathrm{Hom}_{R(\mathfrak{g},K)}(R(\mathfrak{g},K)\otimes_{R(\mathfrak{g},H)}(U\otimes_{\mathbb{C}} Y), X) \qquad \text{by (5.1a) for } P^{\mathfrak{g},K}_{\mathfrak{g},H}$$

$$\cong \mathrm{Hom}_{R(\mathfrak{g},H)}(U\otimes_{\mathbb{C}} Y, \mathrm{Hom}_{R(\mathfrak{g},K)}(R(\mathfrak{g},K),X)_H) \qquad \text{by (2.8)}$$

$$\cong \mathrm{Hom}_{R(\mathfrak{g},H)}(Y, \mathrm{Hom}_{\mathbb{C}}(U,\mathrm{Hom}_{R(\mathfrak{g},K)}(R(\mathfrak{g},K),X)_H)_H) \text{ by (3.8).}$$

By a standard argument in homological algebra, we conclude

$$\mathrm{Hom}_{R(\mathfrak{g},K)}(R(\mathfrak{g},K), \mathrm{Hom}_{\mathbb{C}}(U,X)_K)_H \cong \mathrm{Hom}_{\mathbb{C}}(U, \mathrm{Hom}_{R(\mathfrak{g},K)}(R(\mathfrak{g},K),X)_H)_H \tag{5.2}$$

naturally. The map from left to right is the member of the last Hom above that corresponds to 1 in the first Hom when $Y = \mathrm{Hom}_{R(\mathfrak{g},K)}(R(\mathfrak{g},K),\mathrm{Hom}_{\mathbb{C}}(U,X)_K)_H$. Therefore we have natural isomorphisms

$$\mathrm{Hom}_{R(\mathfrak{g},K)}(U\otimes_{\mathbb{C}}(R(\mathfrak{g},K)\otimes_{R(\mathfrak{q},H)} V),X)$$

$$\cong \mathrm{Hom}_{R(\mathfrak{g},K)}(R(\mathfrak{g},K)\otimes_{R(\mathfrak{q},H)} V,\mathrm{Hom}_{\mathbb{C}}(U,X)_K) \qquad \text{by (3.8)}$$

$$\cong \mathrm{Hom}_{R(\mathfrak{q},H)}(V,\mathrm{Hom}_{R(\mathfrak{g},K)}(R(\mathfrak{g},K),\mathrm{Hom}_{\mathbb{C}}(U,X)_K)_H) \qquad \text{by (2.8)}$$

$$\cong \mathrm{Hom}_{R(\mathfrak{q},H)}(V,\mathrm{Hom}_{\mathbb{C}}(U,\mathrm{Hom}_{R(\mathfrak{g},K)}(R(\mathfrak{g},K),X)_H)_H) \qquad \text{by (5.2)}$$

$$\cong \mathrm{Hom}_{R(\mathfrak{q},H)}(U\otimes_{\mathbb{C}} V,\mathrm{Hom}_{R(\mathfrak{g},K)}(R(\mathfrak{g},K),X)_H) \qquad \text{by (3.8)}$$

$$\cong \mathrm{Hom}_{R(\mathfrak{g},K)}(R(\mathfrak{g},K)\otimes_{R(\mathfrak{q},H)}(U\otimes_{\mathbb{C}} V),X) \qquad \text{by (2.8),}$$

and the isomorphism (5.1a) follows, again by standard homological algebra.

Next let us show that (5.1b) follows in general for the map constructed above if (5.1b) holds for the map involved for $P^{\mathfrak{g},K}_{\mathfrak{g},H}$. Let

$\Phi_1, \ldots, \Phi_6$ be corresponding members of the Hom's that precede (5.2). For y in Y, r in $R(\mathfrak{g},K)$, c in C_K, and f in U, we have

$$\Phi_1(y)(r) = \Phi_2(r\otimes y) \qquad \text{by (3.3)}$$

$$\Phi_2(r\otimes y)(f) = \Phi_3(f\otimes r\otimes y)$$

$$\Phi_4(c\otimes f\otimes y) = \Phi_3\Big(\sum_i f_i\otimes(\pi(\cdot)f,f_i)c(\cdot)\otimes y\Big) \qquad \text{by (5.1b) for } P_{\mathfrak{g},H}^{\mathfrak{g},K}$$

$$\Phi_4(r\otimes f\otimes y) = \Phi_5(f\otimes y)(r) \qquad \text{by (3.3)}$$

$$\Phi_5(f\otimes y) = \Phi_6(y)(f).$$

Put $Y = \mathrm{Hom}_{R(\mathfrak{g},K)}(R(\mathfrak{g},K), \mathrm{Hom}_{\mathbb{C}}(U,X)_K)_H$ and $\Phi_1 = 1$. Then we obtain

$$\begin{aligned}
\Phi_6(y)(f)(c) &= \Phi_5(f\otimes y)(c) = \Phi_4(c\otimes f\otimes y)\\
&= \Phi_3\Big(\sum_i f_i\otimes(\pi(\cdot)f,f_i)c(\cdot)\otimes y\Big)\\
&= \sum_i \Phi_2((\pi(\cdot)f,f_i)c(\cdot)\otimes y)(f_i)\\
&= \sum_i \Phi_1(y)((\pi(\cdot)f,f_i)c(\cdot))(f_i)\\
&= \sum_i \; y((\pi(\cdot)f,f_i)c(\cdot))(f_i) \quad \text{since } \Phi_1 = 1, \qquad (5.3)
\end{aligned}$$

and Φ_6 is the map that implements (5.2). Let $\Psi_1, \ldots, \Psi_6$ be corresponding members of the Hom's that follow (5.2). For r in $R(\mathfrak{g},K)$, c in C_K, f in U, and v in V, we have

$$\Psi_1(f\otimes r\otimes v) = \Psi_2(r\otimes v)(f)$$

$$\Psi_2(r\otimes v) = \Psi_3(v)(r) \qquad \text{by (3.3)}$$

$$\Psi_4(v)(f)(r) = \sum_i \Psi_3(v)((\pi(\cdot)f,f_i)c(\cdot))(f_i) \qquad \text{by (5.3) with } y = \Psi_3(v)$$

$$\Psi_4(v)(f) = \Psi_5(f\otimes v)$$

$$\Psi_6(r\otimes f\otimes v) = \Psi_5(f\otimes v)(r) \qquad \text{by (3.3).}$$

Put $X = U\otimes_{\mathbb{C}} (R(\mathfrak{g},K)\otimes_{R(\mathfrak{q},H)} V)$ and $\Psi_1 = 1$. Then we obtain

$$\begin{aligned}\Psi_6(c\otimes f\otimes v) &= \Psi_5(f\otimes v)(c) = \Psi_4(v)(f)(c)\\ &= \sum_i \Psi_3(v)((\pi(\cdot)f,f_i)c(\cdot))(f_i)\\ &= \sum_i \Psi_2((\pi(\cdot)f,f_i)c(\cdot)\otimes v)(f_i)\\ &= \sum_i \Psi_1(f_i\otimes(\pi(\cdot)f,f_i)c(\cdot)\otimes v)\\ &= \sum_i f_i\otimes(\pi(\cdot)f,f_i)c(\cdot)\otimes v \quad \text{since } \Psi_1 = 1,\end{aligned}$$

and Ψ_6 is our required map Φ satisfying (5.1b).

Consequently it is enough to prove the existence part of the theorem for $P^{\mathfrak{g},K}_{\mathfrak{g},H}$. This we shall do directly. Since Proposition 2.7 gives

$$\begin{aligned}P^{\mathfrak{g},K}_{\mathfrak{g},H}(U\otimes_{\mathbb{C}} V) &= R(\mathfrak{g},K)\otimes_{R(\mathfrak{g},H)}(U\otimes_{\mathbb{C}} V)\\ &\cong C_K\otimes_{R(\mathfrak{k},H)} R(\mathfrak{g},H)\otimes_{R(\mathfrak{g},H)}(U\otimes_{\mathbb{C}} V) \cong C_K\otimes_{R(\mathfrak{k},H)}(U\otimes_{\mathbb{C}} V),\end{aligned}$$

equation (5.1b) defines Φ globally, although not obviously consistently. For consistency, let X be in $\mathfrak{k}$ and h be in H. In obvious notation we have

$$\begin{aligned}&\Phi(r(-X)c\otimes(f\otimes v)) - \Phi(c\otimes X(f\otimes v))\\ &\quad= \sum f_i\otimes((\pi(\cdot)f,f_i)r(-X)c(\cdot)\otimes v)\\ &\qquad- \sum f_i\otimes((\pi(\cdot)\pi(X)f,f_i)c(\cdot)\otimes v)\\ &\qquad- \sum f_i\otimes((\pi(\cdot)f,f_i)c(\cdot)\otimes Xv)\end{aligned}$$

$$= \sum f_i \otimes (r(-X)\{(\pi(\cdot)f, f_i)c(\cdot)\} \otimes v)$$

$$- \sum f_i \otimes ((\pi(\cdot)f, f_i)c(\cdot) \otimes Xv) \equiv 0$$

and

$$\Phi(r(h)^{-1}c \otimes (f \otimes v)) - \Phi(c \otimes h(f \otimes v))$$

$$= \sum f_i \otimes ((\pi(\cdot)f, f_i)c(\cdot h^{-1})) \otimes v$$

$$- \sum f_i \otimes ((\pi(\cdot h)f, f_i)c(\cdot)) \otimes hv \equiv 0 .$$

This proves consistency.

Let us verify that Φ is indeed a $(\mathfrak{g}, K)$ map. The argument is not totally transparent, and we include it. If Y is in $\mathfrak{g}$, then

$$\Phi(Y(c \otimes (f \otimes v))) = \Phi((Y \otimes c) \otimes (f \otimes v))$$

$$= \Phi((c \otimes \mathrm{Ad}(\cdot)^{-1}Y) \otimes (f \otimes v))$$

$$= \sum \Phi(c(\cdot)(\mathrm{Ad}(\cdot)^{-1}Y, Y_i) \otimes Y_i) \otimes (f \otimes v)$$

$$= \sum \Phi(c(\cdot)(\mathrm{Ad}(\cdot)^{-1}Y, Y_i) \otimes Y_i(f \otimes v))$$

$$= \sum_{i,j} f_j \otimes (c(\cdot)(\mathrm{Ad}(\cdot)^{-1}Y, Y_i)(\pi(\cdot)\pi(Y_i)f, f_j) \otimes v)$$

$$+ \sum_{i,j} f_j \otimes (c(\cdot)(\mathrm{Ad}(\cdot)^{-1}Y, Y_i)(\pi(\cdot)f, f_j) \otimes Y_i v) , \qquad (5.4)$$

while

$$Y(\Phi(c \otimes (f \otimes v))) = \sum Y(f_i \otimes (c(\cdot)(\pi(\cdot)f, f_i) \otimes v))$$

$$= \sum \pi(Y)f_i \otimes (c(\cdot)(\pi(\cdot)f, f_i) \otimes v)$$

$$+ \sum f_i \otimes ((Y \otimes c(\cdot)(\pi(\cdot)f, f_i)) \otimes v)$$

$$= \sum_i \pi(Y)f_i \otimes (c(\cdot)(\pi(\cdot)f, f_i) \otimes v)$$

$$+ \sum_{i,j} f_i \otimes (c(\cdot)(\pi(\cdot)f, f_i)(\mathrm{Ad}(\cdot)^{-1}Y, Y_j) \otimes Y_j v) . \qquad (5.5)$$

The second term of (5.4) matches the second term of (5.5). We rewrite the first term of (5.4) by changing from the basis Y_i to the basis $\mathrm{Ad}(\cdot)^{-1}Y_i$, and the term becomes

$$= \sum_{i,j} f_j \otimes (c(\cdot)(Y,Y_i)(\pi(\cdot)\pi(\mathrm{Ad}(\cdot)^{-1}Y_i)f,f_j)\otimes v)$$

$$= \sum_{i,j} f_j \otimes (c(\cdot)(Y,Y_i)(\pi(Y_i)\pi(\cdot)f,f_j)\otimes v)$$

$$= \sum_{j} f_j \otimes (c(\cdot)(\pi(Y)\pi(\cdot)f,f_j)\otimes v)$$

$$= \sum_{i,j} f_j \otimes (c(\cdot)(\pi(Y)f_i,f_j)(\pi(\cdot)f,f_i)\otimes v)$$

$$= \sum_{i} \pi(Y)f_i \otimes (c(\cdot)(\pi(\cdot)f,f_i)\otimes v)\,.$$

This matches the first term of (5.5) and shows Φ is a $\mathfrak{g}$ map. Finally if k is in K, then

$$\Phi(k(c\otimes(f\otimes v))) = \Phi(c(k^{-1}\cdot)\otimes(f\otimes v))$$
$$= \sum f_i \otimes (c(k^{-1}\cdot)(\pi(\cdot)f,f_i)\otimes v)$$
$$= \sum \pi(k)f_i \otimes (c(k^{-1}\cdot)(\pi(\cdot)f,\pi(k)f_i)\otimes v) \quad \text{by change of basis}$$
$$= \sum \pi(k)f_i \otimes (\ell(k)\{c(\cdot)(\pi(\cdot)f,f_i)\}\otimes v)$$
$$= k(\Phi(c\otimes(f\otimes v)))\,.$$

Thus Φ is a $(\mathfrak{g},K)$ map.

Finally we prove that Φ is one-one onto by constructing a two-sided inverse. The argument is more general, providing a two-sided inverse to (5.1b) for $P^{\mathfrak{g},K}_{\mathfrak{q},H}$, and we shall use this generality in §7. Put

$$\Psi(f\otimes(c\otimes v)) = \sum_i c(\cdot)(\pi(\cdot)^{-1}f,f_i)\otimes(f_i\otimes v)\,. \tag{5.6}$$

We check that Ψ is a well defined two-sided inverse for Φ. As with Φ, Ψ is well defined if we check consistency of $c\otimes v$ for $\mathfrak{k}$ and H. If X is in $\mathfrak{k}$ and h is in H, we have

$$\begin{aligned}
&\Psi(f\otimes(r(-X)c\otimes v)) - \Psi(f\otimes(c\otimes Xv))\\
&\qquad = \Sigma\ (r(-X)c(\cdot))(\pi(\cdot)^{-1}f,f_i)\otimes(f_i\otimes v)\\
&\qquad\quad - \Sigma\ c(\cdot)(\pi(\cdot)^{-1}f,f_i)\otimes(f_i\otimes Xv)\\
&\qquad = \Sigma\ r(-X)\{c(\cdot)(\pi(\cdot)^{-1}f,f_i)\}\otimes(f_i\otimes v)\\
&\qquad\quad - \Sigma\ c(\cdot)r(-X)\{(\pi(\cdot)^{-1}f,f_i)\}\otimes(f_i\otimes v)\\
&\qquad\quad - \Sigma\ c(\cdot)(\pi(\cdot)^{-1}f,f_i)\otimes(f_i\otimes Xv)\\
&\qquad \equiv \Sigma\ c(\cdot)(\pi(\cdot)^{-1}f,f_i)\otimes(\pi(X)f_i\otimes v)\\
&\qquad\quad - \Sigma\ c(\cdot)r(-X)\{(\pi(\cdot)^{-1}f,f_i)\}\otimes(f_i\otimes v) \quad \text{since } X \text{ moves across } \otimes\\
&\qquad = \sum_{i,j} c(\cdot)(\pi(\cdot)^{-1}f,f_i)(\pi(X)f_i,f_j)\otimes(f_j\otimes v)\\
&\qquad\quad - \sum_i c(\cdot)(\pi(X)\pi(\cdot)^{-1}f,f_i)\otimes(f_i\otimes v) = 0
\end{aligned}$$

and

$$\begin{aligned}
&\Psi(f\otimes(r(h)^{-1}c\otimes v)) - \Psi(f\otimes(c\otimes hv))\\
&\qquad = \Sigma\ c(\cdot h^{-1})(\pi(\cdot)^{-1}f,f_i)\otimes(f_i\otimes v) - \Sigma\ c(\cdot)(\pi(\cdot)^{-1}f,f_i)\otimes(f_i\otimes hv)\\
&\qquad = \Sigma\ r(h)^{-1}\{c(\cdot)(\pi(\cdot h)^{-1}f,f_i)\}\otimes(f_i\otimes v)\\
&\qquad\quad - \Sigma\ c(\cdot)(\pi(\cdot)^{-1}f,f_i)\otimes(f_i\otimes hv)\\
&\qquad \equiv \Sigma\ c(\cdot)(\pi(\cdot)^{-1}f,\pi(h)f_i)\otimes(\pi(h)f_i\otimes hv)\\
&\qquad\quad - \Sigma\ c(\cdot)(\pi(\cdot)^{-1}f,f_i)\otimes(f_i\otimes hv) = 0\ ,
\end{aligned}$$

the last equality holding since $\{\pi(h)f_i\}$ is another orthonormal basis. Hence Ψ is well defined. In addition,

$$\Psi(\Phi(c\otimes(f\otimes v))) = \sum_i \Psi(f_i\otimes((\pi(\cdot)f,f_i)c(\cdot)\otimes v))$$

$$= \sum_{i,j} (\pi(\cdot)f,f_i)c(\cdot)(\pi(\cdot)^{-1}f_i,f_j)\otimes(f_j\otimes v)$$

$$= \sum_j (f,f_j)c(\cdot)\otimes(f_j\otimes v) = c\otimes(f\otimes v)$$

and

$$\Phi(\Psi(f\otimes(c\otimes v))) = \sum_i \Phi(c(\cdot)(\pi(\cdot)^{-1}f,f_i)\otimes(f_i\otimes v))$$

$$= \sum_{i,j} f_j\otimes(c(\cdot)(\pi(\cdot)^{-1}f,f_i)(\pi(\cdot)f_i,f_j)\otimes v)$$

$$= \sum_j f_j\otimes(c(\cdot)(f,f_j)\otimes v) = f\otimes(c\otimes v),$$

and thus Ψ is a two-sided inverse to Φ.

