Notes on Covering Spaces

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A. Fundamental group.

Notation: X denotes a separable metric space, i.e., a regular Hausdorff space with a countable base.

Paths and loops:

Path in X: continuous function $a : [0, \|a\|] \to X$.

$\|a\|$ = stopping time.

$a(0)$ = initial point.

$a(\|a\|)$ = final point or endpoint.

Loop in X: path with $a(0) = a(\|a\|)$; $a(0)$ is the base point for the loop.

Identity path: a path with $\|a\| = 0$.

Constant path: a path with $a(t) \equiv a(0)$.

Inverse path a^{-1}: $a^{-1}(t) = a(\|a\| - t)$ for $0 \le t \le \|a\|$.

Multiplication of paths: If a and b are paths with $a(\|a\|) = b(0)$, then their product $c = a \cdot b$ is defined to be the path traced out by a and then b:

$$c(t) = \begin{cases} a(t) & \text{for } 0 \le t \le \|a\| \\ b(t - \|a\|) & \text{for } \|a\| \le t \le \|a\| + \|b\| . \end{cases}$$

Properties:

(1) The product c is a path with stopping time $\|a\| + \|b\|$.

(2) If $a \cdot b$ and $b \cdot c$ are defined, then $(a \cdot b) \cdot c$ and $a \cdot (b \cdot c)$ are defined and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

(3) If i is an identity path, then $i \cdot a = a$ whenever $i \cdot a$ is defined and $b \cdot i = b$ whenever $b \cdot i$ is defined.

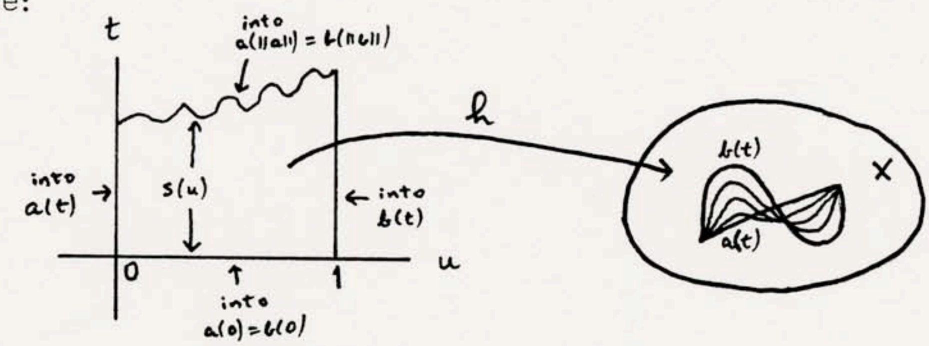
(4) $a \cdot a^{-1}$ and $a^{-1} \cdot a$ are always defined.

Equivalence: Two paths a and b with the same initial points and same final points are <u>equivalent</u>, written $a \simeq b$, if there are continuous functions $s : [0,1] \to [0,\infty)$ and $h : \{(u,t)\} \to X$ (for $u \in [0,1]$ and $t \in [0,s(u)]$) such that

$$h(0,t) = a(t), \qquad h(u,0) = a(0) = b(0)$$

$$h(1,t) = b(t), \qquad h(u,s(u)) = a(\|a\|) = b(\|b\|).$$

Picture:



Properties:

(1) "Equivalent" is an equivalence relation; denote a class by $[a]$.

(2) If $a \simeq a'$ and $b \simeq b'$ and if $a \cdot b$ is defined, then $a' \cdot b'$ is defined and $a \cdot b \simeq a' \cdot b'$. Also $a^{-1} \simeq a'^{-1}$.

(3) Constant paths are equivalent with identity paths.

(4) $a \cdot a^{-1}$ and $a^{-1} \cdot a$ are equivalent with constant paths.

Class multiplication:

If $a \cdot b$ is defined, set

$$[a][b] = [ab] \quad \text{and} \quad [a]^{-1} = [a^{-1}].$$

Both are well defined by (2).

Properties:

(1) If $[a][b]$ and $[b][c]$ are defined, then $([a][b])[c]$ and $[a]([b][c])$ are defined and equal.

(2) $[1][a] = [a]$, where $1 =$ identity path $a(0)$

$[a][1] = [a]$, where $1 =$ identity path $a(\|a\|)$.

(3) $[a][a]^{-1} = [1]$, where 1 = identity path $a(0)$
$[a]^{-1}[a] = [1]$, where 1 = identity path $a(\|a\|)$.

Proposition 1. Fix p in X. The set of classes of loops with base point p is a group under class multiplication, denoted $\pi(X,p)$. If also q is in X, then any path ξ from p to q canonically defines an isomorphism of $\pi(X,p)$ with $\pi(X,q)$.

Proof. In the properties above, all products are now defined. So $\pi(X,p)$ is a group, by the properties. Let a be a loop based at p. Then $\xi^{-1}\cdot a\cdot \xi$ is a loop based at q, and this correspondence defines the isomorphism.

Definition: $\pi(X,p)$ = fundamental group of X with base point p.

If X is pathwise connected, then $\pi(X,p)$ as an abstract group is independent of p. We say a pathwise connected X is simply connected if $\pi(X) = 1$.

Let $f : X \to Y$ be continuous. Then f induces a homormorphism

$$f_* : \pi(X,p) \to \pi(Y,f(p))$$

by $f_*([a]) = [f\circ a]$, which is independent of the representative. The induced homomorphism has the important property

$$(f\circ g)_* = f_* g_* .$$

B. Properties of covering spaces.

Notation: X and Y denote separable metric spaces that are pathwise connected and locally pathwise connected (i.e, each point has arbitrarily small pathwise connected neighborhoods. Then every open set has open connected components.

Let $e : X \to Y$ be continuous, and let V be open in Y. We say V is evenly covered by e if each connected component of $e^{-1}(V)$ is mapped by e homeomorphically onto V. (Note this implies V is connected.)

Let $e : X \to Y$ be continuous. We say e is a covering map if each y in Y has an open neighborhood V_y that is evenly covered by e. (Note that this implies e is onto Y.) In this case Y is called the base space and X is the covering space.

Proposition 2 (Path-lifting theorem). Suppose $e : X \to Y$ is a covering map. If $y(t)$, $0 \leq t \leq 1$, is a path in Y and if x_o is in $e^{-1}(y(0))$, then there exists a unique path $x(t)$, $0 \leq t \leq 1$, in X with $x(0) = x_o$ and $e(x(t)) = y(t)$.

