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Szegö Kernels Associated with Discrete Series

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1. Introduction

In [7] Harish-Chandra gave a parametrization of the discrete series of a connected semisimple Lie group G with finite center. For each discrete series representation we shall give an integral formula that provides an explicit quotient mapping from a suitable nonunitary principal series representation, realized as a space of functions on a maximal compact subgroup K, onto a concrete analytic realization of the discrete series [21, 22]. By duality one can obtain an explicit imbedding of the discrete series representation as a subrepresentation in the nonunitary principal series.

The simplest realization of the discrete series is as the space of square-integrable smooth functions on G that satisfy a transformation law on one side under K and are annihilated by an appropriate first-order elliptic differential operator. We shall use the operator $\mathscr{D}$ introduced in [18, 19] as the first-order operator in question (see § 2). Since the kernel of $\mathscr{D}$ is contained in the kernels of some more familiar operators, such as the pair $\bar{\partial}$ and $\bar{\partial}^*$ and the Dirac operator (see § 3), our quotient mapping can be regarded as an integral formula carrying functions on K to solutions of a familiar first-order elliptic system.

It is for this reason that we refer to the kernel in the integral formula as a Szegö kernel. In fact, it is possible to arrange our parameters in a limiting case so that our kernel is indeed the classical Szegö kernel for the unit ball in $\mathbf{C}^n$, carrying functions on the boundary to holomorphic functions in the interior.

We shall state our main result more precisely, referring to later sections for some of the definitions. By [7] G has a discrete series if and only if rank $G=$ rank K. Thus we may assume that G has a compact Cartan subgroup $T\subseteq K$. For this section we shall assume also, possibly by passing to a double covering of G, that G is acceptable in the sense of [7]. To each nonsingular integral form Λ on the Lie algebra of T, Harish-Chandra associates in Theorem 2 of [6] an

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invariant eigendistribution Θ_Λ, and he proves in [7] the existence of a discrete series representation π_Λ on a space H^Λ with character $\pm\Theta_\Lambda$. These representations exhaust the discrete series, and two such are equivalent if and only if their parameters Λ are conjugate under the Weyl group of K.

Let $\mathfrak{g}$ be the Lie algebra of G. Given Λ, let the positive roots be those for which $\langle\Lambda,\alpha\rangle>0$, and select a fundamental sequence $\alpha_1,\ldots,\alpha_m$ of positive noncompact roots (see Definition 4.1). This sequence determines an Iwasawa decomposition $G=ANK$, and we let M be the centralizer of A in K. Let $\lambda=\Lambda+\delta_n-\delta_k$ be the lowest K-type in π_Λ, and let τ_λ be an irreducible representation of K with highest weight λ, with representation space V_λ, and with nonzero highest weight vector ϕ_λ. Let σ_λ be the representation of M obtained by restricting $\tau_\lambda(M)$ to the M-cyclic subspace H_λ of V_λ generated by ϕ_λ. Then σ_λ is irreducible (Proposition 5.5). Define

$$C^\infty(K,\sigma_\lambda)=\{f\in C^\infty(K,H_\lambda)\mid f(mk)=\sigma_\lambda(m)f(k),\ m\in M,\ k\in K\},$$

$$C^\infty(G,\tau_\lambda)=\{F\in C^\infty(G,V_\lambda)\mid F(kg)=\tau_\lambda(k)F(g),\ k\in K,\ g\in G\}.$$

The Lie algebra of A has basis $E_{\alpha_j}+E_{-\alpha_j}$ with normalizations as in § 2, and we let $\nu=\nu(\lambda)$ be the linear functional determined by

$$\nu(E_{\alpha_j}+E_{-\alpha_j})=\frac{2\langle\lambda+n_j\alpha_j,\alpha_j\rangle}{\langle\alpha_j,\alpha_j\rangle}$$

where n_j is the integer defined in (6.5b). (This expression will be interpreted in Proposition 8.2.) Extend the functions in $C^\infty(K,\sigma_\lambda)$ to be the smooth functions in the representation space of the nonunitary principal series representation $W(\sigma_\lambda,2\rho^+-\nu)$, as in § 6.