<u>Theorem 5.2.</u>[2] Let $(\mathfrak{g},K)$ and $(\mathfrak{q},H)$ be pairs with $\mathfrak{g}\supseteq\mathfrak{q}$ and $K\supseteq H$ compatibly, let U be in $\mathcal{C}(\mathfrak{g},K)$ with action π, and let V be in $\mathcal{C}(\mathfrak{q},H)$. Then there exists a unique isomorphism of $\mathcal{C}(\mathfrak{g},K)$ modules

$$\Phi : \operatorname{Hom}_{\mathbb{C}}(U, I_{\mathfrak{q},H}^{\mathfrak{g},K}(V))_K \to I_{\mathfrak{q},H}^{\mathfrak{g},K}(\operatorname{Hom}_{\mathbb{C}}(U,V)_H) \tag{5.7a}$$

such that

$$\Phi(\psi)(c)(f) = \sum_i \psi(f_i)((\pi(\cdot)^{-1}f,f_i)c(\cdot)) \tag{5.7b}$$

for ψ in $\operatorname{Hom}_{\mathbb{C}}(U,I_{\mathfrak{q},H}^{\mathfrak{g},K}(V))_K$, c in $C_K\subseteq R(\mathfrak{g},K)$, and f in U; here f lies in some finite-dimensional K-invariant subspace of U, $(\cdot,\cdot)$ is a K-invariant inner product for this space, and $\{f_i\}$ is an

---

[2] This theorem is a corrected version of an unproved statement in the proof of Lemma 3.3 of [8].

orthonormal basis of this space. The isomorphism is natural with respect to U and V.

Remark. On the right side of (5.7b), $\psi(f_i)$ is a member of $I_{\mathfrak{q},H}^{\mathfrak{g},K}(V)$, which is a space of maps on $R(\mathfrak{g},K)$, and $(\pi(\cdot)^{-1}f,f_i)c(\cdot)$ is the member of $R(\mathfrak{g},K)$ on which $\psi(f_i)$ is to be evaluated.

Proof. Again uniqueness is clear since the facts that Φ is a $\mathfrak{g}$ map and C_K generates $R(\mathfrak{g},K)$ mean that $\Phi(\psi)$ is determined by its values on C_K. The existence argument proceeds in the same style as with Theorem 5.1 but is much easier since no special case needs attention. For X in $\mathcal{C}(\mathfrak{g},K)$, we have natural isomorphisms

$$\operatorname{Hom}_{R(\mathfrak{g},K)}(X,\operatorname{Hom}_{\mathbb{C}}(U,I_{\mathfrak{q},H}^{\mathfrak{g},K}(V))_K) \tag{5.8a}$$

$$\cong \operatorname{Hom}_{R(\mathfrak{g},K)}(U\otimes_{\mathbb{C}} X,\ I_{\mathfrak{q},H}^{\mathfrak{g},K}(V)) \quad \text{by (3.8)} \tag{5.8b}$$

$$\cong \operatorname{Hom}_{R(\mathfrak{q},H)}(U\otimes_{\mathbb{C}} X,\ V) \quad \text{by (3.2)} \tag{5.8c}$$

$$\cong \operatorname{Hom}_{R(\mathfrak{q},H)}(X,\ \operatorname{Hom}_{\mathbb{C}}(U,V)_H) \quad \text{by (3.8)} \tag{5.8d}$$

$$\cong \operatorname{Hom}_{R(\mathfrak{g},K)}(X,\ I_{\mathfrak{q},H}^{\mathfrak{g},K}(\operatorname{Hom}_{\mathbb{C}}(U,V)_H)) \quad \text{by (3.2).} \tag{5.8e}$$

Then the existence of Φ as in (5.7a) follows by a standard argument in homological algebra if we take $X = \operatorname{Hom}_{\mathbb{C}}(U,I_{\mathfrak{q},H}^{\mathfrak{g},K}(V))_K$ and take Φ to be the member of (5.8e) that corresponds to the identity in (5.8a).

Let us verify that this Φ satisfies (5.7b). Let $\Phi_1 = 1$, Φ_2, Φ_3, Φ_4, and Φ be corresponding members of (5.8a) through (5.8e). For x in X, c and c' in C_K, and f in U, the relevant identities are

$$\Phi(x)(c)(f) = \Phi_4(cx)(f) \quad \text{by (3.4)}$$

$$\Phi_4(x)(f) = \Phi_3(f\otimes x)$$

$$\Phi_2(f\otimes x)(c') = \Phi_3(c'(f\otimes x)) \qquad \text{by (3.4)}$$

$$\Phi_1(x)(f) = \Phi_2(f\otimes x) .$$

Fix ψ in $\mathrm{Hom}_{\mathbb{C}}(U, I^{\mathfrak{g},K}_{\mathfrak{q},H}(V))_K$ and c in C_K and f in U. If e_F is chosen so that $e_F(f\otimes c\psi) = f\otimes c\psi$, then the above identities give

$$\Phi(\psi)(c)(f) = \Phi_4(c\psi)(f) = \Phi_3(f\otimes c\psi)$$

$$= \Phi_3(e_F(f\otimes c\psi)) = \Phi_2(f\otimes c\psi)(e_F)$$

$$= \Phi_1(c\psi)(f)(e_F)$$

$$= (c\psi)(f)(e_F)$$

$$= \int_K (k(\psi(\pi(k)^{-1}f)))(e_F)c(k)\,dk$$

$$= \int_K \psi(\pi(k)^{-1}f)(r(k)^{-1}e_F)c(k)\,dk$$

$$= \sum_i \psi(f_i)\Big(\int_K (\pi(k)^{-1}f, f_i)e_F(\cdot k^{-1})c(k)\,dk\Big)$$

$$= \sum_i \psi(f_i)(e_F * (\pi(\cdot)^{-1}f, f_i)c(\cdot))$$

$$= \sum_i \psi(f_i)((\pi(\cdot)^{-1}f, f_i)c(\cdot)) ,$$

the last equality holding if F is sufficiently large. This proves (5.7b) and completes the proof of Theorem 5.2.

<u>Corollary 5.3</u>. Let $(\mathfrak{g},K)$ and $(\mathfrak{q},H)$ be pairs with $\mathfrak{g}\supseteq\mathfrak{q}$ and $K\supseteq H$ compatibly, let F be a finite-dimensional member of $\mathcal{C}(\mathfrak{g},K)$ with action π, and let V be in $\mathcal{C}(\mathfrak{q},H)$. Then there exists a unique isomorphism of $\mathcal{C}(\mathfrak{q},K)$ modules

$$\Phi : F \otimes_{\mathbb{C}} I^{\mathfrak{g},K}_{\mathfrak{q},H}(V) \to I^{\mathfrak{g},K}_{\mathfrak{q},H}(F \otimes_{\mathbb{C}} V) \tag{5.9a}$$

such that

$$\Phi(f \otimes \varphi)(c) = \sum_i f_i \otimes \varphi((\pi(\cdot)f, f_i)c(\cdot)) \tag{5.9b}$$

for φ in $I^{\mathfrak{q},K}_{\mathfrak{q},H}(V)$, f in F, and c in $C_K \subseteq R(\mathfrak{g},K)$. The isomorphism is natural with respect to F and V.

Proof. We put $U = F^*$ in Theorem 5.2. We have

$$F \otimes_{\mathbb{C}} I(V) \cong \mathrm{Hom}_{\mathbb{C}}(F^*, I(V))$$

under $f \otimes \varphi \to \psi$ with $\psi(f^*) = f^*(f)\varphi$ and under $\sum f_j \otimes \psi(f_j^*) \leftarrow \psi$ if $\{f_j^*\}$ is a dual basis to $\{f_j\}$. Then this ψ maps by (5.7b) to

$$\Phi(\psi)(c)(f^*) = \sum f_i^*(f)\varphi((\pi^*(\cdot)^{-1}f^*, f_i^*)c(\cdot)),$$

which maps into $I(\mathrm{Hom}_{\mathbb{C}}(F^*, V))$ as

$$\sum_j f_j \otimes \sum_i f_i^*(f)\varphi((\pi^*(\cdot)^{-1}f_j^*, f_i^*)c(\cdot)).$$

Sorting out this expression, we arrive at (5.9b).

6. Zuckerman duality

Throughout this section we fix pairs $(\mathfrak{g},K)$ and $(\mathfrak{g},H)$ with $K \supseteq H$ compatibly. The Zuckerman functor $\Gamma = \Gamma^{\mathfrak{g},K}_{\mathfrak{g},H}$ and the "dual" functor $\Pi = \Pi^{\mathfrak{g},K}_{\mathfrak{g},H}$ carry $\mathcal{C}(\mathfrak{g},H)$ to $\mathcal{C}(\mathfrak{g},K)$ and are the covariant functors defined by

$$\Gamma(V) = \mathrm{Hom}_{R(\mathfrak{g},H)}(R(\mathfrak{g},K),V)_K \tag{6.1}$$

$$\Pi(V) = R(\mathfrak{g},K) \otimes_{R(\mathfrak{g},H)} V. \tag{6.2}$$

Thus $\Gamma = I^{\mathfrak{g},K}_{\mathfrak{g},H}$ and $\Pi = P^{\mathfrak{g},K}_{\mathfrak{g},H}$.

Since Γ and Π are special cases of I and P, we can read off a number of properties by specializing the results of §3. The functor Γ is left exact, while Π is right exact. Thus the derived functors Γ^i of Γ are obtained from an injective resolution, while the derived functors Π_i of Π are obtained from a projective resolution. From (3.1) and (3.2), we have adjoint formulas

$$\mathrm{Hom}_{R(\mathfrak{g},K)}(\Pi(V),W) \cong \mathrm{Hom}_{R(\mathfrak{g},H)}(V,\mathcal{F}^{\vee}(W)) \tag{6.3}$$

$$\mathrm{Hom}_{R(\mathfrak{g},K)}(W,\Gamma(V)) \cong \mathrm{Hom}_{R(\mathfrak{g},H)}(\mathcal{F}(W),V)\,, \tag{6.4}$$

where $\mathcal{F}^{\vee} = (\mathcal{F}^{\vee})^{\mathfrak{g},H}_{\mathfrak{g},K}$ and $\mathcal{F} = \mathcal{F}^{\mathfrak{g},H}_{\mathfrak{g},K}$. By Corollary 3.3,

Π carries projectives to projectives (6.5)

Γ carries injectives to injectives. (6.6)

By Proposition 3.5,

Π and Γ can be computed in stages. (6.7)

And by Proposition 3.6,

Π and Γ are exact if H has finite index in K. (6.8)

Our objective is to prove the Duality Theorem below. Let $m = \dim(\mathfrak{k}/\mathfrak{h})$, and write $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{s}$ with $\mathfrak{s}$ stable under $\mathrm{Ad}(H)$. Let $\wp$ be the projection of $\mathfrak{k}$ onto $\mathfrak{s}$. Since K is compact, $\mathfrak{k}$ is unimodular. Thus $\Lambda^m\mathfrak{s}$ becomes a member of $\mathcal{C}(\mathfrak{k},H)$ with trivial $\mathfrak{k}$ action and the adjoint H action.

Theorem 6.1. For $0 \leq i \leq m = \dim(\mathfrak{k}/\mathfrak{h})$ and V in $\mathcal{C}(\mathfrak{g},H)$, there are $(\mathfrak{g},K)$ isomorphisms

$$\Gamma^i(V^c) \cong \Pi_i(V)^c \tag{6.9}$$

and

$$\Gamma^i(V) \cong \Pi_{m-i}(V \otimes_{\mathbb{C}} (\Lambda^m \mathfrak{s})^*) \tag{6.10}$$

natural in V. Consequently

$$\Gamma^i(V^c) \cong \Gamma^{m-i}(V \otimes_{\mathbb{C}} \Lambda^m \mathfrak{s})^c \tag{6.11}$$

naturally in V.

Remark. In the special case that K acts trivially on $\Lambda^m \mathfrak{s}$ and V has finite-dimensional H-isotypic subspaces, (6.11) is the result conjectured by Zuckerman [17] and proved incompletely by Enright and Wallach [8].

The proof of Theorem 6.1 will occupy the remainder of this section. We begin by disposing of the easy parts of the theorem.

The isomorphism (6.11) comes by substituting (6.10) into (6.9). Let us prove (6.9). We have

$$\Pi(V)^c = \mathrm{Hom}_{\mathbb{C}}(\Pi(V),\mathbb{C})_K = \mathrm{Hom}(R(\mathfrak{g},K) \otimes_{R(\mathfrak{g},H)} V, \mathbb{C})_K .$$

On the right side, we regard V as a left $R(\mathfrak{g},H)$ module and a right $\mathbb{C}$ module, and we can then rewrite this Hom, via (2.8c), as

$$\cong \mathrm{Hom}_{R(\mathfrak{g},H)}(R(\mathfrak{g},K), \mathrm{Hom}_{\mathbb{C}}(V,\mathbb{C})_H)_K$$

$$= \mathrm{Hom}_{R(\mathfrak{g},H)}(R(\mathfrak{g},K), V^c)_K = \Gamma(V^c) .$$

In short, we have

$$\Gamma \cdot \mathrm{dual}(V) \cong \mathrm{dual} \cdot \Pi(V) . \qquad (6.12)$$

Formula (6.12) is completely analogous to (4.11), and we obtain (6.9) from (6.12) in the same way that (4.8) is obtained from (4.11).