Proof. Let T be the set of t in $[0,1]$ such that $y|_{[0,t]}$ lifts to a path from x_o. The set T is nonempty since 0 is in T. T is open. [In fact, let t_o be in T. Form the connected component U_o of $e^{-1}(V_{y(t_o)})$ containing $x(t_o)$, so that $e^{-1} : V_{y(t_o)} \to U_o$ is continuous. Extend $x(t)$ by the definition $x(t) = e^{-1}(y(t))$; then we see that T is open. (Note that this definition is forced since $x(t)$ for t near

t_o must be a connected subset of $e^{-1}(V_{y(t_o)})$ containing $x(t_o)$ and so must be in the component U_o.)] T is closed. [In fact, let t_o be a limit point of T not in T. Form $V_{y(t_o)}$ and choose $\delta > 0$ so that $y(t)$ is in $V_{y(t_o)}$ for $t_o - \delta \leq t \leq t_o$. Find the component of $e^{-1}(V_{y(t_o)})$ containing $x(t_o - \delta)$ and lift y to this component. As above, t_o is in T. Thus T is closed.] Since $[0,1]$ is connected, we conclude $T = [0,1]$.

For uniqueness let T' be the set of t in $[0,1]$ such that all lifts of $y|_{[0,t]}$ starting at x_o agree. Then T' is nonempty, and it is closed by continuity. T' is open by the argument for T above. By connectedness, $T' = [0,1]$.

<u>Lemma</u>. Let $e : X \to Y$ be a covering map and let V be an open subset of Y that is evenly covered. Then the components of $e^{-1}(V)$ are open.

Proof. Let U_o be a component of $e^{-1}(V)$ and let x_o be in U_o. Since X is locally connected, choose a connected open neighborhood U of x_o contained in the open set $e^{-1}(V)$. Then $U \subseteq U_o$ since U_o is a component, and so U_o is open.

<u>Proposition 3</u> (Covering homotopy theorem). Let $e : X \to Y$ be a covering map, let K be a compact space, and let $f_o : K \to X$ be continuous. If $g : K \times [0,1] \to Y$ is continuous and satisfies $g(\cdot,0) = ef_o$, then there is a unique continuous $f : K \times [0,1] \to X$ such that $f(\cdot,0) = f_o$ and $g = ef$.

Proof. For each k in K, Proposition 2 shows there is a unique path $f|_{k \times [0,1]}$ starting at $f_o(k)$ and covering the path $g|_{K\times[0,1]}$. This defines f and proves uniqueness. We must prove f is continuous as a function of two variables. Let T be the set of t_o in $[0,1]$ such that $f(k,t)$ is continuous at (k,t) for all k in K and all $t \leq t_o$.

Then 0 is in T. [In fact, fix k_o in K. Form the component U_o of $e^{-1}(V_{g_o(k_o)})$ containing $f_o(k_o)$, so that $e^{-1} : V_{g_o(k_o)} \to U_o$ is continuous. U_o is open by the lemma. Choose a neighborhood N_{k_o} of k_o so that $f_o(N_{k_o}) \subseteq U_o$ and $g(N_{k_o} \times [0,\epsilon)) \subseteq V_{g_o(k_o)}$. Then $f(N_{k_o} \times [0,\epsilon)) \subseteq U_o$. Hence $f = e^{-1}g$ on $N_{k_o} \times [0,\epsilon)$ and is continuous.]

T is open. [In fact, let t_o be in T. Fix k_o in K. Form the component U_o of $e^{-1}(V_{g(k_o,t_o)})$ containing $f(k_o,t_o)$; then $e^{-1} : V_{g(k_o,t_o)} \to U_o$ is continuous. Choose a neighborhood N_{k_o} of k_o and an ϵ_{k_o} so that $f(N_{k_o} \times (t_o - \epsilon_{k_o}, t_o + \epsilon_{k_o})) \subseteq U_o$. Then $f = e^{-1}g$ on $N_{k_o} \times (t_o - \epsilon_{k_o}, t_o + \epsilon_{k_o})$ and so is continuous on this set. The N's cover K. Extract a finite subcover and use the minimum of the ϵ's to see that T is open.]

T is closed. [In fact, let t_o be a limit point of T not in T. Fix k_o in K. Form $V = V_{g(k_o,t_o)}$, and choose N' (a neighborhood of k_o) and ϵ so that $g(N' \times [t_o - \epsilon, t_o + \epsilon]) \subseteq V$. Form the component U_o of $e^{-1}(V)$ containing $f(k_o, t_o - \epsilon)$; then $e^{-1} : V \to U_o$ is continuous. Choose a neighborhood $N \subset N'$ so that $f(N, t_o - \epsilon) \subseteq U_o$. Then $f(N \times [t_o - \epsilon, t_o + \epsilon]) \subseteq U_o$, $f = e^{-1}g$ on this set, and f is continuous at (k_o,t_o). So T is closed.] By connectedness, $T = [0,1]$.

Proposition 4. If $e : X \to Y$ is a covering map, then a path $x(t)$ in X is a contractible loop (i.e., a loop equivalent with a constant map) if and only if the projected path $ex(t)$ is a contractible loop. Consequently e_* is one-one.

Proof. $\Longrightarrow$ is trivial. For $\Longleftarrow$, put $f_o(t) = x(t)$, and suppose $ex(t)$ is contactible. Then we can find a continuous $g(t,s)$ for $s \in [0,1]$ such that $g(\cdot,0) = ex(\cdot)$, $g(0,s) = ex(0)$, $g(1,s) = ex(1) = ex(0)$, and $g(\cdot,1) = ex(0)$. Find f as in Proposition 3. Then $f(\cdot,0) = x(\cdot)$. Also $f(0,s)$ is continuous into the discrete space $e^{-1}(ex(0))$ and so is constant and must be $x(0)$. Similarly $f(\cdot,1) = x(0)$, and then $f(1,s) = x(0)$. So $x(0) = x(1)$, and $x(t)$ is equivalent with the constant $x(0)$.

Proposition 5. If $e : X \to Y$ is a covering, if y_o is in Y, and if x_o is in $e^{-1}(y_o)$, then the lift of a loop $y(t)$ based at y_o is a loop if and only if $[y(t)]$ is in $e_*\pi(X,x_o)$.

Proof. $\Longrightarrow$ is trivial. For $\Longleftarrow$, choose a loop $x'(t)$ based at x_o in X such that $[ex'(t)] = [y(t)]$, and let g exhibit ex' and y as equivalent: $g(\cdot,0) = ex'(\cdot)$, $g(\cdot,1) = y(\cdot)$, $g(0,s) = g(1,s) = y_o$. Produce f as in Proposition 3. As above, $f(\cdot,1)$ is a loop based at x_o, and it is the lift of $y(t)$.

Theorem 1 (Map-lifting theorem). If $e : X \to Y$ is a covering, if P is a pathwise connected, locally pathwise connected separable metric space, if $g : P \to Y$ is continuous, and if p_o is in $g^{-1}(y_o)$ and x_o is in $e^{-1}(y_o)$, then there exists a continuous $f : P \to X$ with $f(p_o) = x_o$ and $g = ef$ if and only if $g_*(\pi(P,p_o)) \subseteq e_*(\pi(X,x_o))$. When f exists, it is unique.