(1.1) **Theorem.** *With $\lambda=\Lambda+\delta_n-\delta_k$ integral and with Λ nonsingular and G-dominant, the operator*

$$S(f)(x)=\int_K\tau_\lambda(k)^{-1}f(kx)\,dk=\int_K e^{\nu H(lx^{-1})}\tau_\lambda(\kappa(lx^{-1}))^{-1}f(l)\,dl$$

carries $C^\infty(K,\sigma_\lambda)$ into the kernel of the operator $\mathcal{D}$ on $C^\infty(G,\tau_\lambda)$, and, under the identification of $C^\infty(K,\sigma_\lambda)$ with the space of the nonunitary principal series $W(\sigma_\lambda,2\rho^+-\nu)$, it carries the K-finite vectors of $W(\sigma_\lambda,2\rho^+-\nu)$ in a $\mathfrak{g}$-equivariant fashion onto the K-finite vectors of the discrete series π_Λ.

The authors came to work on this problem from different directions, starting with [13] and [22]. The work of the first author grew out of an attempt with E. M. Stein to obtain explicit solutions to the $\bar\partial$ and $\bar\partial^*$ system in the case that G/K is Hermitian symmetric.

Often a discrete series representation appears as a quotient of more than one nonunitary principal series representation. Indeed, Theorem 1.1 can produce more than one quotient mapping for a given discrete series; a particular mapping is determined once the fundamental system is fixed. However, the theorem does not provide all quotient mappings in $SU(2,1)$, for example, for the class of discrete series that have three quotient mappings, since there are only two possibilities for the fundamental system in this case.

We have attempted to minimize the number of deep results about discrete series that we use in this paper. The proof that $\mathcal{D}$ annihilates the image of the integral operator is elementary and occupies §§ 4–7. To identify the image with the discrete series requires the deeper theorems and is done in two stages. First, in § 9, we assume the parameter Λ is "far from the walls." In this case we use three facts, which will be stated more precisely in § 9:

(1) (Schmid [19], Hotta-Parthasarathy [10]). Far from the walls the dimension of the space of C^∞ solutions of $\mathscr{D}F=0$ of a given K-type is bounded above by the Blattner multiplicity.

(2) (Hecht-Schmid [8], Enright [3]). Far from the walls the discrete series satisfies the Blattner conjecture.

(3) (Schmid [19], Hotta-Parthasarathy [10]). Far from the walls the L^2 solutions of $\mathscr{D}F=0$ give a realization of the discrete series π_Λ.

The second stage is to pass to the remaining parameters by means of tensor products with finite-dimensional representations and suitable projections. The idea that problems about discrete series could be handled by this approach is due to Zuckerman and is based on a key lemma of his [25]. The corresponding machinery that we need about nonunitary principal series is based on [16] and is assembled in § 10. The actual proof in the second stage of the argument is carried out in Proposition 10.7 and Theorem 10.8.

The Szegö kernel in Theorem 1.1 is defined under the more general assumption that τ_λ makes sense. Under slightly wider conditions than in Theorem 1.1, conditions that are made precise in Theorem 10.8, the image of the Szegö kernel is still irreducible. In the limiting cases, one obtains so-called "limits of discrete series." These representations will be used in § 12 to exhibit all the reducibility that occurs in the unitary principal series of a group of real-rank one.

We should mention that Casselman [2] has announced an abstract subrepresentation theorem for a much wider class of representations than discrete series. His argument is based on asymptotic expansions and does not give values for the parameters of the imbedding.

We have learned that Schmid has independently obtained explicit imbeddings of discrete series representations in representations induced from suitable maximal parabolic subgroups. His work is based on [20]. Schmid has informed us that iteration of his result leads to our formulas.