Now we turn to (6.10), which is the heart of the matter for us. It was recognized by Zuckerman [17] and Enright and Wallach [8] that the proof of (6.11), which comes down to (6.10) for us, should be divided into two steps. The first step is to prove the isomorphism in $\mathcal{C}(\mathfrak{k},K)$, and the second step is to see that the isomorphism actually occurs in $\mathcal{C}(\mathfrak{g},K)$. In [17] and [8], the first step was seen as the easier one and was readily handled by Poincaré duality in $\mathcal{C}(\mathfrak{k},H)$. The difficult step was the passage to $\mathfrak{g}$. Zuckerman [17] looked for formulas that would make the isomorphism clear, while Enright and Wallach [8] wanted to prove a few simple properties of the $\mathfrak{k}$ isomorphism and to deduce the $\mathfrak{g}$ isomorphism from general principles. Our approach combines these ideas. An argument along these lines to get (6.11) directly is not too difficult and is indicated in §8. But (6.10), which involves a direct relationship between Γ and Π, requires a more careful analysis. We choose to do this analysis at the first stage——that of the $\mathfrak{k}$ isomorphisms. As our analysis proceeds, we shall see why we cannot carry along the $\mathfrak{g}$ action explicitly in our computations, and we shall therefore see the role of the Enright-Wallach idea more clearly.

Both [17] and [8] use the relationship between Γ computed in $\mathcal{C}(\mathfrak{g},H)$ and Γ computed in $\mathcal{C}(\mathfrak{k},H)$. Our approach will not use this relationship explicitly, but the relationship does lie behind some of our thinking. To make this relationship precise, let us temporarily write $\Gamma_{\mathfrak{g}}$ for Γ as in (6.1), and let us write $\Gamma_{\mathfrak{k}}$ for the version on $\mathfrak{k} : \Gamma_{\mathfrak{k}} = I_{\mathfrak{k},H}^{\mathfrak{k},K}$. We write $\Pi_{\mathfrak{g}}$ and $\Pi_{\mathfrak{k}}$ similarly. Then the relevant natural isomorphisms are

$$\mathcal{F}_{\mathfrak{g},K}^{\mathfrak{k},K} \circ \Gamma_{\mathfrak{g}}^{i}(V) \cong \Gamma_{\mathfrak{k}}^{i} \circ \mathcal{F}_{\mathfrak{g},H}^{\mathfrak{k},H}(V) \tag{6.13a}$$

and

$$\mathcal{F}_{\mathfrak{g},K}^{\mathfrak{k},K} \circ (\Pi_{\mathfrak{g}})_{i}(V) \cong (\Pi_{\mathfrak{k}})_{i} \circ \mathcal{F}_{\mathfrak{g},H}^{\mathfrak{k},H}(V) \tag{6.13b}$$

for $0 \leq i \leq m$. These formulas for Γ and Π themselves are immediate consequences of (2.8); passage to derived functors is accomplished by using Proposition 3.7 and the same argument that obtains (4.8) from (4.11).

Now let us describe our approach. It follows from (6.13) that Γ^{i} and Π_{i} are 0 for $i > m$. Thus we look for a resolution in $C(\mathfrak{g},H)$ that terminates at the m^{th} step and allows us to compute Γ^{i} or Π_{i}. For $\Gamma^{i}(V)$, we are dealing with $\operatorname{Hom}_{R(\mathfrak{g},H)}(R(\mathfrak{g},K),V)_{K}$, and the normal thing is to replace V by the members of an injective resolution and compute the cohomology of the resulting complex. But if we disregard the surviving $(\mathfrak{g},K)$ action, the situation is analogous to that for the functor $\operatorname{Ext}_{R}^{i}(U,V)$. This functor may be computed by replacing in $\operatorname{Hom}_{R}(U,V)$ either U by a projective resolution or V by an injective resolution, and the cohomology will be the same.

Our approach to Γ^{i} in similar fashion involves considering what happens when we replace $R(\mathfrak{g},K)$ in $\operatorname{Hom}_{R(\mathfrak{g},H)}(R(\mathfrak{g},K),V)_{K}$ by a projective resolution in $C(\mathfrak{g},H)$. This we can take to have length m. The difficulty is that we must keep track of the surviving $(\mathfrak{g},K)$ action.

Let us be more precise. Let X_n be the Koszul resolution (of $\mathbb{C}$) in the category $\mathcal{C}(\mathfrak{k},H)$:

$$X_n = R(\mathfrak{k},H) \otimes_{R(\mathfrak{h},H)} \Lambda^n(\mathfrak{k}/\mathfrak{h}) \,, \tag{6.14a}$$

and let χ denote the action of $R(\mathfrak{k},H)$ on X_n by left multiplication in the first factor. By (3.5) we can write X_n also as

$$X_n = U(\mathfrak{k}) \otimes_{U(\mathfrak{h})} \Lambda^n(\mathfrak{k}/\mathfrak{h}) \,. \tag{6.14b}$$

Proposition 3.4a shows that $C_K \otimes_{\mathbb{C}} X_n$ with action $\ell \otimes \chi$ gives a projective resolution of C_K in $\mathcal{C}(\mathfrak{k},H)$:

$$0 \longleftarrow C_K \xleftarrow{1\otimes\epsilon} C_K \otimes_{\mathbb{C}} X_0 \xleftarrow{1\otimes\partial} \cdots \xleftarrow{1\otimes\partial} C_K \otimes_{\mathbb{C}} X_m \longleftarrow 0 \,. \tag{6.15}$$

The members of (6.15) have a $(\mathfrak{k},H)$-commuting action by K given as $r \otimes 1$, and this action commutes with the boundary operators $1 \otimes \partial$ since $r \otimes 1$ and $1 \otimes \partial$ act in different coordinates. We call this the <u>external K action</u> on (6.15). Next we apply the functor $P_{\mathfrak{k},H}^{\mathfrak{g},H}$ to (6.15), obtaining by Proposition 2.7

$$0 \longleftarrow R(\mathfrak{g},K) \longleftarrow R_0 \longleftarrow \cdots \longleftarrow R_m \longleftarrow 0 \,, \tag{6.16a}$$

where

$$R_n = R(\mathfrak{g},H) \otimes_{R(\mathfrak{k},H)} (C_K \otimes_{\mathbb{C}} X_n) \,. \tag{6.16b}$$

This is a projective resolution of (the representation L on) $R(\mathfrak{g},K)$ in the category $\mathcal{C}(\mathfrak{g},H)$, by Corollary 3.3 and Proposition 3.6. The external K action on (6.15) yields an <u>external K action</u> $1 \otimes r \otimes 1$ on (6.16), and this commutes with the $(\mathfrak{g},H)$ action and the boundary operators $1 \otimes 1 \otimes \partial$.

<u>Lemma 6.2</u>. With X_n written as in (6.14b), the map

$$f_n(t\otimes c\otimes(u\otimes\eta)) = (R\otimes\ell\otimes 1)(u^{tr})(t\otimes c\otimes\eta)$$

is an $R(\mathfrak{g},H)$ isomorphism of R_n onto

$$R_n' = R(\mathfrak{g},H)\otimes_{R(\mathfrak{h},H)}(C_K\otimes_{\mathbb{C}}\Lambda^n(\mathfrak{k}/\mathfrak{h}))$$

respecting the external K actions $1\otimes r\otimes 1$ on R_n and R_n'. Under this identification the boundary operators of (6.16) may be reinterpreted in R_n' as follows:

$$\varepsilon : R(\mathfrak{g},H)\otimes_{R(\mathfrak{h},H)}(C_K\otimes_{\mathbb{C}}\Lambda^0\mathfrak{s}) \to R(\mathfrak{g},H)\otimes_{R(\mathfrak{h},H)}C_K \cong R(\mathfrak{g},K)$$

$$\partial : R(\mathfrak{g},H)\otimes_{R(\mathfrak{h},H)}(C_K\otimes_{\mathbb{C}}\Lambda^n\mathfrak{s}) \to R(\mathfrak{g},H)\otimes_{R(\mathfrak{h},H)}(C_K\otimes_{\mathbb{C}}\Lambda^{n-1}\mathfrak{s})$$

$$\varepsilon(t\otimes(c\otimes 1)) = t\otimes c \cong t(c) \quad \text{(action in } R(\mathfrak{g},K))$$

$$\begin{aligned}\partial(t\otimes c\otimes Y_1\wedge\ldots\wedge Y_n) &= \sum_{i=1}^{n}(-1)^i(R(Y_i)t\otimes c\otimes Y_1\wedge\ldots\wedge\hat{Y}_i\wedge\ldots\wedge Y_n)\\ &+\sum_{i=1}^{n}(-1)^i(t\otimes\ell(Y_i)c\otimes Y_1\wedge\ldots\wedge\hat{Y}_i\wedge\ldots\wedge Y_n)\\ &+\sum_{r<s}(-1)^{r+s}(t\otimes c\otimes\rho[Y_r,Y_s]\wedge Y_1\wedge\ldots\wedge\hat{Y}_r\wedge\ldots\wedge\hat{Y}_s\wedge\ldots\wedge Y_n)\end{aligned} \tag{6.17}$$

Proof. First we write down the isomorphisms, and then we explain them:

$$\begin{aligned}R_n &= R(\mathfrak{g},H)\otimes_{R(\mathfrak{k},H)}(C_K\otimes_{\mathbb{C}}X_n)\\ &\cong (R(\mathfrak{g},H)\otimes_{\mathbb{C}}C_K)\otimes_{R(\mathfrak{k},H)}X_n && \text{by (4.1)}\\ &\cong (R(\mathfrak{g},H)\otimes_{\mathbb{C}}C_K)\otimes_{R(\mathfrak{k},H)}(R(\mathfrak{k},H)\otimes_{R(\mathfrak{h},H)}\Lambda^n(\mathfrak{k}/\mathfrak{h})) && \text{by (6.14a)}\end{aligned}$$

$$\cong ((R(\mathfrak{g},H)\otimes_{\mathbb{C}} C_K)\otimes_{R(\mathfrak{k},H)} R(\mathfrak{k},H))\otimes_{R(\mathfrak{h},H)} \Lambda^n(\mathfrak{k}/\mathfrak{h}) \quad \text{by (2.8a)}$$

$$\cong (R(\mathfrak{g},H)\otimes_{\mathbb{C}} C_K)\otimes_{R(\mathfrak{h},H)} \Lambda^n(\mathfrak{k}/\mathfrak{h}) \quad \text{by Proposition 2.2}$$

$$\cong R(\mathfrak{g},H)\otimes_{R(\mathfrak{h},H)} (C_K\otimes_{\mathbb{C}} \Lambda^n(\mathfrak{k}/\mathfrak{h})) \quad \text{by (4.1).}$$

All of these isomorphisms are simple regroupings; no Mackey isomorphisms are involved. The action that is tensored out over $R(\mathfrak{k},H)$ is $R\otimes\ell\otimes\chi$ in the first two lines, $R\otimes\ell\otimes L\otimes 1$ in the third line, and $R\otimes\ell\otimes L$ in the fourth line. The $(\mathfrak{g},H)$ action by L on $R(\mathfrak{g},H)$ is respected at each step, and the external K action by r on C_K is respected at each step. Reinterpretation of the boundary operator from (3.7) is straightforward and is omitted.

In (6.16a), the $R(\mathfrak{g},K)$ term has two obvious $\mathfrak{g}$ actions, but the terms R_n have only one. The action L on $R(\mathfrak{g},K)$ is the one that extends compatibly to the R_n's. Although we shall not need to do so, it is possible to carry information about the action R on $R(\mathfrak{g},K)$ to the R_n's. There is no difficulty with the K action; it already appears as the external action by K in the C_K factor of R_n. In place of a $\mathfrak{g}$ action on R_n, one can settle for an "action up to homotopy." Namely we invoke [6, p. 76] for each X in $\mathfrak{g}$ (or each X in a basis) to produce a chain map $R_n(X): R_n \to R_n$ over $R(X): R(\mathfrak{g},K)\to R(\mathfrak{g},K)$, and the $R_n(\cdot)$ have the properties of a representation, but with equalities replaced by homotopies. The fact that $R_n(X)$ is not given by explicit formulas (and is not even given uniquely) is what makes the passage from $\mathfrak{k}$ to $\mathfrak{g}$ in the Duality Theorem difficult.

We turn to the computation of the derived functors. Our opening description was in terms of Γ, but we prefer to begin with Π, which is a little easier in technical terms. To keep the notation manageable, we shall work with $\Pi_n(V)$ for a while, rather than $\Pi_n(V\otimes(\wedge^m\mathfrak{s})^*)$. In order to take advantage of the same resolution of $R(\mathfrak{g},K)$ to handle both Γ and Π, we shall use the naturally equivalent definition of Π as

$$\Pi(V) = V\otimes_{R(\mathfrak{g},H)} R(\mathfrak{g},K)\,.$$

Here the action on V is tensored out with the action L on $R(\mathfrak{g},K)$ and $\Pi(V)$ gets its action from R on $R(\mathfrak{g},K)$. (Recall the discussion following the definitions of P and I in §2.)

Let V be in $\mathcal{C}(\mathfrak{g},H)$, and let V_n be a projective resolution of V in $\mathcal{C}(\mathfrak{g},H)$ with action ν_n. By Proposition 3.4, $V_n\otimes_{\mathbb{C}} R(\mathfrak{g},K)$ with action $\nu_n\otimes L$ and $V\otimes_{\mathbb{C}} R_n$ with its tensor product action are both projective resolutions of $V\otimes_{\mathbb{C}} R(\mathfrak{g},K)$ in $\mathcal{C}(\mathfrak{g},H)$, and we place them in the diagram below. By [6, p. 76], there exist chain maps $f'_{n,V}$ and $g'_{n,V}$ over the identity as indicated

$$\begin{array}{ccccccccc}
0 & \longleftarrow & V\otimes_{\mathbb{C}} R(\mathfrak{g},K) & \xleftarrow{\varepsilon'\otimes 1} & V_0\otimes_{\mathbb{C}} R(\mathfrak{g},K) & \xleftarrow{\partial\otimes 1} & V_1\otimes_{\mathbb{C}} R(\mathfrak{g},K) & \xleftarrow{\partial\otimes 1} & \cdots \\
 & & \updownarrow 1 & & f'_{0,V}\downarrow\ \uparrow g'_{0,V} & & f'_{1,V}\downarrow\ \uparrow g'_{1,V} & & \\
0 & \longleftarrow & V\otimes_{\mathbb{C}} R(\mathfrak{g},K) & \xleftarrow{1\otimes\varepsilon} & V\otimes_{\mathbb{C}} R_0 & \xleftarrow{1\otimes\partial} & V\otimes_{\mathbb{C}} R_1 & \xleftarrow{1\otimes\partial} & \cdots \\
 & & \updownarrow 1 & & \downarrow 1\otimes f_0 & & \downarrow 1\otimes f_1 & & \\
0 & \longleftarrow & V\otimes_{\mathbb{C}} R(\mathfrak{g},K) & \xleftarrow{1\otimes\varepsilon} & V\otimes_{\mathbb{C}} R'_0 & \xleftarrow{1\otimes\partial} & V\otimes_{\mathbb{C}} R'_1 & \xleftarrow{1\otimes\partial} & \cdots
\end{array} \qquad (6.18)$$

Moreover, any two versions of $\{f'_{n,V}\}$ are homotopic, and any two versions of $\{g'_{n,V}\}$ are homotopic. In addition, $g'_{n,V}f'_{n,V}$ is homotopic to the identity, and so is $f'_{n,V}g'_{n,V}$.