Proof. If f exists, then

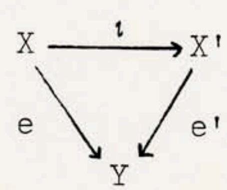
$$g_*\pi(P,p_o) = e_*f_*\pi(P,p_o) \subseteq e_*\pi(X,x_o).$$

Conversely, suppose the inclusion holds. Let p be in P and let w be a path from p_o to p. Then gw is a path from y_o to $y = g(p)$, and we let u be the lift of gw to a path from x_o to some x in X. Define $f(p) = x$. Another choice of w leads to a loop in P that is mapped under g to a loop in Y. Since the class of the loop in Y is in $g_*\pi(P,p_o) \subseteq e_*\pi(X,x_o)$, Proposition 5 shows the loop lifts to a loop in X; therefore x is uniquely determined by this definition.

To see that f is continuous at p, let U_o be the component of $e^{-1}(V_{g(p)})$ containing $f(p)$, so that $e^{-1} : V_{g(p)} \to U_o$ is continuous. Choose a pathwise connected neighborhood N of p in P so that $g(N) \subseteq V_{g(p)}$. Then a path from p within N is mapped by g and lifts to a path from $f(p)$ within U_o. Hence $f(N) \subseteq U_o$. Then $f = e^{-1}g$ on N, and f is continuous on N.

If f exists, then the conditions in the definition above must be satisfied. So f is unique. This completes the proof.

Notation for uniqueness theorem: Let $e : X \to Y$ and $e' : X' \to Y$ be coverings. We say e and e' are <u>equivalent coverings</u> if there is a homeomorphism ι of X onto X' such that $e'\iota = e$.



<u>Theorem 2</u> (Uniqueness theorem). Let $e : X \to Y$ and $e' : X' \to Y$ be coverings, and let y_o be in Y. Then e and e' are equivalent coverings if and only if base points x_o in X and x_o' in X' can be chosen so that $e(x_o) = e'(x_o') = y_o$ and

$$e_*(\pi(X,x_o)) = e_*'(\pi(X',x_o')).$$

Proof. If e and e' are equivalent, choose x_o in $e^{-1}(y_o)$ and define $x_o' = \iota(x_o)$. Then $e_*\pi(X,x_o) = e_*'\iota_*\pi(X,x_o) = e_*'\pi(X',x_o')$.

Conversely suppose $e_*\pi(X,x_o) = e_*'\pi(X',x_o')$. By Theorem 1, we can cover $e : X \to Y$ by $\iota : X \to X'$ and we can cover $e' : X' \to Y$ by $j : X' \to X$. Now $j\iota$ leaves x_o fixed and $e(j\iota) = (ej)\iota = e'\iota = e$. So $j\iota$ covers the identity map of Y into itself and by the uniqueness in Theorem 1 is the identity map of X. Similarly ιj is the identity, and therefore ι is a homeomorphism. By construction $e'\iota = e$, and therefore ι defines an equivalence.

C. Existence of covering spaces.

Notation: X and Y are separable metric spaces, pathwise connected, and locally pathwise connected.

In proving existence of covering spaces, the next proposition will guide the choice of open sets in Y that are to be evenly covered by e. In particular, we shall want to assume that such sets exist.

Proposition 6. If $e : X \to Y$ is a covering, then any pathwise connected open subset Q of Y such that any loop in Q is contactible in Y is evenly covered.

Proof. Fix y_o in Q, x_o in $e^{-1}(y_o)$. Lift the paths in Q from y_o to paths in $e^{-1}(Q)$ from x_o. Let P_{x_o} be the union of the images of these paths, and let $e' = e|_{P_{x_o}}$. Then e' is continuous from P_{x_o} onto Q.

We show e' is one-one. Thus let p_1 and p_2 be in $e'^{-1}(q)$. Connect x_o to p_1 and p_2 by paths in $e^{-1}(Q)$. The projections to Q yield a loop in Q, which is in $e_*(\pi(X,x_o))$ since it is contactible in Y (by hypothesis). By Proposition 5 the lift is a loop and $p_1 = p_2$. Thus e' is one-one.

We show e' is onto Q. If q is given, join y_o to q by a path and lift to a path from x_o to some x. Then $e'(x) = q$.

We show e'^{-1} is continuous. If y is in Q, let

$V \subseteq Q$ be a pathwise connected evenly covered neighborhood of y, let x be in $P_{x_o} \cap e^{-1}(y)$, and let U be the component of $e^{-1}(V)$ containing x, so that $e^{-1} : V \to U$ is continuous. Then $e'^{-1} = e^{-1}$ on V, and e'^{-1} is thus continuous at y. Hence e' is a homeomorphism of P_{x_o} onto Q.

Clearly P_{x_o} is connected. If $x_o \neq x$, then P_{x_o} and P_x coincide or are disjoint since a point of intersection joins x_o to x. Also P_{x_o} is open: In fact, let x be in P_{x_o} and let $U \subseteq e^{-1}(Q)$ be a pathwise connected open neighborhood of x. Then U is path connected to P_{x_o} and so $U \subseteq P_{x_o}$. Thus P_{x_o} is open.

Finally the union of all P_x for x in $e^{-1}(Q)$ is $e^{-1}(Q)$, and so P_{x_o} is a component of $e^{-1}(Q)$. Thus Q is evenly covered.

We say Y is <u>locally simply connected</u> if each y in Y has an open pathwise connected, simply connected neighborhood. In this case, each y in Y has arbitrarily small open pathwise connected neighborhoods with the following property: Any loop in the neighborhood is contractible in Y.

<u>Lemma</u>. If Y is locally simply connected, then $\pi(Y,y_o)$ is countable.

Proof. Let $\mathcal{V}$ be a countable base of open sets of Y that are pathwise connected and contain only loops that are contractible in Y. If V_j and V_k are in $\mathcal{V}$, then $V_j \cap V_k$ has a countable number of components (since Y is separable and the components are open). Let $\mathcal{V}^*$ be the collection of all components of all $V_j \cap V_k$. By the separability we can cover each V_j by countably many open sets V'_ℓ such that $\overline{V}'_\ell \subseteq V_j$. Let $f(t)$, $0 \leq t \leq 1$, be a loop based at y_o. Cover $[0,1]$ with open intervals such that f of each interval is in some V'_ℓ, and, by compactness of $[0,1]$, extract a finite subcover. If $\overline{V}'_\ell \subseteq V_j$ and if f maps some open interval into V'_ℓ, then f maps the closure of the interval into V_j, by continuity. As a result, we can choose points t_i and sets V_i, $0 \leq i \leq n$, with $t_o = 0$, $t_n = 1$, $f(t_i) \in V_i \cap V_{i-1}$ for $i \geq 1$, and $f(t) \in V_i$ for $t_i \leq t \leq t_{i+1}$. Let V_i^* be the component of $V_i \cap V_{i-1}$ to which $f(t_i)$ belongs. To f we can associate the finite sequence

$$V_1, V_1^*, V_2, V_2^*, \ldots, V_{n-1}^*, V_n$$

and this sequence determines $[f]$. (To see this, suppose g is given with the same sequence. We may assume without loss of generality that $g(t_i) = f(t_i)$ for all i since V_i^* is pathwise connected. Then $f \simeq g$ for each interval $t_i \leq t \leq t_{i+1}$, because taken in succession they give a loop in V_i.) The set of sequences is countable, and so $\pi(Y,y_o)$ is countable.