2. The Operator $\mathscr{D}$

The following notation will be in force for §§ 2–10. We let G be a connected semisimple Lie group with finite center and fix a Cartan decomposition $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ of the Lie algebra. We assume that rank $\mathfrak{g}=$ rank $\mathfrak{k}$. The Cartan involution is denoted θ, and bar denotes conjugation of $\mathfrak{g}^{\mathbf{C}}$ with respect to $\mathfrak{g}$. Then $X\to\theta\bar{X}$ is the conjugation of $\mathfrak{g}^{\mathbf{C}}$ with respect to the compact form $\mathfrak{k}\oplus i\mathfrak{p}$.

Let $\mathfrak{t}\subseteq\mathfrak{k}$ be a compact Cartan subalgebra. Let Δ be the set of roots of $(\mathfrak{g}^{\mathbf{C}}, \mathfrak{t}^{\mathbf{C}})$, and let Δ_k and Δ_n be the sets of compact and noncompact roots, respectively. The Weyl groups of Δ and Δ_k are W and W_K.

We shall need to make computations with root vectors, and we fix a normalization of them. Namely by [9, pp. 155–156] we can select root vectors E_α in such a way that

$$B(E_\alpha, E_{-\alpha}) = 2/\langle\alpha, \alpha\rangle, \tag{2.1}$$

where B is the Killing form, and

$$\theta \bar{E}_\alpha = -E_{-\alpha}. \tag{2.2}$$

Then it follows that H_α defined by

$$H_\alpha = [E_\alpha, E_{-\alpha}] \tag{2.3}$$

satisfies $\alpha(H_\alpha) = 2$ and that

$E_\alpha + E_{-\alpha}$, $i(E_\alpha - E_{-\alpha})$ are in $\mathfrak{g}$ if α is noncompact,

$E_\alpha - E_{-\alpha}$, $i(E_\alpha + E_{-\alpha})$ are in $\mathfrak{g}$ if α is compact.

The Hermitian form

$$\langle U, V\rangle = -B(U, \theta \bar{V}) \tag{2.4}$$

is a positive definite inner product on $\mathfrak{g}^{\mathbf{C}}$.

For functions on G, we use vector field notation for differentiation, letting $Xf(g) = \frac{d}{dt} f((\exp tX)^{-1} g)|_{t=0}$ if X is in $\mathfrak{g}$. If X and Y are in $\mathfrak{g}$ and $Z = X + iY$, let $Zf = Xf + iYf$. Then

$$Z\bar{f} = \overline{\bar{Z}f}. \tag{2.5}$$

After we introduce a notion of positivity on the roots of $(\mathfrak{g}^{\mathbf{C}}, t^{\mathbf{C}})$, we let Δ^+, Δ_k^+, and Δ_n^+ be the obvious sets of positive roots. Define δ to be half the sum of the positive roots, δ_k to be half the sum of the positive compact roots, and δ_n to be $\delta - \delta_k$. Let K and T be the analytic subgroups corresponding to $\mathfrak{k}$ and $\mathfrak{t}$. The *integral forms* on $\mathfrak{t}^{\mathbf{C}}$ are those that lift to T. As mentioned in § 1, the discrete series of G is parametrized, according to [7], by a set of nonsingular forms Λ, modulo the action of W_K. It will be important to note that this parametrization is valid without reference to positivity of roots; then in considering a particular parameter we can introduce a positive system Δ^+ to suit our convenience.

Let λ be an integral form on $\mathfrak{t}^{\mathbf{C}}$, and fix a choice of the positive system Δ^+ that makes λ dominant with respect to Δ_k^+. Let τ_λ be an irreducible representation of K with highest weight λ, and let V_λ be the representation space.