The top row of (6.18) has an external $(\mathfrak{g}, K)$ action π'_n given by $1 \otimes R$, and π'_n commutes with the boundary operators, which act in the second coordinate. The middle row has an external K action π_n, which comes from the R_n coordinate, and we know that this action commutes with the boundary operators. Both actions reduce to $1 \otimes R$ on $V \otimes_{\mathbb{C}} R(\mathfrak{g}, K)$. Because of this, we have homotopy relations

$$f'_{n,V}\pi'_n(k) \simeq \pi_n(k) f'_{n,V} \quad \text{and} \quad g'_{n,V}\pi_n(k) \simeq \pi'_n(k) g'_{n,V} \qquad (6.19)$$

for each k in K.

The maps f_n in (6.18) are those in Lemma 6.2. The bottom row has an external K action, and it commutes with the boundary operators, by the lemma. Moreover, the maps f_n are compatible with the K actions on the middle and bottom rows.

Lemma 6.3. For V in $\mathcal{C}(\mathfrak{g}, H)$, the map

$$f''_{n,V}(v \otimes t \otimes c \otimes \eta) = \eta \otimes t^{tr} v \otimes c$$

is a vector space isomorphism of $V \otimes_{R(\mathfrak{g},H)} R'_n$ onto

$$(\wedge^n(\mathfrak{k}/\mathfrak{h}) \otimes_{\mathbb{C}} V) \otimes_H C_K \qquad (6.20)$$

that respects the external action by K given by r on the C_K factor. Under this identification the boundary operators $1 \otimes \partial$ on $V \otimes_{R(\mathfrak{g},H)} R'_n$ may be reinterpreted on (6.20) as follows:

$$\partial : (\wedge^n \mathfrak{s} \otimes_{\mathbb{C}} V) \otimes_H C_K \to (\wedge^{n-1} \mathfrak{s} \otimes_{\mathbb{C}} V) \otimes_H C_K$$

$$\partial(Y_1\wedge\cdots\wedge Y_n\otimes v\otimes c)=\sum_{i=1}^{n}(-1)^i(Y_1\wedge\cdots\wedge\hat{Y}_i\wedge\cdots\wedge Y_n\otimes Y_i v\otimes c)$$
$$+\sum_{i=1}^{n}(-1)^i(Y_1\wedge\cdots\wedge\hat{Y}_i\wedge\cdots\wedge Y_n\otimes v\otimes \ell(Y_i)c)$$
$$+\sum_{r<s}(-1)^{r+s}(P[Y_r,Y_s]\wedge Y_1\wedge\cdots\wedge\hat{Y}_r\wedge\cdots\wedge\hat{Y}_s\wedge\cdots\wedge Y_n\otimes v\otimes c). \tag{6.21}$$

Proof. The relevant isomorphisms are

$$\begin{aligned} V\otimes_{R(\mathfrak{g},H)}R_n' &= V\otimes_{R(\mathfrak{g},H)}(R(\mathfrak{g},H)\otimes_{R(\mathfrak{h},H)}(C_K\otimes_{\mathbb{C}}\Lambda^n(\mathfrak{k}/\mathfrak{h}))) \\ &\cong (V\otimes_{R(\mathfrak{g},H)}R(\mathfrak{g},H))\otimes_{R(\mathfrak{h},H)}(C_K\otimes_{\mathbb{C}}\Lambda^n(\mathfrak{k}/\mathfrak{h})) && \text{by (2.8a)} \\ &\cong V\otimes_{R(\mathfrak{h},H)}(C_K\otimes_{\mathbb{C}}\Lambda^n(\mathfrak{k}/\mathfrak{h})) && \text{by Proposition 2.2} \\ &\cong (\Lambda^n(\mathfrak{k}/\mathfrak{h})\otimes_{\mathbb{C}}V)\otimes_H C_K && \text{by (4.1),} \end{aligned}$$

and the right side is (6.20). The composite map is $f''_{n,V}$, and no Mackey isomorphisms are involved. The rest of the argument is straightforward and is omitted.

We now apply the functor $(\cdot)\otimes_{R(\mathfrak{g},H)}\mathbb{C}$ to the diagram (6.18) and to the associated maps and homotopies. Retaining the same names for the maps as in (6.18), dropping the first column, invoking (4.1), and adjoining a row to take Lemma 6.3 into account, we obtain the commutative diagram

$$
\begin{array}{ccccccc}
0 \longleftarrow & V_0 \otimes_{R(\mathfrak{g},H)} R(\mathfrak{g},K) & \longleftarrow & V_1 \otimes_{R(\mathfrak{g},H)} R(\mathfrak{g},K) & \longleftarrow \cdots \\
 & {\scriptstyle f'_{0,V}} \downarrow \uparrow {\scriptstyle g'_{0,V}} & & {\scriptstyle f'_{1,V}} \downarrow \uparrow {\scriptstyle g'_{1,V}} & \\
0 \longleftarrow & V \otimes_{R(\mathfrak{g},H)} R_0 & \longleftarrow & V \otimes_{R(\mathfrak{g},H)} R_1 & \longleftarrow \cdots \\
 & \downarrow {\scriptstyle 1\otimes f_0} & & \downarrow {\scriptstyle 1\otimes f_1} & \\
0 \longleftarrow & V \otimes_{R(\mathfrak{g},H)} R'_0 & \longleftarrow & V \otimes_{R(\mathfrak{g},H)} R'_1 & \longleftarrow \cdots \\
 & \downarrow {\scriptstyle f''_{0,V}} & & \downarrow {\scriptstyle f''_{1,V}} & \\
0 \longleftarrow & (\Lambda^0(\mathfrak{k}/\mathfrak{h}) \otimes_{\mathbb{C}} V) \otimes_H C_K & \xleftarrow{\ \partial\ } & (\Lambda^1(\mathfrak{k}/\mathfrak{h}) \otimes_{\mathbb{C}} V) \otimes_H C_K & \longleftarrow \cdots
\end{array}
\tag{6.22}
$$

of vector spaces. Because of the external K actions in (6.18) and Lemma 6.3, the rows are complexes in $\mathcal{C}(\mathfrak{k},K)$. The vertical maps $1\otimes f_n$ and $f''_{n,V}$ are $(\mathfrak{k},K)$ maps, but the first row of vertical maps need not be $(\mathfrak{k},K)$ maps. The homologies of the top and bottom complexes are, respectively, $\Pi_n(V)$ and something to which we give a separate name $\Pi'_n(V)$. The diagram (6.22) induces vector space isomorphisms on homology

$$f''_{n,V,*} \circ (1\otimes f_n)_* \circ f'_{n,V,*} : \Pi_n(V) \to \Pi'_n(V) \tag{6.23}$$

with $(f'_{n,V,*})^{-1} = g'_{n,V,*}$. Relations (6.19) say that the isomorphisms (6.23) are $(\mathfrak{k},K)$ maps.

What we have just done for the functor Π we now repeat for the functor Γ. We begin with the analog of (6.18). There is one additional complication that comes from having to take the H-finite or K-finite vectors of each Hom space. We must distinguish two kinds of finiteness conditions—those relative to H that refer to actions in the original category $\mathcal{C}(\mathfrak{g},H)$ and those relative to K that refer to the imposed "external" action. The analog of (6.18) is

$$\begin{array}{ccccccc}
0 \to \mathrm{Hom}_{\mathbb{C}}(R(\mathfrak{g},K),V)_H & \xrightarrow{\mathrm{Hom}(1,\varepsilon')} & \mathrm{Hom}_{\mathbb{C}}(R(\mathfrak{g},K),V_0)_H & \xrightarrow{\mathrm{Hom}(1,\partial')} & \mathrm{Hom}_{\mathbb{C}}(R(\mathfrak{g},K),V_1)_H & \longrightarrow \\
\updownarrow 1 & & a'_{0,V}\downarrow\uparrow b'_{0,V} & & a'_{1,V}\downarrow\uparrow b'_{1,V} & \\
0 \to \mathrm{Hom}_{\mathbb{C}}(R(\mathfrak{g},K),V)_H & \xrightarrow{\mathrm{Hom}(\varepsilon,1)} & \mathrm{Hom}_{\mathbb{C}}(R_0,V)_H & \xrightarrow{\mathrm{Hom}(\partial,1)} & \mathrm{Hom}_{\mathbb{C}}(R_1,V)_H & \longrightarrow \\
\updownarrow 1 & & \uparrow \mathrm{Hom}(f_0,1) & & \uparrow \mathrm{Hom}(f_1,1) & \\
0 \to \mathrm{Hom}_{\mathbb{C}}(R(\mathfrak{g},K),V)_H & \xrightarrow{\mathrm{Hom}(\varepsilon,1)} & \mathrm{Hom}_{\mathbb{C}}(R'_0,V)_H & \xrightarrow{\mathrm{Hom}(\partial,1)} & \mathrm{Hom}_{\mathbb{C}}(R'_1,V)_H & \longrightarrow
\end{array} \tag{6.24}$$

We construct the chain maps $a'_{n,V}$ and $b'_{n,V}$ initially by [6, p. 76], but then we have to modify them. Namely each Hom is a direct product of K types relative to the external action (which is by right translation in the first variable). On each K type, we redefine our map by following it by projection to the same K type, so that the composition takes each K type in its domain to the corresponding K type in the range. The diagram (6.24) still commutes. In addition, when homotopies are constructed, we project them in the same way, so that they too carry K type to K type.

As in (6.18), the K actions in the middle and bottom rows are compatible, but those in the top and middle rows are compatible only up to homotopy. All the K actions commute with the relevant coboundary operators.

Lemma 6.4. For V in $\mathcal{C}(\mathfrak{g},H)$, the map

$$a''_{n,V}(\varphi) = \varphi', \quad \varphi'(t\otimes c\otimes\eta) = t(\varphi(c)(\eta))$$

is a vector space isomorphism of

$$\mathrm{Hom}_H(C_K, \mathrm{Hom}_{\mathbb{C}}(\Lambda^n(\mathfrak{k}/\mathfrak{h}),V))_K \tag{6.25}$$

onto $\mathrm{Hom}_{R(\mathfrak{g},H)}(R'_n,V)_K$ that respects the external action by K given by r on the C_K factor. Under this identification the coboundary operators $\mathrm{Hom}(\partial,1)$ on $\mathrm{Hom}_{R(\mathfrak{g},H)}(R'_n,V)_K$ may be reinterpreted on (6.25) as follows:

$$d : \mathrm{Hom}_H(C_K, \mathrm{Hom}_{\mathbb{C}}(\Lambda^n\mathfrak{s},V))_K \to \mathrm{Hom}_H(C_K,\mathrm{Hom}_{\mathbb{C}}(\Lambda^{n+1}\mathfrak{s},V))_K$$

$$d\varphi(c)(Y_1\wedge\ldots\wedge Y_{n+1}) = \sum_{i=1}^{n+1}(-1)^{i+1}Y_i(\varphi(c)(Y_1\wedge\ldots\wedge\widehat{Y}_i\wedge\ldots\wedge Y_{n+1}))$$

$$+\sum_{i=1}^{n+1}(-1)^i\varphi(\ell(Y_i)c)(Y_1\wedge\ldots\wedge\widehat{Y}_i\wedge\ldots\wedge Y_{n+1})$$

$$+\sum_{r<s}(-1)^{r+s}\varphi(c)(\wp[Y_r,Y_s]\wedge Y_1\wedge\ldots\wedge\widehat{Y}_r\wedge\ldots\wedge\widehat{Y}_s\wedge\ldots\wedge Y_{n+1}). \tag{6.26}$$

Proof. The relevant isomorphisms are

$$\mathrm{Hom}_{R(\mathfrak{g},H)}(R'_n,V)_K = \mathrm{Hom}_{R(\mathfrak{g},H)}(R(\mathfrak{g},H)\otimes_{R(\mathfrak{h},H)}(C_K\otimes_{\mathbb{C}}\Lambda^n(\mathfrak{k}/\mathfrak{h})),V)_K$$

$$\cong \mathrm{Hom}_{R(\mathfrak{h},H)}(C_K\otimes_{\mathbb{C}}\Lambda^n(\mathfrak{k}/\mathfrak{h}),\mathrm{Hom}_{R(\mathfrak{g},H)}(R(\mathfrak{g},H),V)_H)_K \quad \text{by (2.8b)}$$

$$\cong \mathrm{Hom}_{R(\mathfrak{h},H)}(C_K\otimes_{\mathbb{C}}\Lambda^n(\mathfrak{k}/\mathfrak{h}),V)_K \quad \text{by Proposition 2.5}$$

$$\cong \mathrm{Hom}_{R(\mathfrak{h},H)}(C_K,\mathrm{Hom}_{\mathbb{C}}(\Lambda^n(\mathfrak{k}/\mathfrak{h}),V))_K \quad \text{by (3.8).}$$

We omit the details.

The next step is to apply the invariants functor $(\cdot)^{R(\mathfrak{g},H)}$ to the diagram (6.24), taking the K-finite part and adjoining another row to the diagram from Lemma 6.4. We obtain the commutative diagram

$$
\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{Hom}_{R(\mathfrak{g},H)}(R(\mathfrak{g},K),V_0)_K & \longrightarrow & \mathrm{Hom}_{R(\mathfrak{g},H)}(R(\mathfrak{g},K),V_1)_K & \longrightarrow \cdots \\
& a'_{0,V}\uparrow\downarrow b'_{0,V} & & a'_{1,V}\uparrow\downarrow b'_{1,V} & \\
0 \longrightarrow & \mathrm{Hom}_{R(\mathfrak{g},H)}(R_0,V)_K & \longrightarrow & \mathrm{Hom}_{R(\mathfrak{g},H)}(R_1,V)_K & \longrightarrow \cdots \\
& \uparrow \mathrm{Hom}(f_0,1) & & \uparrow \mathrm{Hom}(f_1,1) & \\
0 \longrightarrow & \mathrm{Hom}_{R(\mathfrak{g},H)}(R'_0,V)_K & \longrightarrow & \mathrm{Hom}_{R(\mathfrak{g},H)}(R'_1,V)_K & \longrightarrow \cdots \\
& \uparrow a''_{0,V} & & \uparrow a''_{1,V} & \\
0 \longrightarrow & \mathrm{Hom}_H(C_K,\mathrm{Hom}_{\mathbb{C}}(\wedge^0(\mathfrak{k}/\mathfrak{h}),V))_K & \xrightarrow{d} & \mathrm{Hom}_H(C_K,\mathrm{Hom}_{\mathbb{C}}(\wedge^1(\mathfrak{k}/\mathfrak{h}),V))_K & \longrightarrow \cdots
\end{array}
\tag{6.27}
$$

of vector spaces. Each row is a complex in $\mathcal{C}(\mathfrak{k},K)$, and the maps $\mathrm{Hom}(f_n,1)$ and $a''_{n,V}$ are $(\mathfrak{k},K)$ maps; but the first row of vertical maps need not be $(\mathfrak{k},K)$ maps. The cohomologies of the top and bottom complexes are, respectively, $\Gamma^n(V)$ and something to which we give a separate name $\Gamma'^n(V)$. The diagram (6.27) induces vector space isomorphisms on cohomology

$$a'^{*}_{n,V} \circ \mathrm{Hom}(f_n,1) \circ a''^{*}_{n,V} : \Gamma'^n(V) \to \Gamma^n(V) \tag{6.28}$$

with $(a'^{*}_{n,V})^{-1} = b'^{*}_{n,V}$, and we see that these isomorphisms are $(\mathfrak{k},K)$ maps.