<u>Theorem 3</u> (Existence theorem). If Y is locally simply connected, if y_o is in Y, and if H is a subgroup of $\pi(Y,y_o)$, then there exists a covering space X with covering map $e : X \to Y$ and with point x_o in X such that $e(x_o) = y_o$ and $e_*(\pi(X,x_o)) = H$.

Proof. Let X be the set of equivalence classes of paths in Y from y_o under the equivalence relation that $f \sim f'$ if

(1) f and f' have the same final point

(2) $[f' \cdot f^{-1}]$ is in H.

This is an equivalence relation because H is a group. Typical equivalence classes will be denoted x or $\{f\}$. Let x_o be the class of the constant path at y_o. For x in X, let $e(x)$ be the endpoint of a path in the class x; then $e(x_o) = y_o$.

Let $\mathcal{V}$ be a base of open sets in Y that are pathwise connected and contain only loops that are contractible in Y. For V in $\mathcal{V}$ and x in $e^{-1}(V)$, define $U(x,V) \subseteq X$ by

$$U(x,V) = \{x' \in e^{-1}(V) \mid x' = \text{class of some } f \cdot h, \text{ where } \{f\} = x \text{ and } h \text{ is a path starting at } e(x) \text{ and remaining within } V\}.$$

The sets $U(x,V)$ have the following properties:

(1) A class x' in $U(x,V)$ is not affected by using a different representative f, nor is it affected by using a different h as long as the endpoint of h is not changed. [In fact, $\{f \cdot h\} = \{f' \cdot h\}$ because $f \cdot h$ and $f' \cdot h$ have the same endpoint and $[f \cdot h \cdot h^{-1} \cdot f'^{-1}] = [f \cdot f'^{-1}] \in H$. Also, if h and h' both have the same endpoint, then so do $f \cdot h$ and $f \cdot h'$; since $h \cdot h'^{-1}$ is contractible in Y, we have

$$[f \cdot h \cdot h'^{-1} \cdot f^{-1}] = [f \cdot f^{-1}] = [1] \in H$$

and hence $\{f \cdot h\} = \{f \cdot h'\}$.]

(2) If x' is in $U(x,V)$, then $U(x',V) = U(x,V)$. [In fact, let x'' be in $U(x,V)$, and write $x' = \{f_o \cdot h_o\}$ and $x'' = \{f \cdot h\}$. Applying (1), we have

$$x'' = \{f\cdot h\} = \{f\cdot h_o\cdot h_o^{-1}\cdot h\} = \{(f\cdot h_o)\cdot(h_o^{-1}\cdot h)\}. \quad (*)$$

On the right side, $\{f\cdot h_o\} = x'$ because $f\cdot h_o$ has endpoint the same as for $f_o\cdot h_o$ and because $x = \{f\} = \{f_o\}$ implies

$$[f_o\cdot h_o\cdot h_o^{-1}\cdot f^{-1}] = [f_o\cdot f^{-1}] \in H.$$

Also $h_o^{-1}\cdot h$ is a path within V starting at $e(x')$. Thus (*) shows that x'' is in $U(x',V)$. Hence $U(x,V) \subseteq U(x',V)$. In particular, x is in $U(x',V)$. Then we can repeat the above argument with x and x' interchanged to conclude $U(x',V) \subseteq U(x,V)$.]

(3) If x is in $U(x_1,V_1) \cap U(x_2,V_2)$, choose V in so that $e(x)$ is in V and $V \subseteq V_1 \cap V_2$. Then

$$x \text{ is in } U(x,V) \quad \text{and} \quad U(x,V) \subseteq U(x_1,V_1) \cap U(x_2,V_2).$$

[In fact, $U(x,V) \subseteq U(x,V_1) = U(x_1,V_1)$, with the equality holding by (2). Similarly, $U(x,V) \subseteq U(x_2,V_2)$, and the assertion follows.]

Now let $\mathcal{U} = \{U(x,V)\}$. $\mathcal{U}$ is a base for a topology of X, according to (3), and e is continuous since $e^{-1}(V)$ is the union of all $U(x,V)$ for x in $e^{-1}(V)$. Next, e is one-one on $U(x,V)$. [In fact, suppose x and x' are in $U(x,V)$, $e(x) = e(x')$, and $\{f\} = x$. The path h that exhibits x' as in $U(x,V)$ is then a loop in V, hence contractible. Thus $x = \{f\} = \{f\cdot h\} = x'$.] Also e maps $U(x,V)$ onto V. [In fact, let v be given in V, and join $e(x)$ to v by a path h in V. Then $f\cdot h$ defines a point x' of X with $e(x') = v$ and exhibits x' as in $U(x,V)$.] Moreover, $e^{-1} : V \to U(x,V)$ is continuous because $(e^{-1})^{-1}(U(x',V')) = V'$ is open. Thus $e : U(x,V) \to V$ is a homeomorphism. Since V is connected, $U(x,V)$ is connected. The set $U(x,V)$ is open by definition, and $e^{-1}(V)$ is the union of the $U(x,V)$ for x in $e^{-1}(V)$, with the $U(x,V)$ disjoint or equal, by (2). Consequently the sets $U(x,V)$ are the connected components of $e^{-1}(V)$. Hence V is evenly covered by e.

Now we prove the appropriate topological properties of X. Since $U(x,V)$ is open and is homeomorphic with V, X is locally pathwise connected. To see that X is pathwise connected, let x in X be given. Then x is a class of paths in Y starting at y_o. Pick such a path f. Then for $0 \leq t \leq \|f\|$, $\tilde{f}(t) = f|_{[0,t]}$ is a path from x_o to x in X. The space X is Hausdorff and regular because these properties are local properties and they hold in Y. To complete the argument that X is a separable metric space, it is enough, in view of the metrization theorem, to prove that X has a countable base.

To prove that X has a countable base, we may assume that $\mathcal{V}$ is countable. For each V in $\mathcal{V}$, select one y in V. Then the number of sets $U(x,V)$ with $e(x) = y$ is the same as the number of elements of $e^{-1}(y)$, which is the number of classes of paths from y_o to y, modulo H. Since $\pi(Y,y_o)$ is countable by the lemma, $\mathcal{U}$ is countable.