With such a λ fixed, we introduce, following Schmid [19], the differential operator $\mathscr{D}$ with which we shall work. First let $C^\infty(G, \tau_\lambda)$ be the space of all C^∞ functions $F\colon G \to V_\lambda$ such that $F(kg) = \tau_\lambda(k) F(g)$ for all k in K and g in G. Next, since the K-representation $\mathrm{Ad}(K)|_{\mathfrak{p}^{\mathbf{C}}}$ has all weights of multiplicity one, it follows that we have a decomposition

$$\tau_\lambda \otimes \mathrm{Ad}|_{\mathfrak{p}^{\mathbf{C}}} = \sum_{\beta \in \Delta_n} m_\beta \tau_{\lambda+\beta}$$

with each m_β equal to 0 or 1. Let τ_λ^- be the subrepresentation of this tensor product given by

$$\tau_\lambda^- = \sum_{\beta \in \Delta_n^+} m_{-\beta} \tau_{\lambda-\beta}. \tag{2.6}$$

Then τ_λ^- acts in a subspace V_λ^- of $V_\lambda \otimes \mathfrak{p}^{\mathbf{C}}$, and we let

$$P\colon V_\lambda \otimes \mathfrak{p}^{\mathbf{C}} \to V_\lambda^- \tag{2.7}$$

be the orthogonal projection. Note that

$$\tau_\lambda^-(k) P(X) = P((\tau_\lambda(k) \otimes \mathrm{Ad}(k)) X) \tag{2.8}$$

for all X in $V_\lambda \otimes \mathfrak{p}^{\mathbf{C}}$ and that $P(X)=0$ if X is a weight vector whose weight is not of the form $\lambda - Q$ with Q a nonempty sum of positive roots.

The differential operator $\mathscr{D}$ carries $C^\infty(G, \tau_\lambda)$ into $C^\infty(G, \tau_\lambda^-)$ and is given by

$$\mathscr{D} f(g) = \sum_{i=1}^{2n} P(X_i f(g) \otimes \bar{X}_i), \tag{2.9}$$

where $X_1, \ldots, X_{2n}$ is an orthonormal basis of $\mathfrak{p}^{\mathbf{C}}$ with respect to the inner product (2.4). The operator $\mathscr{D}$ is independent of the choice of basis and is equivariant with respect to right translation by G on $C^\infty(G, \tau_\lambda)$ and $C^\infty(G, \tau_\lambda^-)$.

We shall use both the general formula (2.9) and the specialized version obtained by using the particular orthonormal basis $(\frac{1}{2}|\beta|^2)^{1/2} E_\beta$, $\beta \in \Delta_n$. For this basis the formula is

$$\mathscr{D} f(g) = \sum_{\beta \in \Delta_n} \tfrac{1}{2} |\beta|^2 P(E_\beta f(g) \otimes E_{-\beta}). \tag{2.10}$$

3. Other Possible Differential Operators

The operator $\mathscr{D}$ is particularly adapted to studying discrete series. However, some other first-order differential operators with geometric interpretations quite apart from group representations have been used in realizing discrete series. Since we are going to give an integral formula that produces functions in the kernel of $\mathscr{D}$, some discussion of the relationships among these operators is in order. We shall apply the corollary of this section in § 10.

Other operators that have been used are the complex $\bar{\partial}$ and $\bar{\partial}^*$, the de Rham d and d^*, and the Dirac D. Our point is that $\mathscr{D}$ is more primitive than all of these in the sense that the kernel of $\mathscr{D}$, suitably interpreted, is contained in the kernels of the other operators.

Schmid has pointed out to us that the realizability of discrete series in the kernels of such operators is not so much a property of differential operators as it is an algebraic property of discrete series – specifically the lowest K-type result. Thus the sharpest results with differential operators should be expected to come from the operator that most closely mirrors lowest K-type properties in its definition; this operator is $\mathscr{D}$.