<u>Lemma 6.5</u>. Define ∂ on $(\Lambda^n\mathfrak{s}\otimes_{\mathbb{C}} V\otimes_{\mathbb{C}} (\Lambda^m\mathfrak{s})^*)\otimes_{\mathbb{C}} C_K$ by (6.21), and define d on $\mathrm{Hom}_{\mathbb{C}}(C_K, \mathrm{Hom}_{\mathbb{C}}(\Lambda^n\mathfrak{s}, V))_K$ by (6.26). Then the map

$$\mathfrak{D}(\xi\otimes v\otimes \epsilon\otimes c) = \lambda_{\xi\otimes v\otimes\epsilon\otimes c} \tag{6.29a}$$

with

$$\lambda_{\xi\otimes v\otimes\epsilon\otimes c}(c')(\gamma) = \epsilon(\xi\wedge\gamma)\left(\int_K cc'\,dk\right)v \tag{6.29b}$$

passes to the chain/cochain level and then to the homology/cohomology level, yielding a $(\mathfrak{k}, K)$ isomorphism

$$\mathfrak{D}_* : \Pi'_n(V\otimes_{\mathbb{C}} (\Lambda^m\mathfrak{s})^*) \to \Gamma'^{m-n}(V). \tag{6.29c}$$

Proof. It is easy to see that ∂ commutes with the H action and hence descends to the chain level of (6.20). Also it is trivial that ∂ and d commute with the external K action coming from the action of r on C_K.

Also the map (6.29b) gives a linear isomorphism of $(\Lambda^n\mathfrak{s}\otimes_{\mathbb{C}} V\otimes (\Lambda^m\mathfrak{s})^*)\otimes_{\mathbb{C}} C_K$ onto $\mathrm{Hom}_{\mathbb{C}}(C_K, \mathrm{Hom}_{\mathbb{C}}(\Lambda^{m-n}\mathfrak{s}, V))_K$; to see that this map is onto, it is helpful to use Lemma 2.3 to see first that $C_K{}^c \cong C_K$. The linear isomorphism (6.29b) respects the H actions

$$h(\xi\otimes v\otimes\epsilon\otimes c) = h\xi\otimes hv\otimes h\epsilon\otimes \ell(h)c$$

$$(h\lambda)(c')(\gamma) = h(\lambda(h^{-1}c'))(\gamma) = h(\lambda(h^{-1}c')(h^{-1}\gamma))$$

because

$$\begin{aligned}
\lambda_{h(\xi\otimes v\otimes\epsilon\otimes c)}(c')(\gamma) &= \lambda_{h\xi\otimes hv\otimes h\epsilon\otimes\ell(h)c}(c')(\gamma) \\
&= (h\epsilon)(h\xi\wedge\gamma)\left(\int_K (\ell(h)c)c'\,dk\right)hv \\
&= \epsilon(\xi\wedge h^{-1}\gamma)\left(\int_K c\cdot(\ell(h)^{-1}c')\,dk\right)hv \\
&= h\left(\epsilon(\xi\wedge h^{-1}\gamma)\left(\int_K c\cdot(\ell(h)^{-1}c')\,dk\right)v\right) \\
&= h(\lambda_{\xi\otimes v\otimes\epsilon\otimes c}(h^{-1}c')(h^{-1}\gamma)) = (h\lambda_{\xi\otimes v\otimes\epsilon\otimes c})(c')(\gamma).
\end{aligned}$$

Restricted to the H invariants, our map then gives a linear isomorphism of (6.20) (with $V \otimes (\Lambda^m \mathfrak{s})^*$ in place) onto (6.25). Our map respects the external K actions

$$k(\xi \otimes v \otimes \epsilon \otimes c) = \xi \otimes v \otimes \epsilon \otimes r(k)c$$

$$k\lambda(c')(\gamma) = \lambda(r(k)^{-1}c')(\gamma)$$

by a similar computation, and hence the end product is going to be a $(\mathfrak{k}, K)$ map. Thus the lemma will follow if we show that

$$d\lambda_{\xi\otimes v\otimes\epsilon\otimes c} = (-1)^{|\xi|}\lambda_{\partial(\xi\otimes v\otimes\epsilon\otimes c)}, \tag{6.30}$$

where $|\xi|$ is the degree n of ξ.

Equation (6.30) has the initial appearance of (4.14) with V replaced by $C_K \otimes_{\mathbb{C}} V$. But actually (6.30) is obtained from such an identity only after an integration. Let us write out the details. To prove (6.30), we are to show that

$$d\lambda_{\xi\otimes v\otimes\epsilon\otimes c}(c')(\gamma) = (-1)^n \lambda_{\partial(\xi\otimes v\otimes\epsilon\otimes c)}(c')(\gamma) \tag{6.31}$$

for all c' in C_K and γ in $\Lambda^{m-n}\mathfrak{s}$. Introduce ξ_i, ξ_{ij}, γ_i, and γ_{ij} as is (4.20). From (6.26) and (6.29b), we find

$$d\lambda_{\xi\otimes v\otimes\epsilon\otimes c}(c')(\gamma)$$

$$= \sum_{i=0}^{m-n} (-1)^i Y_i(\lambda_{\xi\otimes v\otimes\epsilon\otimes c}(c')(\gamma_i)) + \sum_{i=0}^{m-n} (-1)^{i+1}\lambda_{\xi\otimes v\otimes\epsilon\otimes c}(\ell(Y_i)c')(\gamma_i)$$

$$+ \sum_{i<j} (-1)^{i+j}\lambda_{\xi\otimes v\otimes\epsilon\otimes c}(c')(P[Y_i, Y_j] \wedge \gamma_{ij})$$

$$= \sum_{i=0}^{m-n} (-1)^i \epsilon(\xi \wedge \gamma_i)\Big[\big(\int_K cc'\,dk\big) Y_i v - \big(\int_K c\cdot \ell(Y_i) c'\,dk\big) v\Big]$$

$$+ \sum_{i<j} (-1)^{i+j} \epsilon(\xi \wedge \rho[Y_i, Y_j] \wedge \gamma_{ij}) \big(\int_K cc'\,dk\big) v$$

$$= \sum_{i=0}^{m-n} (-1)^i \epsilon(\xi \wedge \gamma_i) \int_K Y_i (c \otimes v)\cdot c'\,dk$$

$$+ \sum_{i<j} (-1)^{i+j} \epsilon(\xi \wedge \rho[Y_i, Y_j] \wedge \gamma_{ij}) \int_K (c \otimes v)\cdot c'\,dk$$

$$= \int_K d\lambda_{\xi\otimes(c\otimes v)\otimes\epsilon}(\gamma)\cdot c'\,dk\ , \tag{6.32}$$

while from (6.21) and (6.29b) we find

$$\lambda_{\partial(\xi\otimes v\otimes\epsilon\otimes c)}(c')(\gamma)$$

$$= \sum_{r=1}^{n} (-1)^r \epsilon(\xi_r \wedge \gamma) \big(\int_K cc'\,dk\big) X_r v + \sum_{r=1}^{n} (-1)^r \epsilon(\xi_r \wedge \gamma) \big(\int_K \ell(X_r) c\cdot c'\,dk\big) v$$

$$+ \sum_{r<s} (-1)^{r+s} \epsilon(\rho[X_r, X_s] \wedge \xi_{rs} \wedge \gamma) \big(\int_K cc'\,dk\big) v$$

$$= \sum_{r=1}^{n} (-1)^r \epsilon(\xi_r \wedge \gamma) \int_K X_r (c \otimes v)\cdot c'\,dk$$

$$+ \sum_{r<s} (-1)^{r+s} \epsilon(\rho[X_r, X_s] \wedge \xi_{rs} \wedge \gamma) \int_K (c \otimes v)\cdot c'\,dk$$

$$= \int_K \lambda_{\partial(\xi\otimes(c\otimes v)\otimes\epsilon)}(\gamma)\cdot c'\,dk\ . \tag{6.33}$$

The equality of (6.32) with $(-1)^n$ times (6.33) follows from (4.14) by multiplying by c' and integrating over K. Thus (6.31) holds, and so does (6.30). This proves Lemma 6.5.

Now we have the ingredients to define the isomorphism in Theorem 6.1; it is the composition T_V of the maps

$$\Pi_n(V \otimes_{\mathbb{C}} (\wedge^m \mathfrak{s})^*) \to \Pi'_n(V \otimes_{\mathbb{C}} (\wedge^m \mathfrak{s})^*) \to \Gamma'^{m-n}(V) \to \Gamma^{m-n}(V) \qquad (6.34)$$

given in (6.23), (6.29), and (6.28). So far, these are $(\mathfrak{k}, K)$ isomorphisms, and we have not verified any naturality for them. We want to conclude that the composition T_V is a $(\mathfrak{g}, K)$ isomorphism.

7. Passage from $\mathfrak{k}$ to $\mathfrak{g}$

We retain the notation of §6. The abstract device for concluding that the composition T_V in (6.34) is a $(\mathfrak{g}, K)$ isomorphism is Proposition 7.1 below, which gives the idea of Enright and Wallach [8].

Proposition 7.1.[3] Let G and H be two covariant functors from $\mathcal{C}(\mathfrak{g}, H)$ to $\mathcal{C}(\mathfrak{g}, K)$ such that $\mathfrak{F}^{\mathfrak{k},K}_{\mathfrak{g},K} \circ G \cong \mathfrak{F}^{\mathfrak{k},K}_{\mathfrak{g},K} \circ H$, i.e., such that there exists a natural family $T_V : G(V) \to H(V)$ of $(\mathfrak{k}, K)$ isomorphisms for V in $\mathcal{C}(\mathfrak{g}, H)$. Then the maps T_V are $(\mathfrak{g}, K)$ isomorphisms if the following conditions are satisfied:

(a) For each V in $\mathcal{C}(\mathfrak{g}, H)$, there exists a $(\mathfrak{k}, K)$ map $\psi_V : \mathfrak{g} \otimes_{\mathbb{C}} G(V) \to G(\mathfrak{g} \otimes_{\mathbb{C}} V)$ such that

$$\begin{array}{ccc} \mathfrak{g} \otimes G(V) & \xrightarrow{\ \psi_V\ } & G(\mathfrak{g} \otimes V) \\ & {\scriptstyle m} \searrow \quad \swarrow {\scriptstyle G(m)} & \\ & G(V) & \end{array}$$

---

[3] This is a corrected version of Proposition 3.7 of [8].

is commutative. Here m denotes multiplication, which is a $(\mathfrak{g},K)$ map and a $(\mathfrak{g},H)$ map in the respective places it occurs in the diagram.

(b) For each V in $\mathcal{C}(\mathfrak{g},H)$, there exists a $(\mathfrak{k},K)$ map $\varphi_V : \mathfrak{g}\otimes_{\mathbb{C}} H(V) \to H(\mathfrak{g}\otimes_{\mathbb{C}} V)$ such that

$$\begin{array}{ccc} \mathfrak{g}\otimes H(V) & \xrightarrow{\ \varphi_V\ } & H(\mathfrak{g}\otimes V) \\ & {\scriptstyle m}\searrow \quad \swarrow{\scriptstyle H(m)} & \\ & H(V) & \end{array}$$

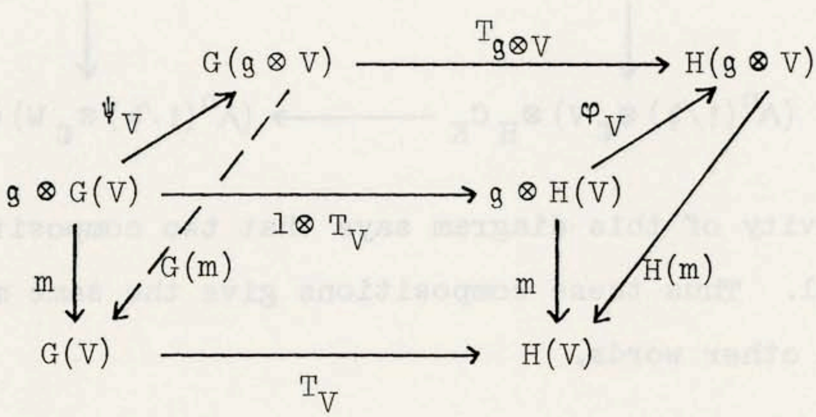
is commutative.

(c) For each V in $\mathcal{C}(\mathfrak{g},H)$, the maps ψ_V and φ_V of (a) and (b) make the diagram

$$\begin{array}{ccc} \mathfrak{g}\otimes G(V) & \xrightarrow{\ \psi_V\ } & G(\mathfrak{g}\otimes V) \\ {\scriptstyle 1\otimes T_V}\downarrow & & \downarrow{\scriptstyle T_{\mathfrak{g}\otimes V}} \\ \mathfrak{g}\otimes H(V) & \xrightarrow{\ \varphi_V\ } & H(\mathfrak{g}\otimes V) \end{array}$$

commutative.

Proof. We set up a 3-dimensional diagram in $\mathcal{C}(\mathfrak{k},K)$ that takes the form of a triangular prism



The triangles at the ends are commutative by (a) and (b), and the rectangle on top is commutative by (c). The rectangle in the back is commutative by naturality of T_V with respect to V. A little diagram chase shows therefore that the front rectangle is commutative, i.e., that T_V is a $\mathfrak{g}$ map.

We shall apply Proposition 7.1 with $G = \Pi_n((\cdot)\otimes_{\mathbb{C}}(\Lambda^m\mathfrak{s})^*)$, $H = \Gamma^{m-n}$, and T_V as in (6.34). The first step is to prove that T_V is natural. Thus let $\varphi : V \to W$ be a $(\mathfrak{g},H)$ map. We treat the factors of (6.34) separately. In treating Π, let us drop the tensoring with $(\Lambda^m\mathfrak{s})^*$ to simplify the notation. We construct two copies of (6.18), one for V in one layer and one for W in a second layer. Then we put in place maps between the layers in obvious fashion for the second and third rows. To get maps between the layers for the first rows, we use [6, p. 76]. Next we pass to the corresponding version of (6.22). We can consistently adjoin the bottom row to each layer with the obvious maps between layers. Then we discard the middle two rows from each layer, using composite maps between rows. The result is four parallel complexes, the n^{th} section of which is the commutative diagram

$$\begin{array}{ccc} V_n \otimes_{R(\mathfrak{g},H)} R(\mathfrak{g},K) & \longrightarrow & W_n \otimes_{R(\mathfrak{a},H)} R(\mathfrak{g},K) \\ \big\downarrow & & \big\downarrow \\ (\Lambda^n(\mathfrak{k}/\mathfrak{h}) \otimes_{\mathbb{C}} V) \otimes_H C_K & \longrightarrow & (\Lambda^n(\mathfrak{k}/\mathfrak{h}) \otimes_{\mathbb{C}} W) \otimes_H C_K \end{array}$$

The commutativity of this diagram says that two compositions of chain maps are equal. Thus these compositions give the same maps on homology. In other words,

$$\begin{array}{ccc} \Pi_n(V) & \longrightarrow & \Pi_n(W) \\ \downarrow & & \downarrow \\ \Pi'_n(V) & \longrightarrow & \Pi'_n(W) \end{array}$$

is commutative. This proves that (6.23) is natural. Similarly (6.28) is natural. Finally it is clear from the definitions that (6.29) is natural. Hence T_V is natural.