Finally we are to show that $e_*(\pi(X,x_o)) = H$. Let $\tilde{f}$ be a loop based at x_o and let $f = e\tilde{f}$. If x is the point in X corresponding to f, then $f'(t) = f|_{[0,t]}$ is a path from x_o to x covering f, and so $f' = \tilde{f}$. Thus x is the endpoint of $\tilde{f}$, which is x_o. Consequently f and the constant path represent the same point in X, and [f] must be in H. Thus $e_*(\pi(X,x_o)) \subseteq H$. In the reverse direction, let [f] be in H, and lift f to $\tilde{f}$. Again f represents x_o since [f] is in H, and it represents the endpoint of $\tilde{f}$. Thus $\tilde{f}$ is a loop, and Proposition 5 shows that [f] is in $e_*(\pi(X,x_o))$. Hence $e_*(\pi(X,x_o)) = H$. This completes the proof.

By Theorems 3 and 2, if Y is locally simply connected, Y has a simply connected covering space that is unique up to equivalence. This space is called the <u>universal covering space</u> of Y.

D. Computation of fundamental groups.

We shall establish formulas for $\pi(X)$ for some basic spaces X and then show how $\pi(Y)$ can often be computed when a covering $e : X \to Y$ is given and $\pi(X)$ is known. This will allow us to compute $\pi(X)$ in all cases of interest for Lie group theory.

Proposition 7. $\mathbb{R}^n$ is simply connected.

Proof. Without loss of generality, let the base point be 0. Let $f(t)$ be a loop based at 0. Then $h(u,t) = (1-u)f(t)$, $0 \leq u \leq 1$, exhibits $f(t)$ as equivalent with a constant path.

Proposition 8. If X and Y are pathwise connected separable metric spaces with $x_0 \in X$ and $y_0 \in Y$, then $\pi(X \times Y, (x_0, y_0))$ is canonically isomorphic with $\pi(X, x_0) \oplus \pi(Y, y_0)$.

Proof. We map $\pi(X \times Y, (x_0, y_0))$ to $\pi(X, x_0) \oplus \pi(Y, y_0)$ by mapping a loop $(f(t), g(t))$ based at (x_0, y_0) to $([f(t)], [g(t)])$. If p_X and p_Y denote the projection maps of $X \times Y$ onto X and Y, respectively, then this map can be written as $((p_X)_*, (p_Y)_*)$; hence it is a well-defined map on the fundamental group and is a group homomorphism. To see it is onto, let $[f(t)] \in \pi(X, x_0)$ and $[g(t)] \in \pi(Y, y_0)$. Without loss of generality, we may assume $0 \leq t \leq 1$ in both cases. Then $[(f(t), g(t))]$ maps onto $([f(t)], [g(t)])$. Thus the map is onto. To see the map is one-one, suppose $[f(t)] = 1$ and $[g(t)] = 1$. Again suppose $0 \leq t \leq 1$ in both cases. Find $h_f(u,t)$ with $h(u,0) = h(u,1) = x_0$, $h(0,t) = f(t)$, and $h(1,t) = x_o$. Find $h_g(u,t)$ similarly. Then $h_{(f,g)}(u,t) = (h_f(u,t), h_g(u,t))$ exhibits $[(f(t),g(t)]$ as equal to 1.

<u>Proposition 9</u>. Let $X = \mathbb{R}^1$ = line and $Y = \{z \in \mathbb{C} \mid |z| = 1\}$ = circle. Let $e : X \to Y$ be the map $e(x) = e^{ix}$. Then e is a covering map, $\pi(Y) = \mathbb{Z}$, and $y(t) = e^{it}$, $0 \leq t \leq 2\pi$, is a generator of $\pi(Y)$.

Proof. Let $e^{i\theta}$ in Y be given, $0 \leq \theta \leq 2\pi$. Choose $V_\theta = \{e^{i\varphi} \mid |\varphi - \theta| < \pi/2\}$. Then

$$e^{-1}(V_\theta) = \bigcup_{n=-\infty}^{+\infty} \{\varphi \in X \mid |\varphi - \theta - 2\pi n| < \pi/2\}$$

disjointly, with each set homeomorphic to V_θ. Hence e is a covering map.

We use $x_o = 0$ as base point in X. By Proposition 7, $\pi(X,0) = \{1\}$. Let $y_n(t) = e^{it}$, $0 \leq t \leq 2\pi n$. The lift of y_n is $x_n(t) = t$, $0 \leq t \leq 2\pi n$, which is not a loop. By Proposition 5, $[y_n]$ is not in $e_*\pi(X,0) = \{1\}$. By definition $[y_n] = [y_1]^n$. Hence y_1 generates an infinite cyclic group contained in $\pi(Y,1)$.

Now suppose $y^*(t)$, $0 \leq t \leq 1$, is any loop based at 1 in Y. Lift to a path $x^*(t)$, $0 \leq t \leq 1$, based at 0. Then $e(x^*(1)) = 1$ implies $x^*(1) = 2\pi n$ for some integer n. Form $y^* \cdot y_n^{-1}$. This path lifts to a loop in X (necessarily contractible) and so is contractible in Y, by Proposition 4. Thus $[y^*] = [y_n] = [y_1]^n$. Hence y_1 generates all of $\pi(Y,1)$.

Proposition 10. The n-sphere S^n is simply connected if $n > 1$.

Proof. This follows from the next lemma if we take

$$X_1 = S^n - \{(1,0,\ldots,0)\} \text{ and } X_2 = S^n - \{(-1,0,\ldots,0)\}.$$

Lemma. Let X be pathwise connected and locally pathwise connected. If there exist connected, simply connected open subsets X_1 and X_2 of X with $X = X_1 \cup X_2$ and with $X_1 \cap X_2$ connected, then $\pi(X) = 1$.

Proof. Fix x_0 in $X_1 \cap X_2$. (If $X_1 \cap X_2$ is empty, one of X_1 and X_2 must be empty, and the result follows.) Let $a(t)$ be a loop based at x_0. Without loss of generality, we may assume $0 \leq t \leq 1$. To each point in $(0,1)$, we can associate an open interval centered at the point such that a maps the closure of this interval completely into X_1 or completely into X_2. For $t = 0$, similarly, a maps the closure of some (relatively) open interval $[0,\epsilon)$ into one of X_1 and X_2; and for $t = 1$, we similarly obtain a (relatively) open interval $(\epsilon',1]$. By compactness of $[0,1]$, we extract a finite subcover and obtain a partition $0 = t_0 < t_1 < \ldots < t_n = 1$ so that $a([t_{i-1},t_i]) \subseteq X_{k_i}$ for $1 \leq i \leq n$. Construct a path b_i, $0 \leq i \leq n$, from x_0 to $a(t_i)$ such that b_i remains in whatever X_j's $a(t_i)$ is in. This is possible since $X_1 \cap X_2$ is pathwise connected. Let $a_i(t) = a(t + t_{i-1})$ for $0 \leq t \leq t_i$, $1 \leq i \leq n$. Then

$$[a] = [b_0 a_1 b_1^{-1}][b_1 a_2 b_2^{-1}]\cdots[b_{n-1} a_n b_n^{-1}] .$$

Fix i. Then $a_i(t) \subseteq X_{k_i}$, in particular for $t = 0$ and $t = t_i - t_{i-1}$. So $b_i^{-1} \subseteq X_{k_i}$ by construction. Thus

$b_{i-1}a_ib_i^{-1}$ lies in X_{k_i}. Since X_{k_i} is simply connected, $[b_{i-1}a_ib_i^{-1}] = 1$. Therefore $[a] = 1$.