By way of illustration we shall relate $\mathscr{D}$ to the Dirac operator, used by Parthasarathy [17] and Hotta and Parthasarathy [10]. For this discussion we shall assume that δ_k and δ_n are integral; this integrality can be achieved by passing to a covering group. Then the representation $\mathrm{Ad}(K)|_{\mathfrak{p}^{\mathbf{C}}} \subseteq SO(\mathfrak{p}^{\mathbf{C}})$ on $\mathfrak{p}^{\mathbf{C}}$ lifts to a representation

$$L(K) \subseteq \mathrm{Spin}(\mathfrak{p}^{\mathbf{C}})$$

on the Clifford algebra of $\mathfrak{p}^{\mathbb{C}}$ and then restricts to a representation L_0 of K on a spin module $\mathscr{H}$ that is invariant and irreducible under Clifford multiplication Cliff on the left. The representation L_0 is not irreducible, but its highest weight is δ_n and has multiplicity one, and the other weights are of the form $\delta_n - Q$ with Q a sum of distinct members of Δ_n^+.

Suppose that λ is integral and that $\lambda-\delta_n$ (and not merely λ) is dominant with respect to K. Then we have a canonical inclusion

$$\tau_\lambda \subseteq \tau_{\lambda-\delta_n} \otimes L_0$$

since L_0 has a highest weight vector with weight δ_n, and we can regard $C^\infty(G, \tau_\lambda)$ as contained in $C^\infty(G, \tau_{\lambda-\delta_n} \otimes L_0)$. The Dirac operator

$$D\colon C^\infty(G, \tau_{\lambda-\delta_n} \otimes L_0) \to C^\infty(G, \tau_{\lambda-\delta_n} \otimes L_0)$$

is given by

$$Df(g) = \sum_{i=1}^{2n} (I \otimes \mathrm{Cliff}(X_i))\, \bar{X}_i f(g),$$

where $X_1, \ldots, X_{2n}$ is an orthonormal basis of $\mathfrak{p}^{\mathbb{C}}$. Thus we can regard the domain of $\mathscr{D}$ as included in the domain of D.

(3.1) **Proposition.** *If f in $C^\infty(G, \tau_\lambda)$ satisfies $\mathscr{D} f = 0$, then $Df = 0$.*

Proof. Write $\tau_\lambda \otimes \mathrm{Ad}|_{\mathfrak{p}^{\mathbb{C}}} = \tau_\lambda^- \oplus \tau_\lambda^+$, and let P and P^+ be the orthogonal projections on the two constituent spaces V_λ^- and V_λ^+. Define

$$D_1 f(g) = \sum X_i f(g) \otimes \bar{X}_i,$$

so that $D_1 f(g)$ is in $V_\lambda \otimes \mathfrak{p}^{\mathbb{C}} \subseteq V_{\lambda-\delta_n} \otimes L_0 \otimes \mathfrak{p}^{\mathbb{C}}$. Let D_2 be the mapping of $V_{\lambda-\delta_n} \otimes (L_0 \otimes \mathfrak{p}^{\mathbb{C}})$ into $V_{\lambda-\delta_n} \otimes L_0$ given by $I \otimes \mathrm{Cliff}$. Then $\mathscr{D} f(g) = P D_1 f(g)$ and $Df(g) = D_2 D_1 f(g)$. We claim $D_2 P^+ = 0$. If so, then

$$D_1 f(g) = \mathscr{D} f(g) + P^+ D_1 f(g)$$

and

$$Df(g) = D_2 D_1 f(g) = D_2 \mathscr{D} f(g) + D_2 P^+ D_1 f(g) = D_2 \mathscr{D} f(g),$$

and the proposition follows. To see that $D_2 P^+ = 0$, simply observe that D_2 and $P^+$are K-equivariant, that the highest weights of V_λ^+ are of the form $\lambda + \beta$ with β in Δ_n^+, and that all weights of the target space $V_{\lambda-\delta_n} \otimes L_0$ are of the form $\lambda - Q$ with Q a sum of positive roots.

(3.2) **Corollary.** *If Ω is the Casimir operator, if f in $C^\infty(G, \tau_\lambda)$ is in the kernel of $\mathscr{D}$, and if $\lambda - \delta_n$ is dominant with respect to K, then*

$$\Omega f = (|\Lambda|^2 - |\delta|^2) f,$$

where $\Lambda = \lambda + \delta_k - \delta_n$.