Next let us verify hypothesis (a). Again let us work with V in place of $V \otimes_{\mathbb{C}} (\wedge^m \mathfrak{s})^*$. We have

$$\mathfrak{g} \otimes \Pi(V_n) = \mathfrak{g} \otimes_{\mathbb{C}} (V_n \otimes_{R(\mathfrak{g},H)} R(\mathfrak{g},K)) \tag{7.1a}$$

and

$$\Pi(\mathfrak{g} \otimes V_n) = (\mathfrak{g} \otimes_{\mathbb{C}} V_n) \otimes_{R(\mathfrak{g},H)} R(\mathfrak{g},K) . \tag{7.1b}$$

Recall that we are using the alternate definition in §2 of the P functor to obtain these equations. If we were to reverse the factors in the tensor products over $R(\mathfrak{g},H)$ and map t to t^{tr} in $R(\mathfrak{g},K)$, then the two members of (7.1) would be respectively $\mathfrak{g} \otimes_{\mathbb{C}} P^{\mathfrak{g},K}_{\mathfrak{g},H}(V_n)$ and $P^{\mathfrak{g},K}_{\mathfrak{g},H}(\mathfrak{g} \otimes_{\mathbb{C}} V_n)$. Theorem 5.1 gives us a Mackey isomorphism μ_n from the second of these to the first, and we unwind to the setting of (7.1) as μ'_n. Then we set up the diagram

$$\begin{array}{ccc} (\mathfrak{g} \otimes_{\mathbb{C}} V_n) \otimes_{R(\mathfrak{g},H)} R(\mathfrak{g},K) & \xrightarrow{\mu'_n} & \mathfrak{g} \otimes_{\mathbb{C}} (V_n \otimes_{R(\mathfrak{g},H)} R(\mathfrak{g},K)) \\ & {\scriptstyle (m)\otimes 1} \searrow \quad \swarrow {\scriptstyle m} & \\ & V_n \otimes_{R(\mathfrak{g},H)} R(\mathfrak{g},K) & \end{array} \tag{7.2}$$

To see that this diagram commutes, it is enough to check the effect on $(X \otimes v) \otimes c$ in the top left, where X is in $\mathfrak{g}$, v is in V_n, and c is in C_K. From (5.1b) we have

$$\mu_n(c\otimes(X\otimes v)) = \sum_i X_i\otimes((\mathrm{Ad}(\cdot)X,X_i)c(\cdot)\otimes v)\,.$$

Thus

$$\mu_n'((X\otimes v)\otimes c) = \sum_i X_i\otimes(v\otimes\{(\mathrm{Ad}(\cdot)X,X_i)c^{\sim}(\cdot)\}^{\sim})$$

$$= \sum_i X_i\otimes(v\otimes(\mathrm{Ad}(\cdot)^{-1}X,X_i)c(\cdot))\,.$$

Remembering that $\mathfrak{g}$ acts through R on $R(\mathfrak{g},K)$ in the tensor products over $R(\mathfrak{g},H)$, we have

$$m\mu_n'((X\otimes v)\otimes c) = v\otimes\sum R(X_i)\{(\mathrm{Ad}(\cdot)^{-1}X,X_i)c(\cdot)\}$$

$$= v\otimes\sum(-(\mathrm{Ad}(\cdot)^{-1}X,X_i)c(\cdot)\otimes X_i)$$

$$= v\otimes(-X\otimes c)$$

$$= Xv\otimes c \quad \text{through } \otimes_{R(\mathfrak{g},H)}$$

$$= ((m)\otimes 1)((X\otimes v)\otimes c)\,.$$

Thus (7.2) commutes.

To pass to homology, we need to verify that μ_n', $(m)\otimes 1$, and m in (7.2) are chain maps. For μ_n' the relevant diagram is

$$\begin{array}{ccc}
(\mathfrak{g}\otimes_{\mathbb{C}} V_n)\otimes_{R(\mathfrak{g},H)} R(\mathfrak{g},K) & \xrightarrow{1\otimes\partial\otimes 1} & (\mathfrak{g}\otimes_{\mathbb{C}} V_{n-1})\otimes_{R(\mathfrak{g},H)} R(\mathfrak{g},K) \\
\downarrow \mu_n' & & \downarrow \mu_{n-1}' \\
\mathfrak{g}\otimes_{\mathbb{C}}(V_n\otimes_{R(\mathfrak{g},H)} R(\mathfrak{g},K)) & \xrightarrow{1\otimes\partial\otimes 1} & \mathfrak{g}\otimes_{\mathbb{C}}(V_{n-1}\otimes_{R(\mathfrak{g},H)} R(\mathfrak{g},K))
\end{array}$$

and the argument is

$$\mu_{n-1}'(1\otimes\partial\otimes 1)(X\otimes v\otimes c) = \mu_{n-1}'(X\otimes\partial v\otimes c)$$

$$= \sum X_i\otimes\partial v\otimes(\mathrm{Ad}(\cdot)^{-1}X,X_i)c(\cdot) = (1\otimes\partial\otimes 1)\mu_{n-1}'(X\otimes v\otimes c)\,.$$

For $(m)\otimes 1$ the relevant diagram is

$$\begin{array}{ccc}
(\mathfrak{g}\otimes_{\mathbb{C}} V_n)\otimes_{R(\mathfrak{g},H)} R(\mathfrak{g},K) & \xrightarrow{1\otimes\partial\otimes 1} & (\mathfrak{g}\otimes_{\mathbb{C}} V_{n-1})\otimes_{R(\mathfrak{g},H)} R(\mathfrak{g},K) \\
\Big\downarrow (m)\otimes 1 & & \Big\downarrow (m)\otimes 1 \\
V_n\otimes_{R(\mathfrak{g},H)} R(\mathfrak{g},K) & \xrightarrow{\partial\otimes 1} & V_{n-1}\otimes_{R(\mathfrak{g},H)} R(\mathfrak{g},K)
\end{array}$$

and the argument is

$$\begin{aligned}
((m)\otimes 1)(1\otimes\partial\otimes 1)(X\otimes v\otimes c) &= ((m)\otimes 1)(X\otimes\partial v\otimes c) \\
&= X(\partial v)\otimes c = \partial(Xv)\otimes c \quad \text{since } \partial \text{ is a } \mathfrak{g} \text{ map} \\
&= (\partial\otimes 1)(Xv\otimes c) = (\partial\otimes 1)((m)\otimes 1)(X\otimes v\otimes c)\,.
\end{aligned}$$

For m the relevant diagram is

$$\begin{array}{ccc}
\mathfrak{g}\otimes_{\mathbb{C}} (V_n\otimes_{R(\mathfrak{g},H)} R(\mathfrak{g},K)) & \xrightarrow{1\otimes\partial\otimes 1} & \mathfrak{g}\otimes_{\mathbb{C}} (V_{n-1}\otimes_{R(\mathfrak{g},H)} R(\mathfrak{g},K)) \\
\Big\downarrow m & & \Big\downarrow m \\
V_n\otimes_{R(\mathfrak{g},H)} R(\mathfrak{g},K) & \xrightarrow{\partial\otimes 1} & V_{n-1}\otimes_{R(\mathfrak{g},H)} R(\mathfrak{g},K)
\end{array}$$

and the argument is

$$\begin{aligned}
m(1\otimes\partial\otimes 1)(X\otimes v\otimes c) &= m(X\otimes\partial v\otimes c) \\
&= -\partial v\otimes (c\otimes X) = -(\partial\otimes 1)(v\otimes (c\otimes X)) \\
&= (\partial\otimes 1)(1\otimes R(X))(v\otimes c) = (\partial\otimes 1)m(X\otimes v\otimes c)\,.
\end{aligned}$$

Thus (7.2) passes to homology, and (a) in Proposition 7.1 is verified with $\psi_{V\otimes\wedge^m\mathfrak{g}} = (\mu_n')_*^{-1}$.

The verification of (b) is similar except that Theorem 5.2 is involved. We have

$$\mathfrak{g} \otimes \Gamma(V_n) = \mathfrak{g} \otimes_{\mathbb{C}} \mathrm{Hom}_{R(\mathfrak{g},H)}(R(\mathfrak{g},K),V_n)_K = \mathfrak{g} \otimes_{\mathbb{C}} I^{\mathfrak{g},K}_{\mathfrak{g},H}(V_n)$$

and

$$\Gamma(\mathfrak{g} \otimes V_n) = \mathrm{Hom}_{R(\mathfrak{g},H)}(R(\mathfrak{g},K),\mathfrak{g} \otimes_{\mathbb{C}} V_n)_K = I^{\mathfrak{g},K}_{\mathfrak{g},H}(\mathfrak{g} \otimes_{\mathbb{C}} V_n) .$$

Since $\mathfrak{g}$ is finite-dimensional, we can apply Corollary 5.3 to obtain a Mackey isomorphism σ_n from the first line to the second line. Then we set up the diagram

$$\begin{array}{ccc} \mathfrak{g} \otimes_{\mathbb{C}} \mathrm{Hom}_{R(\mathfrak{g},H)}(R(\mathfrak{g},K),V_n)_K & \xrightarrow{\ \sigma_n\ } & \mathrm{Hom}_{R(\mathfrak{g},H)}(R(\mathfrak{g},K),\mathfrak{g} \otimes_{\mathbb{C}} V_n)_K \\ & {\scriptstyle m} \searrow \qquad \swarrow {\scriptstyle \mathrm{Hom}(1,m)} & \\ & \mathrm{Hom}_{R(\mathfrak{g},H)}(R(\mathfrak{g},K),V_n)_K & \end{array} \tag{7.3}$$

To see that this diagram commutes, we use (5.9) to write

$$\sigma_n(X \otimes \varphi)(c) = \sum X_i \otimes \varphi((\mathrm{Ad}(\cdot)X,X_i)c(\cdot))$$

for X in $\mathfrak{g}$, φ in $\mathrm{Hom}_{R(\mathfrak{g},H)}(R(\mathfrak{g},K),V_n)_K$, and c in C_K. Then

$$\begin{aligned} (\mathrm{Hom}(1,m)\sigma_n)(X \otimes \varphi)(c) &= m(\sigma_n(X \otimes \varphi)(c)) \\ &= m(\sum X_i \otimes \varphi((\mathrm{Ad}(\cdot)X,X_i)c(\cdot))) \\ &= \sum X_i (\varphi((\mathrm{Ad}(\cdot)X,X_i)c(\cdot))) \\ &= \sum \varphi(X_i \otimes (\mathrm{Ad}(\cdot)X,X_i)c(\cdot)) \quad \text{since } \varphi \text{ is a } \mathfrak{g} \text{ map} \\ &= \varphi(c \otimes X) = (X\varphi)(c) = m(X \otimes \varphi)(c) . \end{aligned}$$

Thus (7.3) commutes. To pass to cohomology, we need to verify that σ_n, $\mathrm{Hom}(1,m)$, and m are chain maps. We omit these details, which are similar to the ones in the proof of (a). The conclusion is that (b) holds in Proposition 7.1 with $\varphi_V = (\sigma_n)^*$.

We turn to the verification of (c). First let us notice that the construction of ψ_V and φ_V above would have made sense with any finite-dimensional $(\mathfrak{g}, K)$ module F used in place of $\mathfrak{g}$; the Mackey isomorphisms would still apply. Let the action on F be called π. We set up the diagram

$$\begin{array}{ccccccc} F\otimes_{\mathbb{C}}\Pi_n(V\otimes_{\mathbb{C}}(\wedge^m\mathfrak{s})^*) & \longrightarrow & F\otimes_{\mathbb{C}}\Pi'_n(V\otimes_{\mathbb{C}}(\wedge^m\mathfrak{s})^*) & \longrightarrow & F\otimes_{\mathbb{C}}\Gamma'^{m-n}(V) & \longrightarrow & F\otimes_{\mathbb{C}}\Gamma^{m-n}(V) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \Pi_n(F\otimes_{\mathbb{C}}V\otimes_{\mathbb{C}}(\wedge^m\mathfrak{s})^*) & \longrightarrow & \Pi'_n(F\otimes_{\mathbb{C}}V\otimes_{\mathbb{C}}(\wedge^m\mathfrak{s})^*) & \longrightarrow & \Gamma'^{m-n}(F\otimes_{\mathbb{C}}V) & \longrightarrow & \Gamma^{m-n}(F\otimes_{\mathbb{C}}V) \end{array} \tag{7.4}$$

The maps of the bottom row are those of (6.34), the maps of the top row are 1 tensored with the maps of (6.34), and the vertical maps are obtained from Mackey isomorphisms or their inverses. To prove (c), it is enough to prove that each constituent square of (7.4) commutes.

The easy square is the middle one, and we handle that first. We set up the diagram

$$\begin{array}{ccc} F\otimes_{\mathbb{C}}((\wedge^n\mathfrak{s}\otimes_{\mathbb{C}}V\otimes_{\mathbb{C}}(\wedge^m\mathfrak{s})^*)\otimes_H C_K) & \xrightarrow{1\otimes\mathcal{S}_V} & F\otimes_{\mathbb{C}}\mathrm{Hom}_H(C_K,\mathrm{Hom}_{\mathbb{C}}(\wedge^{m-n}\mathfrak{s},V))_K \\ \Big\downarrow \text{Mackey}'^{-1} & & \Big\downarrow \text{Mackey} \\ (\wedge^n\mathfrak{s}\otimes_{\mathbb{C}}F\otimes_{\mathbb{C}}V\otimes_{\mathbb{C}}(\wedge^m\mathfrak{s})^*)\otimes_H C_K & \xrightarrow{\mathcal{S}_{F\otimes V}} & \mathrm{Hom}_H(C_K,\mathrm{Hom}_{\mathbb{C}}(\wedge^{m-n}\mathfrak{s},F\otimes_{\mathbb{C}}V))_K \end{array} \tag{7.5}$$

The top and bottom maps come from (6.29). The map on the left is the inverse of the Mackey isomorphism associated to $P_{\mathfrak{h},H}^{\mathfrak{k},K}$, except that it has to be reinterpreted to take into account the reversal of the order of tensor products over H (cf. μ_n vs. μ'_n in (7.2)). The map on the right is the Mackey isomorphism (5.9) for $I_{\mathfrak{h},H}^{\mathfrak{k},K}$.

Let us see that (7.5) commutes. On the one hand, (5.9) gives

$$\{\text{Mackey}\circ(1\otimes \mathfrak{d}_V)(f\otimes\xi\otimes v\otimes\epsilon\otimes c)\}(c') = \text{Mackey}(f\otimes\lambda_{\xi\otimes v\otimes\epsilon\otimes c})(c')$$

$$= \sum f_i \otimes \lambda_{\xi\otimes v\otimes\epsilon\otimes c}((\pi(\cdot)f,f_i)c'(\cdot)) . \quad (7.6)$$

On the other hand, the operator on the left is to be computed from the inverse formula (5.6), except that the $\pi(\cdot)^{-1}$ gets replaced by $\pi(\cdot)$ because of the reversal of order of tensor products. So

$$\{\mathfrak{d}_{F\otimes V}\circ\text{Mackey}'^{-1}(f\otimes\xi\otimes v\otimes\epsilon\otimes c)\}(c')$$

$$= \mathfrak{d}_{F\otimes V}(\sum \xi\otimes f_i\otimes v\otimes\epsilon\otimes(\pi(\cdot)f,f_i)c(\cdot))(c')$$

$$= \sum\lambda_{\xi\otimes f_i\otimes v\otimes\epsilon\otimes(\pi(\cdot)f,f_i)c(\cdot)}(c') . \quad (7.7)$$

Evaluation of (7.6) and (7.7) on γ by means of (6.29) gives the same result in the two cases. Thus (7.5) commutes.