Let X be the universal covering space of a pathwise connected, locally pathwise connected, locally simply connected, separable metric space Y, and let $e : X \to Y$ be the covering map. A deck transformation of X is a homeomorphism f of X that satisfies $ef = e$.

Theorem 4. Let $e : X \to Y$ be a universal covering map (as above), and let $e(x_o) = y_o$. Then

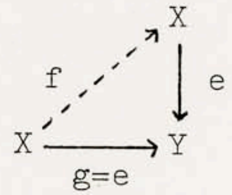
(1) $\pi(Y,y_o)$ is in one-one correspondence with $e^{-1}(y_o)$, the correspondence being that $x_1 \in e^{-1}(y_o)$ corresponds to $[e(\text{any path from } x_o \text{ to } x_1)]$.

(2) the group of deck transformations H of X acts simply transitively on $e^{-1}(y_o)$.

(3) the correspondence that associates to a deck transformation f in H the member of $\pi(Y,y_o)$ corresponding to $f(x_o)$ is a group isomorphism of H onto $\pi(Y,y_o)$.

Proof. (1) Let $[y(t)]$, $y(0) = y_o$, be in $\pi(Y,y_o)$. Let $x(t)$ be its lift with $x(0) = x_o$, and let x_y be the endpoint. We make $y \to x_y$. This map is independent of the representative because a contractible loop based at y_o lifts to a loop in X, by Proposition 4. For the inverse correspondence, let x_1 be given. Then $[e(\text{any path from } x_o \text{ to } x_1)]$ is well defined because any loop based at x_o is contractible, X being simply connected. The result is in $\pi(Y,y_o)$ and defines the inverse map.

(2) If x_1 is in $e^{-1}(y_o)$, we apply Theorem 1 to the diagram



$f(x_o) = x_1$.

Then $g_*(\pi(X,x_o)) = \{1\}$, and so f exists with $ef = g$. Arguing similarly, we see that f is a homeomorphism. This proves transitivity. The transitivity is simple by the uniqueness in Theorem 1.

(3) The map in question is one-one onto by (1) and (2). We show it is a homomorphism. Let f and g be in H. Then $f \circ g$ corresponds to $[ex(t)]$, where x is any path from x_o to $f(g(x_o))$. If we choose $x(t)$ to pass through $f(x_o)$ on the way, we see that the problem is to show that $[eu(t)] = [ev(t)]$ if

$u(t)$ is a path from x_o to $g(x_o)$

$v(t)$ is a path from $f(x_o)$ to $f(g(x_o))$.

Now $[ef(u(t))] = [eu(t)]$ since f is a deck transformation, and $f(u(t))$ is a path from $f(x_o)$ to $f(g(x_o))$. Since X is simply connected, $[ef(u(t))] = [ev(t)]$. Therefore $[eu(t)] = [ev(t)]$.

E. Topological groups.

A topological group is a Hausdorff space that is a group such that multiplication and inversion are continuous.

Properties of a topological group G:

(1) Left and right translations are homeomorphisms.

Proof. The continuity of multiplication implies the continuity of translation, by restriction. Translation is a homeomorphism because translation by the inverse element is continuous.

(2) To each neighborhood V of the identity 1 corresponds a neighborhood U such that $UU^{-1} \subseteq V$.

Proof. This is the statement of continuity of the map $(x,y) \to xy^{-1}$ at $(x,y) = (1,1)$.

(3) G is regular as a topological space.

Proof. If a point and a closed set are given, we may assume the point is 1, by (1). Then find U in (2) for $V = G - F$. We claim that $\overline{U} \subseteq V$. In fact, if x is in $\overline{U} - U$, then xU is a neighborhood of x and so meets U. If y is in $xU \cap U$, then $y = xu$, $x = yu^{-1} \in UU^{-1} \subseteq V$.

(4) Let H be a closed subgroup, and let G/H have the quotient topology. Then the projection $p : G \to G/H$ is open, and G/H is a Hausdorff regular space such that the action of G on G/H is jointly continuous. If G has a countable base, so does G/H.

Proof. The set $E \subseteq G/H$ is open if and only if $p^{-1}(E)$ is open. Let U be open in G. Then $p^{-1}(p(U)) = \bigcup_{h \in H} Uh$, which is open. The action of G on G/H is the composition

$$G \times G \xrightarrow{\text{multiplication}} G \xrightarrow{\text{quotient}} G/H,$$

which is continuous.

If x is in G/H and F is a disjoint closed subset, we may assume $x = 1$ by this continuity. Choose a neighborhood U of 1 in G and a neighborhood N of 1 in G/H such that $UN \subseteq G/H - F$. As in (3), we claim $\overline{N} \subseteq UN$. [In fact, if y is in $\overline{N} - N$, then $U^{-1}y$ is a neighborhood of y since p is open. Hence $U^{-1}y \cap N$ is not empty; let z be a member. Then $z = u^{-1}y$, and $y = uz \in UN$.] Consequently G/H is regular. Next, $p^{-1}(\{x\}) = xH$ is closed, and so $\{x\}$ is closed. Thus G/H is a T_1 regular space and must be Hausdorff. If $\mathcal{U}$ is a base of G, $p\mathcal{U}$ is a base of G/H since p is open. Hence if G has a countable base, so does G/H.

(5) Let H be a closed subgroup. If H and G/H are connected, then G is connected.

Proof. Let $G = U \cup V$ with U and V nonempty, open, and disjoint. For each x in G, xH is connected, by (1). Since $xH \subseteq U \cup V$, we must have $xH \subseteq U$ or $xH \subseteq V$. Let

$$A = \{x \in G \mid xH \subseteq U\} \quad \text{and} \quad B = \{x \in G \mid xH \subseteq V\}.$$

We have just seen that $G = A \cup B$. Since U and V are complements, $A = \{x \in G \mid xH \cap V = \emptyset\}$. It follows that

$$A = \{x \in G \mid xH \cap p^{-1}pV = \emptyset\}.$$

Consequently (4) shows that A is open in G. Similarly

B is open in G. Since $p^{-1}pA = A$ and $p^{-1}pB = B$, we see that $G/H = pA \cup pB$ is a disjoint decomposition of G/H into open sets. By connectedness of G/H, one of pA and pB is empty, say pB. Then B is empty, $A = G$, and $U = G$. We conclude G is connected.