Proof. Under the hypothesis on λ, Parthasarathy [17, p. 16] shows

$$-D^2 = \Omega - (|\Lambda|^2 - |\delta|^2) I \tag{3.1}$$

on $C^\infty(G, \tau_{\lambda-\delta_n} \otimes L_0)$. If f is in $C^\infty(G, \tau_\lambda)$, then (3.1) applies to f. By Proposition 3.1, $\mathscr{D} f = 0$ implies $D^2 f = 0$. The result follows.

4. Construction of Orthogonal Roots

We work with the roots defined relative to the compact Cartan subalgebra of § 2 and with a fixed notion of positive roots. Let $\alpha_1, \ldots, \alpha_m$ be a sequence of strongly orthogonal positive noncompact roots. (That is, no $\alpha_i \pm \alpha_j$ is a root.) The space $\sum_{j=1}^{m} \mathbf{R}(E_{\alpha_j} + E_{-\alpha_j})$ is an abelian subspace of $\mathfrak{p}$. If it is maximal abelian, then the sequence $\alpha_1, \ldots, \alpha_m$ is maximal (but not always conversely). In any case if $\alpha_1, \ldots, \alpha_m$ is maximal, then we can define a function $\gamma \to \alpha(\gamma)$ carrying Δ_n into $\{\alpha_1, \ldots, \alpha_m\}$ by this rule: $\alpha(\gamma)$ is the first α_j such that γ is not strongly orthogonal to α_j.

(4.1) *Definition.* A sequence $\alpha_1, \ldots, \alpha_m$ of positive noncompact roots is a *fundamental sequence* if

(1) the α_j form a strongly orthogonal set,

(2) $\mathfrak{a} = \sum_{j=1}^{m} \mathbf{R}(E_{\alpha_j} + E_{-\alpha_j})$ is maximal abelian in $\mathfrak{p}$,

(3) α_j is a simple root in the subsystem of roots strongly orthogonal to $\alpha_1, \ldots, \alpha_{j-1}$,

(4) for each γ in Δ_n^+ either

(a) $|\alpha(\gamma)| \geqq |\gamma|$, or

(b) $|\alpha(\gamma)| < |\gamma|$ and $\gamma - 3\alpha(\gamma)$ is a root.

By (1) and (2) and the remarks above, $\alpha(\gamma)$ is well defined. Thus requirement (4) is meaningful.

Properties (1) and (2) will allow us in the next section to define an Iwasawa decomposition with the aid of $\alpha_1, \ldots, \alpha_m$, and property (3) should be regarded as a compatibility condition between the ordering of Δ and the ordering of the yet-to-be defined restricted roots. For later use we isolate a consequence of property (3):

$$\gamma > 0 \quad \text{and} \quad \gamma \neq \alpha(\gamma) \quad \text{and} \quad \gamma - n\alpha(\gamma) \in \Delta \qquad \text{imply } \gamma - n\alpha(\gamma) > 0. \tag{4.1}$$

Property (4) accomplishes several things. Its main effect is to make the explicit computation of the Iwasawa decomposition easy on the Lie algebra level. The next lemma shows how it allows us also to control compact roots that are orthogonal to $\alpha_1, \ldots, \alpha_m$. The option of (4b) is introduced to ensure the existence of fundamental sequences, proved below. If we insisted on (4a) always, then the group split G_2 with the short simple root noncompact and the long simple root compact would admit no fundamental sequence. Conversely if $\gamma > 0$ satisfies (4b), then $\alpha(\gamma)$ and $\gamma - 3\alpha(\gamma)$ are the simple roots of a split G_2 factor of $\mathfrak{g}$, $\alpha(\gamma)$ is short and noncompact, and $\gamma - 3\alpha(\gamma)$ is long and compact.