Now let us check that the four maps in (7.5) are chain maps. For the horizontal maps, this is the main content of Lemma 6.5. For the left map, we use (6.21). We have

$$\text{Mackey}'^{-1}\circ(1\otimes\partial)(f\otimes\xi\otimes v\otimes\epsilon\otimes c)$$

$$= \text{Mackey}'^{-1}\Big\{\sum_{i=1}^{n}(-1)^i(Y_1\wedge\ldots\wedge\hat{Y}_i\wedge\ldots\wedge Y_n\otimes f\otimes Y_i v\otimes\epsilon\otimes c)$$

$$+\sum_{i=1}^{n}(-1)^i(Y_1\wedge\ldots\wedge\hat{Y}_i\wedge\ldots\wedge Y_n\otimes f\otimes v\otimes\epsilon\otimes\ell(Y_i)c)$$

$$+\sum_{r<s}(-1)^{r+s}(\mathfrak{p}[Y_r,Y_s]\wedge Y_1\wedge\ldots\wedge\hat{Y}_r\wedge\ldots\wedge\hat{Y}_s\wedge\ldots\wedge Y_n\otimes f\otimes v\otimes\epsilon\otimes c)\Big\}$$

$$= \sum_{i=1}^{n}(-1)^i\sum_j (Y_1\wedge\ldots\wedge\hat{Y}_i\wedge\ldots\wedge Y_n\otimes f_j\otimes Y_i v\otimes\epsilon\otimes(\pi(\cdot)f,f_j)c(\cdot))$$

$$+\sum_{i=1}^{n}(-1)^i\sum_j (Y_1\wedge\ldots\wedge\hat{Y}_i\wedge\ldots\wedge Y_n\otimes f_j\otimes v\otimes\epsilon\otimes(\pi(\cdot)f,f_j)\ell(Y_i)c(\cdot))$$

$$+\sum_{r<s}(-1)^{r+s}\sum_j (\rho[Y_r,Y_s]\wedge Y_1\wedge\cdots\wedge\hat{Y}_r\wedge\cdots\wedge\hat{Y}_s\wedge\cdots\wedge Y_n \otimes f_j\otimes v\otimes\epsilon\otimes(\pi(\cdot)f,f_j)c(\cdot)) \qquad (7.8)$$

and

$$\partial\cdot\mathrm{Mackey}'^{-1}(f\otimes\xi\otimes v\otimes\epsilon\otimes c)$$

$$= \partial(\Sigma\ \xi\otimes f_j\otimes v\otimes\epsilon\otimes(\pi(\cdot)f,f_j)c(\cdot))$$

$$= \sum_{i=1}^{n}(-1)^i\sum_j (Y_1\wedge\cdots\wedge\hat{Y}_i\wedge\cdots\wedge Y_n\otimes Y_i(f_j\otimes v)\otimes\epsilon\otimes(\pi(\cdot)f,f_j)c(\cdot))$$

$$+\sum_{i=1}^{n}(-1)^i\sum_j (Y_1\wedge\cdots\wedge\hat{Y}_i\wedge\cdots\wedge Y_n\otimes(f_j\otimes v)\otimes\epsilon\otimes\ell(Y_i)\{(\pi(\cdot)f,f_j)c(\cdot)\})$$

$$+\sum_{r<s}(-1)^{r+s}\sum_j (\rho[Y_r,Y_s]\wedge Y_1\wedge\cdots\wedge\hat{Y}_r\wedge\cdots\wedge\hat{Y}_s\wedge\cdots\wedge Y_n \otimes f_j\otimes v\otimes\epsilon\otimes(\pi(\cdot)f,f_j)c(\cdot))\,. \qquad (7.9)$$

The difference (7.9) less (7.8) comes from $\pi(Y_i)f_j\otimes v$ in the first term and $\ell(Y_i)(\pi(\cdot)f,f_j)\cdot c(\cdot)$ in the second term. For fixed i, if $\xi_i = Y_1\wedge\cdots\wedge\hat{Y}_i\wedge\cdots\wedge Y_n$, this difference is $(-1)^i$ times the quantity

$$\sum_j (\xi_i\otimes\pi(Y_i)f_j\otimes v\otimes\epsilon\otimes(\pi(\cdot)f,f_j)c(\cdot))$$

$$+\sum_j (\xi_i\otimes f_j\otimes v\otimes\epsilon\otimes(-\pi(Y_i)\pi(\cdot)f,f_j)c(\cdot))$$

$$= \sum_{j,k}(\xi_i\otimes f_k\otimes v\otimes\epsilon\otimes(\pi(\cdot)f,f_j)(\pi(Y_i)f_j,f_k)c(\cdot))$$

$$+\sum_{j,k}(\xi_i\otimes f_j\otimes v\otimes\epsilon\otimes(\pi(\cdot)f,f_k)(-\pi(Y_i)f_k,f_j)c(\cdot)) = 0\,.$$

Thus the left map in (7.5) is a chain map, and similarly the right map is a chain map. Therefore the commutativity of (7.5) passes to homology/cohomology, and the center square of (7.4) commutes.

Now we consider the left square. This is the hard step. Again let us work with V in place of $V \otimes_{\mathbb{C}} (\Lambda^m \mathfrak{s})^*$. We set up the diagram

$$\begin{array}{ccc} F \otimes_{\mathbb{C}} (V_n \otimes_{R(\mathfrak{g},H)} R(\mathfrak{g},K)) & \xrightarrow{1 \otimes \rho_{n,V}} & F \otimes_{\mathbb{C}} ((\Lambda^n \mathfrak{s} \otimes_{\mathbb{C}} V) \otimes_H C_K) \\ \uparrow \mu_n' & & \uparrow \mu_n'' \\ (F \otimes_{\mathbb{C}} V_n) \otimes_{R(\mathfrak{g},H)} R(\mathfrak{g},K) & \xrightarrow{\rho_{n,F\otimes V}} & (\Lambda^n \mathfrak{s} \otimes_{\mathbb{C}} F \otimes_{\mathbb{C}} V) \otimes_H C_K \end{array} \qquad (7.10)$$

in which μ_n' is the Mackey isomorphism of (7.2) (involving Theorem 5.1 and $P^{\mathfrak{g},K}_{\mathfrak{g},H}$), μ_n'' is the isomorphism "Mackey' " in (7.5) (involving Theorem 5.1 and $P^{\mathfrak{k},K}_{\mathfrak{h},H}$), and $\rho_{n,V}$ is the composite vertical map in (6.22). We know from above that μ_n'' is a chain map, from the verification of (a) that μ_n' is a chain map, and from (6.22) that the horizontal maps are chain maps. We are to prove that (7.10) becomes commutative upon passage to homology.

The difficulty is that (7.10) need not be commutative as it stands. We therefore subdivide the problem into two parts——one where a homotopy argument works and one where an exact computation works——by interpolating a middle vertical corresponding to the second row of (6.22). The new diagram is

$$\begin{array}{ccccc} F \otimes_{\mathbb{C}} (V_n \otimes_{R(\mathfrak{g},H)} R(\mathfrak{g},K)) & \xrightarrow{1 \otimes f'_{n,V}} & F \otimes_{\mathbb{C}} (V \otimes_{R(\mathfrak{g},H)} R_n) & \xrightarrow{1 \otimes f''_{n,V}(1 \otimes f_n)} & F \otimes_{\mathbb{C}} ((\Lambda^n \mathfrak{s} \otimes_{\mathbb{C}} V) \otimes_H C_K) \\ \uparrow \mu_n' & & \uparrow \mu_n''' & & \uparrow \mu_n'' \\ (F \otimes_{\mathbb{C}} V_n) \otimes_{R(\mathfrak{g},H)} R(\mathfrak{g},K) & \xrightarrow{f'_{n,F\otimes V}} & (F \otimes_{\mathbb{C}} V) \otimes_{R(\mathfrak{g},H)} R_n & \xrightarrow{f''_{n,F\otimes V}(1 \otimes f_n)} & (\Lambda^n \mathfrak{s} \otimes_{\mathbb{C}} F \otimes_{\mathbb{C}} V) \otimes_H C_K \end{array}$$

(7.11)

Note that three of the four maps in the right square are $(\mathfrak{k},K)$ isomorphisms and chain maps, and thus we can define μ_n''' by them. Then μ_n''' is a $(\mathfrak{k},K)$ isomorphism and a chain map, and the right square commutes. Since $R_n = R(\mathfrak{g},H) \otimes_{R(\mathfrak{k},H)} (C_K \otimes_{\mathbb{C}} X_n)$, the domain of μ_n''' is

$$(F \otimes_{\mathbb{C}} V) \otimes_{R(\mathfrak{g},H)} (R(\mathfrak{g},H) \otimes_{R(\mathfrak{k},H)} (C_K \otimes_{\mathbb{C}} X_n)) .$$

In principle we could drop the $R(\mathfrak{g},H)$ in two places here, but it is convenient not to do so. But it is enough to use domain elements whose $R(\mathfrak{g},H)$ coordinate is in C_H. Then direct calculation shows that the following formula gives a well defined linear mapping for which the right square commutes (so that it must be the correct formula):

$$\begin{aligned} &\mu_n'''(f \otimes v \otimes c_H \otimes c_K \otimes x) \\ &\quad = \sum_{i,m} f_m \otimes v \otimes (\pi(\cdot_H)^{-1} f, f_i) c_H(\cdot) \otimes (\pi(\cdot_K)^{-1} f_i, f_m) c_K(\cdot) \otimes x . \end{aligned} \tag{7.12}$$

With this formula for μ_n''', we are to prove that the left square of (7.11) commutes on the homology level.

We shall construct a corresponding square on the resolution level. We use superscripts $\mathfrak{g}$ and K on the appropriate sides to indicate which modules are contributing to the $(\mathfrak{g},H)$ and K actions, respectively:

$$\begin{array}{ccc} {}^K F \otimes_{\mathbb{C}} ({}^{\mathfrak{g}} V_n \otimes_{\mathbb{C}} {}^{\mathfrak{g}} R(\mathfrak{g},K)^K) & \xrightarrow{1 \otimes f'_{n,V}} & {}^K F \otimes_{\mathbb{C}} ({}^{\mathfrak{g}} V \otimes_{\mathbb{C}} ({}^{\mathfrak{g}} R(\mathfrak{g},H) \otimes_{R(\mathfrak{k},H)} (C_K^K \otimes_{\mathbb{C}} X_n))) \\ \Big\uparrow \sigma_n' & & \Big\uparrow \sigma_n''' \\ ({}^{\mathfrak{g}} F \otimes_{\mathbb{C}} {}^{\mathfrak{g}} V_n) \otimes_{\mathbb{C}} {}^{\mathfrak{g}} R(\mathfrak{g},K)^K & \xrightarrow{f'_{n,F\otimes V}} & ({}^{\mathfrak{g}} F \otimes_{\mathbb{C}} {}^{\mathfrak{g}} V) \otimes_{\mathbb{C}} ({}^{\mathfrak{g}} R(\mathfrak{g},H) \otimes_{R(\mathfrak{k},H)} (C_K^K \otimes_{\mathbb{C}} X_n)) \end{array}$$

(7.13)

Here $f'_{n,F\otimes V}$ and $f'_{n,V}$ are our original mappings from (6.18). We shall construct the maps σ'_n and σ'''_n as isomorphisms that respect the indicated $(\mathfrak{g},H)$ and K actions. For σ'_n we write

$$({}^{\mathfrak{g}}F\otimes_{\mathbb{C}}{}^{\mathfrak{g}}V_n)\otimes_{\mathbb{C}}{}^{\mathfrak{g}}R(\mathfrak{g},K)^K \cong ({}^{\mathfrak{g}}F\otimes_{\mathbb{C}}{}^{\mathfrak{g}}V_n)\otimes_{\mathbb{C}}({}^{\mathfrak{g}}R(\mathfrak{g},H)\otimes_{R(\mathfrak{k},H)}C_K^K) \quad \text{by (2.15)}$$

$$\cong {}^{\mathfrak{g}}V_n\otimes_{\mathbb{C}}({}^{\mathfrak{g}}R(\mathfrak{g},H)\otimes_{R(\mathfrak{k},H)}(F\otimes_{\mathbb{C}}C_K^K)) \quad \text{by (5.1) for } P_{\mathfrak{k},H}^{\mathfrak{g},H} \text{ in inverse direction}$$

$$\cong {}^{\mathfrak{g}}V_n\otimes_{\mathbb{C}}((F\otimes_{\mathbb{C}}{}^{\mathfrak{g}}R(\mathfrak{g},H))\otimes_{R(\mathfrak{k},H)}C_K^K) \quad \text{by (4.1)}$$

$$\cong {}^{K}F\otimes_{\mathbb{C}}({}^{\mathfrak{g}}V_n\otimes_{\mathbb{C}}({}^{\mathfrak{g}}R(\mathfrak{g},H)\otimes_{R(\mathfrak{k},H)}C_K^K)) \quad \text{by (5.1) for } P_{\mathfrak{k},H}^{\mathfrak{k},K} \text{ in reverse order}$$

$$\cong {}^{K}F\otimes_{\mathbb{C}}({}^{\mathfrak{g}}V_n\otimes_{\mathbb{C}}{}^{\mathfrak{g}}R(\mathfrak{g},K)^K) \quad \text{by (2.15).}$$

In detail, σ'_n is given by

$$f\otimes v\otimes(c_H *_H c_K) \to f\otimes v\otimes c_H\otimes c_K$$

$$\to \sum_i v\otimes(\pi(\cdot_H)^{-1}f,f_i)c_H(\cdot)\otimes f_i\otimes c_K$$

$$\to \sum_i v\otimes f_i\otimes(\pi(\cdot_H)^{-1}f,f_i)c_H(\cdot)\otimes c_K$$

$$\to \sum_{i,m} f_m\otimes v\otimes(\pi(\cdot_H)^{-1}f,f_i)c_H(\cdot)\otimes(\pi(\cdot_K)^{-1}f_i,f_m)c_K(\cdot)$$

$$\to \sum_{i,m} f_m\otimes v\otimes(\pi(\cdot_H)^{-1}f,f_i)c_H(\cdot) *_H (\pi(\cdot_K)^{-1}f_i,f_m)c_K(\cdot).$$

Let us compute this last convolution:

$$\sum_i(\pi(\cdot_H)^{-1}f,f_i)c_H(\cdot) *_H (\pi(\cdot_K)^{-1}f_i,f_m)c_K(\cdot)$$

$$= \sum_i\int_H(\pi(h)^{-1}f,f_i)c_H(h)(\pi(h^{-1}\cdot)^{-1}f_i,f_m)c_K(h^{-1}\cdot)\,dh$$

$$= \int_H\sum_i(f,\pi(h)f_i)\overline{(\pi(\cdot)f_m,\pi(h)f_i)}c_H(h)c_K(h^{-1}\cdot)\,dh$$

$$= \int_H (f,\pi(\cdot)f_m)c_H(h)c_K(h^{-1}\cdot)\,dh \quad \text{since } \{\pi(h)f_i\} \text{ is orthonormal}$$

$$= (\pi(\cdot)^{-1}f,f_m)c_H *_H c_K(\cdot)\ .$$

Thus σ_n' is a globally defined isomorphism such that

$$\sigma_n'(f\otimes v\otimes c_K) = \sum_m f_m\otimes v\otimes(\pi(\cdot)^{-1}f,f_m)c_K(\cdot)\ . \tag{7.14}$$

For σ_n''' we write

$$({}^{\mathfrak g}F\otimes_{\mathbb C}{}^{\mathfrak g}V)\otimes_{\mathbb C}({}^{\mathfrak g}R(\mathfrak g,H)\otimes_{R(\mathfrak k,H)}(C_K^K\otimes_{\mathbb C}X_n))$$

$$\cong {}^{\mathfrak g}V\otimes_{\mathbb C}({}^{\mathfrak g}R(\mathfrak g,H)\otimes_{R(\mathfrak k,H)}(F\otimes_{\mathbb C}C_K^K\otimes_{\mathbb C}X_n)) \quad \text{by (5.1) for } P^{\mathfrak g,H}_{\mathfrak k,H} \text{ in inverse direction}$$

$$\cong {}^{\mathfrak g}V\otimes_{\mathbb C}((F\otimes_{\mathbb C}X_n\otimes_{\mathbb C}{}^{\mathfrak g}R(\mathfrak g,H))\otimes_{R(\mathfrak k,H)}C_K^K) \quad \text{by (4.1)}$$

$$\cong {}^KF\otimes_{\mathbb C}({}^{\mathfrak g}V\otimes_{\mathbb C}((X_n\otimes_{\mathbb C}{}^{\mathfrak g}R(\mathfrak g,H))\otimes_{R(\mathfrak k,H)}C_K^K)) \quad \text{by (5.1) for } P^{\mathfrak k,K}_{\mathfrak k,H} \text{ in reverse order}$$

$$\cong {}^KF\otimes_{\mathbb C}({}^{\mathfrak g}V\otimes_{\mathbb C}({}^{\mathfrak g}R(\mathfrak g,H)\otimes_{R(\mathfrak k,H)}(C_K^K\otimes_{\mathbb C}X_n))) \quad \text{by (4.1)}.$$