(6) Let H be a closed subgroup. If H and G/H are compact, then G is compact.

Proof. Let $\mathcal{U}$ be an open cover of G. For each x in G, $\mathcal{U}$ is an open cover of xH. Let $\mathcal{V}_x$ be a finite subcover and let

$$V_x = \{y \in G \mid yH \text{ is covered by } \mathcal{V}_x\}.$$

We prove V_x is open. Fix y in V_x. For each $h \in H$ we can find $U_h \in \mathcal{V}_x$ with $yh \in U_{yh}$. By continuity of multiplication, we can then find open neighborhoods M_h of y and N_h of h such that $M_h N_h \subseteq U_h$. The open sets N_h cover H, and we let $\{N_{h_j}\}$ be a finite subcover. Then $M = \cap_j M_{h_j}$ is an open neighborhood of y. If z is in M, we show z is in V_x. Let h be given and let $h \in N_{h_j}$. Then $zh \in M_{h_j} N_{h_j} \subseteq U_{h_j} \in \mathcal{V}_x$. Hence z is in V_x, and we conclude V_x is open. The sets pV_x cover G/H. Since G/H is compact, let $\{pV_{x_1}, \ldots, pV_{x_n}\}$ be a finite subcover. Since $V_{x_j} = p^{-1}pV_{x_j}$ for all j, $\{V_{x_1}, \ldots, V_{x_n}\}$ covers G. Then $\cup_j \mathcal{V}_{x_j}$ is a finite subcover of $\mathcal{U}$.

(7) If H is a closed normal subgroup, then G/H is a topological group.

Proof. Let V be a neighborhood of 1 in G/H. Choose by (4) a neighborhood U of 1 in G and a neighborhood N of 1 in G/H such that $UN \subseteq V$. Then pU and N are neighborhoods of 1 in G/H such that $(pU)N \subseteq V$. Hence multiplication is continuous at $(1,1)$, therefore everywhere. If V is a neighborhood of 1 in G/H and U is an open neighborhood of 1 in G with $U^{-1} \subseteq p^{-1}(V)$, then $pU^{-1} \subseteq V$, and so inversion is continuous at 1, hence everywhere. Finally G/H is Hausdorff by (4).

(8) Any open subgroup is closed.

Proof. If H is the open subgroup, then $H = G - \bigcup_{x \notin H} xH$ shows H is closed.

(9) The identity component G_o of 1 in G is a closed normal subgroup.

Proof. The image of $G_o \times G_o$ under multiplication is a connected set containing 1. Hence $G_oG_o \subseteq G_o$. Similarly $G_o^{-1} \subseteq G_o$. So G_o is a group. It is closed because components are closed in any topological space. It is normal because the same argument shows $xG_ox^{-1} \subseteq G_o$ for each x in G. If G is locally connected, its components are open, and G_o in particular is open. Hence so are the cosets of G_o. Since $p : G \to G/G_o$ is an open mapping, G/G_o has every subset open.

(10) If G is connected, then any neighborhood of 1 generates G.

Proof. Let V be a neighborhood of 1, and choose an open neighborhood U of 1, by continuity of inversion, such that $U = U^{-1} \subseteq V$. Set $H = \bigcup_{n=1}^{\infty} UU\cdots U$, with n factors in the n^{th} term. Then H is nonempty, is open, and is a subgroup of G, since $U = U^{-1}$. By (8), H is closed. Therefore $H = G$. Then V must generate G since $U \subseteq V$.

(11) If H is a discrete subgroup of G (i.e., if every subset of H is relatively open), then H is a closed subgroup.

Proof. Choose a neighborhood V of 1 in G so that $H \cap V = \{1\}$, and choose an open neighborhood U of 1 with $UU \subseteq V$, by (2). If x is in $\overline{H} - H$, then $U^{-1}x$ is a neighborhood of x and so must contain a member h of H. Write $u^{-1}x = h$. Then $u = xh^{-1}$ is in $\overline{H} - H$ and is in U. Since it is a limit point of H, we can find $h' \neq 1$ such that $h' \in Uxh^{-1}$. Then h' is in $U(xh^{-1}) \subseteq UU \subseteq V$, and $h' = 1$, contradiction.

(12) If G is connected, then any discrete normal subgroup H of G lies in the center of G.

Proof. If h is in H, then ghg^{-1} is in H, and by continuity ghg^{-1} is in the same component of H as $1h1^{-1} = h$. Since H is discrete, $ghg^{-1} = h$ and h is central.

Notation for the remainder of the section:

G = pathwise connected, locally pathwise connected, separable metric topological group

H = closed subgroup of G, locally pathwise connected (but not necessarily connected)

<u>Proposition 11</u>. (a) The quotient G/H is pathwise connected and locally pathwise connected.

(b) If H_o is the identity component of H, then the natural map of G/H_o onto G/H is a covering map.

Proof. (a) If x and y are given in G/H, take their preimages in G, connect them by a path, and map back down to G/H to see that G/H is pathwise connected. If x is in an open subset U of G/H, take a preimage $\tilde{x}$ in $p^{-1}(x)$, and choose a pathwise connected open subset $V \subseteq p^{-1}(U)$ with $\tilde{x}$ in V. Then $p(V)$ is a pathwise connected open subneighborhood of U and shows that G/H is locally pathwise connected.

(b) Let $p_o : G \to G/H_o$, $p : G \to G/H$, and $q : G/H_o \to G/H$ be the projection maps. Then $q^{-1}(U) = p_o(p^{-1}(U))$ is open if U is open, and so q is continuous. Also $q(U) = p(p_o^{-1}(U))$ shows that q is open. Since H is locally connected, we can find an open neighborhood U of 1 in G with $U \cap H = H_o$ (i.e., H_o is relatively open in H). Next, find an open connected V about 1 with $V^{-1}V \subseteq U$; this is possible by (2). Then $V^{-1}V \cap H \subseteq H_o$. Form the sets

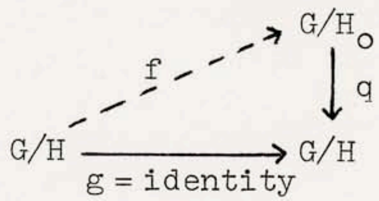
$$VhH_o, \quad h \in H.$$

These sets are open and connected in G/H_o and their union is $q^{-1}(VH)$. If $Vh_1H_o \cap Vh_2H_o$ is not empty, then the same thing is true first of $VH_oh_1 \cap VH_oh_2$, then of $V^{-1}VH_o \cap H_oh_2h_1^{-1}$, and finally of $V^{-1}V \cap H_oh_2h_1^{-1}$. Since $V^{-1}V \cap H \subseteq H_o$, $h_2h_1^{-1}$ is in H_o, and so $Vh_1H_o = Vh_2H_o$. In short, distinct sets VhH_o are disjoint. Thus q is one-one continuous open from each VhH_o onto VH, and VH is evenly covered. By translation we see that each gVH is evenly covered. Hence q is a covering map.