(4.2) **Lemma.** *If β is a compact root orthogonal to each root $\alpha_1, \dots, \alpha_k$ in an initial segment of a fundamental sequence $\alpha_1, \dots, \alpha_m$, then β is strongly orthogonal to $\alpha_1, \dots, \alpha_k$.*

Proof. Assuming the contrary, let $j \leqq k$ be the least index such that β is not strongly orthogonal to α_j. Then $\beta + \alpha_j$ is a noncompact root. Moreover, $\alpha(\beta + \alpha_j) = \alpha_j$. [In fact, if $\beta + \alpha_j + \alpha_i$ is a root, then $\langle \beta + \alpha_j + \alpha_i, \beta \rangle > 0$ says $\alpha_j + \alpha_i$ is a root, contradicting (1).] Since β is orthogonal to α_j,

$$|\beta + \alpha_j| > |\alpha_j| = |\alpha(\beta + \alpha_j)|,$$

and (4a) fails for $\beta + \alpha_j$. Then (4b) holds and β and α_j are orthogonal roots in a common simple factor of type G_2. Since orthogonality implies strong orthogonality in G_2 and $\beta + \alpha_j$ is a root, we have reached a contradiction.

We turn to the question of existence of fundamental sequences. Let $\varepsilon_1, \dots, \varepsilon_n$ be the simple roots of Δ. We begin with a naive attempt at constructing a fundamental sequence. Namely fix a lexicographic ordering $\mathcal{O}$ yielding $\varepsilon_1, \dots, \varepsilon_n$ as simple roots. Define, relative to $\mathcal{O}$,

α_1 = smallest root in Δ_n^+,
α_2 = smallest root in Δ_n^+ strongly orthogonal to α_1,
α_3 = smallest root in Δ_n^+ strongly orthogonal to α_1, α_2
etc.

Call $\alpha_1, \dots, \alpha_m$ the *sequence associated to* $\mathcal{O}$. Properties (1) and (3) in Definition 4.1 are trivially valid. Since $\alpha_1, \dots, \alpha_m$ is clearly maximal, the function $\alpha(\gamma)$ is defined. We shall prove that (4a) implies (2).

(4.3) **Lemma.** *If $\alpha_1, \dots, \alpha_m$ is the sequence associated to $\mathcal{O}$ and if $|\alpha(\gamma)| \geqq |\gamma|$ for all γ in Δ_n, then $\mathfrak{a} = \sum_{j=1}^{m} \mathbf{R}(E_{\alpha_j} + E_{-\alpha_j})$ is maximal abelian in $\mathfrak{p}$.*

Proof. Suppose

$$X = \sum_{\beta \in \Delta_n^+} a_\beta (E_\beta + E_{-\beta}) + \sum_{\beta \in \Delta_n^+} b_\beta i (E_\beta - E_{-\beta}) \qquad (a_\beta, b_\beta \text{ real})$$

centralizes $\mathfrak{a}$. We may assume $a_{\alpha_j} = 0$ for $1 \leqq j \leqq m$ without loss of generality. We form

$$\begin{aligned} 0 = [X, E_{\alpha_j} + E_{-\alpha_j}] &= \sum (\text{root vectors}) + i b_{\alpha_j}([E_{\alpha_j}, E_{-\alpha_j}] - [E_{-\alpha_j}, E_{\alpha_j}]) \\ &= \sum (\text{root vectors}) + 2 i b_{\alpha_j} H_{\alpha_j}. \end{aligned}$$

Thus $b_{\alpha_j} = 0$, $1 \leqq j \leqq m$. Changing notation, we then have

$$X = \sum_{\substack{\beta \in \Delta_n \\ \beta \neq \pm \alpha_j}} c_\beta E_\beta \qquad (c_\beta \text{ complex}).$$

Let j be the least index such that $\alpha(\beta) = \alpha_j$ and $c_\beta \neq 0$ for some β. Form $[X, E_{\alpha_j} + E_{-\alpha_j}]$, suppose $\beta + \alpha_j$ is a root, and consider the term containing $E_{\beta + \alpha_j}$. It is

$$c_\beta [E_\beta, E_{\alpha_j}] + c_{\beta'} [E_{\beta'}, E_{-\alpha_j}] = 0, \qquad \text{where } \beta + \alpha_j = \beta' - \alpha_j.$$