In detail, σ_n''' is given by

$$f\otimes v\otimes c_H\otimes c_K\otimes x$$

$$\to \sum_i v\otimes(\pi(\cdot_H)^{-1}f,f_i)c_H(\cdot)\otimes f_i\otimes c_K\otimes x$$

$$\to \sum_i v\otimes f_i\otimes x\otimes(\pi(\cdot_H)^{-1}f,f_i)c_H(\cdot)\otimes c_K$$

$$\to \sum_{i,m} f_m\otimes v\otimes x\otimes(\pi(\cdot_H)^{-1}f,f_i)c_H(\cdot)\otimes(\pi(\cdot_K)^{-1}f_i,f_m)c_K(\cdot)$$

$$\to \sum_{i,m} f_m\otimes v\otimes(\pi(\cdot_H)^{-1}f,f_i)c_H(\cdot)\otimes(\pi(\cdot_K)^{-1}f_i,f_m)c_K(\cdot)\otimes x\ .$$

Thus σ_n''' is a globally defined isomorphism such that

$$\sigma_n'''(f\otimes v\otimes c_H\otimes c_K\otimes x) = \sum_{i,m} f_m\otimes v\otimes (\pi(\cdot_H)^{-1}f, f_i)c_H(\cdot)\otimes (\pi(\cdot_K)^{-1}f_i, f_m)c_K(\cdot)\otimes x \tag{7.15}$$

Next we observe that the four mappings in (7.13) are chain maps. In fact, the horizontal maps are chain maps by construction. In the case of σ_n', the boundary operator operates just in the V_n factor, and (7.14) shows this is unaffected by σ_n'. In the case of σ_n''', the boundary operator operates just in the X_n factor, and (7.15) shows this is unaffected by σ_n''. These chain maps lie over a diagram

$$\begin{array}{ccc} {}^K F\otimes_{\mathbb{C}} ({}^{\mathfrak{g}}V\otimes_{\mathbb{C}} {}^{\mathfrak{g}}R(\mathfrak{g},K)^K) & \xrightarrow{(2.15)} & {}^K F\otimes_{\mathbb{C}} ({}^{\mathfrak{g}}V\otimes_{\mathbb{C}} ({}^{\mathfrak{g}}R(\mathfrak{g},H)\otimes_{R(\mathfrak{k},H)} C_K^K)) \\ \uparrow \sigma' & & \uparrow \sigma''' \\ ({}^{\mathfrak{g}}F\otimes_{\mathbb{C}} {}^{\mathfrak{g}}V)\otimes_{\mathbb{C}} {}^{\mathfrak{g}}R(\mathfrak{g},K)^K & \xrightarrow{(2.15)} & ({}^{\mathfrak{g}}F\otimes_{\mathbb{C}} {}^{\mathfrak{g}}V)\otimes_{\mathbb{C}} ({}^{\mathfrak{g}}R(\mathfrak{g},H)\otimes_{R(\mathfrak{k},H)} C_K^K) \end{array} \tag{7.16}$$

in which σ' is determined by (7.14) and σ''' is determined by (7.15) with the x eliminated. The map (2.15) is achieved by convolving the c_H and c_K coordinates. Under this identification, σ''' agrees with σ', by the computation of convolution that precedes (7.14). In other words (7.16) commutes. Since (7.13) involves terms in projective resolutions, (7.13) commutes up to homotopy.

Now we apply the functor ${}^{\mathfrak{g}}(\cdot) \to (\cdot)\otimes_{R(\mathfrak{g},H)} \mathbb{C}$ to the diagram (7.13). Using (4.1) to eliminate the terms $\mathbb{C}$, we obtain

$$\begin{array}{ccc} {}^K F\otimes_{\mathbb{C}} (V_n\otimes_{R(\mathfrak{g},H)} R(\mathfrak{g},K)^K) & \xrightarrow{1\otimes f'_{n,V}} & {}^K F\otimes_{\mathbb{C}} (V\otimes_{R(\mathfrak{g},H)} R_n^K) \\ \uparrow & & \uparrow \\ (F\otimes_{\mathbb{C}} V_n)\otimes_{R(\mathfrak{g},H)} R(\mathfrak{g},K)^K & \xrightarrow{f'_{n,F\otimes V}} & (F\otimes_{\mathbb{C}} V)\otimes_{R(\mathfrak{g},H)} R_n^K \end{array} \tag{7.17}$$

The vertical maps here are still given notationally by (7.14) and (7.15). Theorem 5.1 shows that the left map is μ_n', and (7.12) shows that the right map is μ_n'''. Thus (7.17) coincides with the left square of (7.11). As the image of (7.13), which commutes up to homotopy, the left square of (7.11) commutes up to homotopy.

Therefore (7.11) passes to homology as a commutative diagram, and so does (7.10). This proves that the left square of (7.4) commutes, and a completely parallel argument (making repeated use of Corollary 5.3) proves that the right square of (7.4) commutes. Thus (c) holds in Proposition 7.1, and the proof of Theorem 6.1 is complete.

8. Concluding remarks

a. Work of Enright and Wallach

As we mentioned in the introduction, [8] contains a number of minor errors and one serious gap.

The minor errors are the systematic equating of $\mathrm{Hom}_{\mathbb{C}}(U \otimes V, W)$ with $\mathrm{Hom}_{\mathbb{C}}(U, V^* \otimes W)$ for infinite-dimensional vector spaces, improper formulation of Proposition 3.7, and improper operations on resolutions at the beginning of §4. The relation for Hom should in each case be replaced by a valid associativity formula, such as the ones we give in (2.8), (3.8), and (4.1). Compare, for example, the proof of their Lemma 3.1 with our proof of our Proposition 3.7. And compare their proof of Lemma 3.3 with our Proposition 3.4. Wallach has pointed out [15] that $U(\mathfrak{g})$ in their Lemma 3.4 is to be regarded as a representation under the adjoint representation, not the regular representation, and then the proof of Lemma 3.4 relies only on cases where the proof of Lemma 3.3 is valid.

Proposition 3.7 of [8] is misstated, since the commutativity of the triangular diagrams is not included as a hypothesis. We give the correct statement here as Proposition 7.1, with hypotheses about the triangular diagrams included as (a) and (b). When Enright and Wallach apply their Proposition 3.7, the commutativity of the triangular diagrams has already been verified (Lemma 3.4).

The first paragraph of §4 of [8] contains several cancelling errors concerning resolutions. We have included an account of standard resolutions and operations on them between formulas (3.5) and (3.9).

The serious gap appears in the proof of Theorem 4.3 of [8]. In our notation the authors are attempting to verify (c) of Proposition 7.1 when $G(V) = \Gamma^i(V)$, $H(V) = \Gamma^{m-i}(V^c)^c$, and some admissibility conditions are satisfied. As we did in (7.4), they split into three squares the rectangle whose commutativity is to be proved. One of the intermediate functors is (in our notation)

$$\sum_{\gamma \in \hat{K}} H^i(\mathfrak{l}, H; V_\gamma \otimes V) \otimes V_\gamma^*$$

with K action on the V_γ^* alone, and the isomorphism of $\Gamma^i(V)$ with this associates the tuple of cohomology elements to a member of V, K type by K type. Commutativity of each of the three component squares is to be proved. One of these, for example, amounts to

$$\begin{array}{ccc} F \otimes \Gamma^i(V) & \longrightarrow & F \otimes \sum (H^i(\mathfrak{l}, H; V_\gamma \otimes V) \otimes V_\gamma^*) \\ \Big| & & \Big| \\ \Gamma^i(F \otimes V) & \longrightarrow & \sum (H^i(\mathfrak{l}, H; V_\gamma \otimes F \otimes V) \otimes V_\gamma^*) \end{array}$$

with the map on the left given in Lemma 3.4 of [8] and with the horizontal maps given as above. The gap is that the map on the right side of the diagram is not specified in the paper. (There is a similar gap in the proof of commutativity of the other two component squares.) The correct map is a Mackey isomorphism, as we shall see in the next subsection.

b. Sketch of alternate proof of duality

In the context of [8], one considers $(\mathfrak{g},\mathfrak{k})$ modules. The closest thing to that context in the present paper is $(\mathfrak{g},K)$ modules with K connected. Thus let us assume that K and H are connected and that $\Gamma(V)$ is just the subspace of K-finite vectors in V.

Since H is connected, it acts trivially on $\Lambda^m\mathfrak{s}$. Thus let us discard $\Lambda^m\mathfrak{s}$ from (6.11). In order to prove the Duality Theorem in the form of (6.11), it suffices to prove that $(\Gamma^m)_i(V) \cong \Gamma^{m-i}(V)$ and $(\Gamma^m)_i(V)^c \cong \Gamma^i(V^c)$ naturally for V in $\mathcal{C}(\mathfrak{g},H)$. The first of these identities follows by dimension-shifting (cf. [5, p. 221]) once one shows that $\Gamma^i(P) = 0$ for P projective and $i < m$; the latter fact follows from (4.4) of [8] and Poincaré duality in the category $\mathcal{C}(\mathfrak{k},H)$. For the second of these identities, the argument that derives our (4.8) from (4.11) shows that it is enough to prove that

$$\Gamma^m(V)^c \cong \Gamma(V^c) \tag{8.1}$$

naturally for V in $\mathcal{C}(\mathfrak{g},H)$.

To prove (8.1) we introduce $\Gamma'(V) = (C_K \otimes_{\mathbb{C}} V)^{\mathfrak{k}}$ as a K module under $r \otimes 1$, the $\mathfrak{k}$ invariants being computed for the tensor product of ℓ on C_K and the given representation on V. Evaluation e at 1 of the C_K factor gives a map of $\Gamma'(V)$ into V and exhibits a natural K isomorphism $\Gamma'(V) \cong \Gamma(V)$. It follows readily that

$$\Gamma^i(V) \cong H^i(\mathfrak{k},H;C_K \otimes_{\mathbb{C}} V), \quad 0 \leq i \leq m,$$

on the K level, the external action on the right side coming from $r \otimes 1$. (Since $C_K \cong \sum V_\gamma \otimes V_\gamma^*$, the reader will notice a parallel between this formula and formulas in §4 of [8].)

Then it is not too hard to use Poincaré duality to set up a sequence of K isomorphisms that exhibit (8.1) on the K level. To pass to $(\mathfrak{g},K)$ isomorphisms, one needs to verify (c) in Proposition 7.1. There are several steps to this verification, but the key one is to identify the right-hand mapping in the diagram below and to prove the commutativity of the diagram:

$$\begin{array}{ccc} F \otimes_{\mathbb{C}} \Gamma(V) & \xleftarrow{1 \otimes e} & F \otimes_{\mathbb{C}} (C_K \otimes_{\mathbb{C}} V)^{\mathfrak{k}} \\ \downarrow 1 & & \downarrow \\ \Gamma(F \otimes_{\mathbb{C}} V) & \longleftarrow & (C_K \otimes_{\mathbb{C}} (F \otimes_{\mathbb{C}} V))^{\mathfrak{k}} \end{array} \tag{8.2}$$

Now Lemma 2.3 and admissibility of C_K imply that $C_K{}^c \cong C_K$ and that

$$(C_K \otimes_{\mathbb{C}} V)^{\mathfrak{k}} \cong \mathrm{Hom}_{\mathfrak{k}}(C_K, V)_K \cong I_{\mathfrak{k},H}^{\mathfrak{k},K}(V).$$

Hence the isomorphism on the right side of (8.2) is to implement

$$F \otimes_{\mathbb{C}} I_{\mathfrak{k},H}^{\mathfrak{k},K}(V) \cong I_{\mathfrak{k},H}^{\mathfrak{k},K}(F \otimes_{\mathbb{C}} V).$$

Corollary 5.3 says that there is a Mackey isomorphism of this kind, and one checks readily that (8.2) then commutes.

<u>c. Alternating tensors in the Duality Theorem</u>

The presence of $\Lambda^m\mathfrak{s}$ in Theorem 6.1 is annoying but necessary. Nevertheless $\Lambda^m\mathfrak{s}$ disappears in applications. In the context of applications (see [12, p. 344]), there are the following additional ingredients: a global group G, a closed subgroup L of a particular kind, and a θ-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$. It is assumed that $\mathfrak{l}$ is the Levi factor of $\mathfrak{q}$, that $\mathfrak{u}$ is the nilpotent part, and that L has complexified Lie algebra $\mathfrak{l}$. Let $\overline{\mathfrak{u}}$ be the algebra opposite to $\mathfrak{u}$, so that $\mathfrak{g} = \mathfrak{q} \oplus \overline{\mathfrak{u}}$ as vector spaces.

In the present paper we take K maximal compact in G and put $H = L \cap K$. As a vector space, $\mathfrak{s}$ is then $(\mathfrak{u} \cap \mathfrak{k}) \oplus (\overline{\mathfrak{u}} \cap \mathfrak{k})$. The actions of H on the two terms are complex conjugates of one another, and thus H acts trivially on $\Lambda^m \mathfrak{s}$.

References

1. Baldoni-Silva, M. W., and A. W. Knapp, Unitary representations induced from maximal parabolic subgroups, J. Func. Anal.

2. Bien, F., Spherical $\mathcal{D}$-modules and representations of reductive Lie groups, Ph.D. dissertation, Massachusetts Institute of Technology, June 1986.

3. Borel, A., and N. Wallach, Continuous Cohomology, Discrete Subgroups, and Representations of Reductive Groups, Princeton University Press, Princeton, 1980.

4. Bott, R., Homogeneous vector bundles, Ann. of Math. 66 (1957), 203-248.

5. Brown, K. S., Cohomology of Groups, Springer-Verlag, New York, 1982.

6. Cartan, H., and S. Eilenberg, Homological Algebra, Princeton University Press, Princeton, 1956.

7. Enright, T. J., Unitary representations for two real forms of a semisimple Lie algebra: a theory of comparison, "Lie Group Representations I," Springer-Verlag Lecture Notes in Math. 1024 (1983), 1-29.

8. Enright, T. J., and N. R. Wallach, Notes on homological algebra and representations of Lie algebra, Duke Math. J. 47 (1980), 1-15.

9. Hochschild, G., Relative homological algebra, Trans. Amer. Math. Soc. 82 (1956), 246-269.

10. Schmid, W., Homogeneous complex manifolds and representations of semisimple Lie groups, Ph.D. dissertation, University of California at Berkeley, 1967.

11. Schmid, W., L^2-cohomology and the discrete series, Ann. of Math. 103 (1976), 375-394.

12. Vogan, D. A., Representations of Real Reductive Lie Groups, Birkhäuser, Boston, 1981.

13. Vogan, D. A., Unitarizability of certain series of representations, Ann. of Math. 120 (1984), 141-187.

14. Vogan, D. A., The unitary dual of GL(n) over an archimedean field, Invent. Math. 83 (1986), 449-505.

15. Wallach, N. R., Representations constructed by derived functors, lectures at Institute for Advanced Study, Princeton, Fall 1982.

16. Wallach, N. R., On the unitarizability of derived functor modules, Invent. Math. 78 (1984), 131-141.

17. Zuckerman, G. J., Construction of representations via derived functors, lectures at Institute for Advanced Study, Princeton, Spring 1978.

Department of Mathematics
State University of New York
Stony Brook, NY 11794

Department of Mathematics
Massachusetts Institute of Technology
Cambridge, MA 02139