<u>Corollary</u>. (a) If G/H is simply connected, then H is connected.

(b) If H is discrete, then the quotient map of G onto G/H is a covering map.

Proof. (a) If G/H is simply connected, then $q : G/H_o \to G/H$ cannot be a nontrivial covering because the diagram



says that f exists and is q^{-1}. So $H_o = H$, and H is connected.

(b) This is the special case of Proposition 11b in which $H_o = \{1\}$.

<u>Proposition 12</u>. Let G be simply connected and let H be a discrete subgroup of G, so that $p : G \to G/H$ is a covering map. Then the group of deck transformations of G is exactly the group of right translations in G by members of H. Consequently $\pi(G/H, 1\cdot H)$ is canonically isomorphic with H.

Proof. Let $f_h(g) = gh$. Then $pf_h(g) = ghH = gH = p(g)$, so that $pf_h = p$ and f_h is a deck transformation. Since $p^{-1}(1\cdot H) = H$, the group of translations f_h is simply transitive on $p^{-1}(1\cdot H)$ and by Theorem 4 is the full group of deck transformations. Again by Theorem 4, $\pi(G/H, 1\cdot H) \cong H$.

<u>Proposition 13</u>. Let G be locally simply connected, let $\tilde{G}$ be the universal covering space with covering map $e : \tilde{G} \to G$, and let $\tilde{1}$ be in $e^{-1}(1)$. Then there exists a unique multiplication on $\tilde{G}$ that makes $\tilde{G}$ into a topological group in such a way that e is a group homomorphism.

Proof. Let $m : G \times G \to G$ be multiplication, and let $\varphi : \tilde{G} \times \tilde{G} \to G$ be the composition $m \circ (e,e)$. Since

$$\{1\} = \varphi_*(\pi(\tilde{G} \times \tilde{G}, \tilde{1} \times \tilde{1})) \subseteq e_*(\pi(\tilde{G}, \tilde{1})),$$

there exists a unique continuous $\tilde{\varphi} : \tilde{G} \times \tilde{G} \to \tilde{G}$ such that $\varphi = e\tilde{\varphi}$ and $\tilde{\varphi}(\tilde{1},\tilde{1}) = \tilde{1}$. This $\tilde{\varphi}$ is the multiplication on $\tilde{G}$ (and it is the only possible candidate for the multiplication).

It is associative by unique lifting because

$$\begin{aligned} e\tilde{\varphi}(\tilde{\varphi}(x,y),z) &= \varphi(\tilde{\varphi}(x,y),z) = \varphi(x,y)(ez) = ((ex)(ey))ez \\ &= ex((ey)(ez)) = e\tilde{\varphi}(x,\tilde{\varphi}(y,z)) \end{aligned}$$

and $\tilde{\varphi}(\tilde{\varphi}(\tilde{1},\tilde{1}),\tilde{1}) = \tilde{1} = \tilde{\varphi}(\tilde{1},\tilde{\varphi}(\tilde{1},\tilde{1}))$. It has $\tilde{1}$ as identity because $\tilde{\varphi}(\tilde{1},\cdot)$ and $\tilde{\varphi}(\cdot,\tilde{1})$ cover the identity and send $\tilde{1}$ to $\tilde{1}$. To obtain existence of inverses, we lift the composition inversion $\circ\, e : \tilde{G} \to G$ to a map of $\tilde{G}$ to $\tilde{G}$ sending $\tilde{1}$ to $\tilde{1}$. Finally e is a group homomorphism because

$$e(\tilde{\varphi}(x,y)) = \varphi(x,y) = m(ex,ey).$$

The group $\tilde{G}$ of Proposition 13 is called the <u>universal covering group</u> of G.

In the homework, we shall obtain information about rotation groups $SO(n)$ and special unitary groups $SU(n)$ and their fundamental groups. The information will be obtained by induction by means of the following proposition.

<u>Proposition 14</u>. If G/H is simply connected and if G and H are locally simply connected, then $\pi(G,1)$ is isomorphic to a quotient group of $\pi(H,1)$.

Remarks: In the homework, we shall see that the fundamental group of a group is necessarily abelian.

Proof. Let $\tilde{G}$ be the universal covering group of G and let $e : \tilde{G} \to G$ be the covering homormorphism. Set $\tilde{H} = e^{-1}(H)$. Define $e_o : \tilde{G}/\tilde{H} \to G/H$ by $e_o(\tilde{g}\tilde{H}) = e(g)H$. Then e_o is well defined, one-one, and onto. An open set in $\tilde{G}/\tilde{H}$ is mapped under e_o to its preimage in $\tilde{G}$, to its image in G, and to its image in G/H. So e_o is open. Similarly e_o^{-1} is open, and so e_o is a homeomorphism. Therefore $\tilde{G}/\tilde{H}$ is simply connected.

We claim that $\tilde{H}$ is locally pathwise connected. In fact, if U is a connected open neighborhood of $\tilde{1}$ in $\tilde{G}$ mapped homeomorphically by e, then $e(U) \cap H$ contains a relatively open pathwise connected neighborhood V of 1 since H is locally pathwise connected. Then $U \cap e^{-1}(V)$ is homeomorphic with V and is the required neighborhood of 1 in $\tilde{H}$.

By (a) of the Corollary to Proposition 11, $\tilde{H}$ is connected. By (b) of the Corollary, the map $e|_{\tilde{H}} : \tilde{H} \to H$ is a covering map. Now by Proposition 11, $\pi(G,1) \cong \ker e = \ker e|_{\tilde{H}}$. Hence the result follows from the following lemma applied to H.

<u>Lemma</u>. If G is locally simply connected and $e : G' \to G$ is a covering homomorphism and $\tilde{e} : \tilde{G} \to G'$ is the universal covering group and homomorphism, then $\ker e \cong \ker e\tilde{e}/\ker \tilde{e}$.

Proof. The map of $\ker e\tilde{e}/\ker \tilde{e}$ to $\ker e$ is induced by $\tilde{e}$. Namely if $\tilde{g}$ is in $\ker e\tilde{e}$, then $\tilde{e}(\tilde{g}) \in \ker e$ and $\tilde{g}\tilde{k} \in \ker e\tilde{e}$ for $\tilde{k} \in \ker \tilde{e}$ map to the same member of $\ker e$ under $\tilde{e}$. Since $\tilde{e}$ is a homomorphism, so is the induced map. The map is onto because if k is in $\ker e$ and $\tilde{k}$ is in $\tilde{e}^{-1}(k)$, then $\tilde{e}(\tilde{k}) = k$. To see it is one-one, let $\tilde{e}(g) = 1$ in $\ker e$. Then g is in $\ker \tilde{e}$; that is, g is trivial in $\ker e\tilde{e}/\ker \tilde{e}$. So the map is an isomorphism onto.