If $\beta+\alpha_j$ and β' are both roots, then $\beta, \beta+\alpha_j, \beta+2\alpha_j=\beta'$ is a root string. Since $\beta+\alpha_j \neq 0$ and since β or β' must be the end of the full string, we obtain $|\alpha_j|<|\beta|$ or $|\alpha_j|<|\beta'|$. Since $\alpha(\beta)=\alpha(\beta')=\alpha_j$, the result is a contradiction to the inequality $|\alpha(\gamma)| \geqq |\gamma|$ for all γ. We conclude that if $\beta+\alpha_j$ is a root, then β' is not a root. Hence our equation reads $c_\beta[E_\beta, E_{\alpha_j}]=0$, and we conclude $c_\beta=0$.

Thus we may assume $\beta-\alpha_j$ is a root and consider the term containing $E_{\beta-\alpha_j}$, which is

$$c_\beta[E_\beta, E_{-\alpha_j}]+c_{\beta''}[E_{\beta''}, E_{\alpha_j}]=0, \qquad \text{where } \beta-\alpha_j=\beta''+\alpha_j.$$

If $\beta-\alpha_j$, and β'' both exist, then we argue as above with the root string $\beta, \beta-\alpha_j, \beta-2\alpha_j$. Thus β'' is not a root and we conclude $c_\beta=0$. This proves the lemma.

(4.4) **Lemma.** *Unless* $\mathfrak{g}$ *has a factor whose complexification is* F_4 *or* G_2, *every system of simple roots* $\varepsilon_1, \ldots, \varepsilon_n$ *for* Δ *admits a lexicographic ordering* $\mathcal{O}$ *whose associated sequence satisfies* $|\alpha(\gamma)| \geqq |\gamma|$ *for all* γ *in* Δ_n.

Proof. We may assume $\mathfrak{g}$ is simple, and we investigate the various complex Dynkin diagrams separately. For the single-line diagrams, any $\mathcal{O}$ will work trivially. For example, use $\varepsilon_1, \ldots, \varepsilon_n$ as a basis to define the ordering $\mathcal{O}$. In view of our hypotheses, we are left with B_n and C_n.

Case C_n. List the simple roots in the order

$$e_1-e_2, e_2-e_3, \ldots, e_{n-1}-e_n, 2e_n,$$

and let $\mathcal{O}$ be the resulting lexicographic ordering. We claim $|\alpha(\gamma)| \geqq |\gamma|$ for all noncompact γ. There are two subcases. First, if some $2e_i$ is noncompact, then $2e_j$ differs from $2e_i$ by twice a root and hence is noncompact. The sequence associated to $\mathcal{O}$ is then $2e_n, 2e_{n-1}, \ldots, 2e_1$ and $|\alpha(\gamma)| \geqq |\gamma|$ clearly. Second, if no $2e_i$ is noncompact, then all the noncompact roots have the same length, and $|\alpha(\gamma)| \geqq |\gamma|$ trivially.

Case B_n. List the simple roots in the order

$$e_n, e_1-e_2, e_2-e_3, \ldots, e_{n-1}-e_n,$$

and let $\mathcal{O}$ be the resulting lexicographic ordering. We claim $|\alpha(\gamma)| \geqq |\gamma|$ for all noncompact γ. First, e_i+e_j and e_i-e_j differ by twice a root and hence are both compact or else both noncompact. In the associated sequence $\alpha_1, \ldots, \alpha_m$, the first α's are of the form e_i-e_j. Once the roots of this kind stop, there can be no more such roots. Any root e_i+e_j after the first α's is such that e_i-e_j is already in the list, since otherwise e_i-e_j could have been adjoined to the list after the first α's. There can be at most one root e_k in the list since the e_k's are not strongly orthogonal. Also in our ordering any e_k is less than any e_i+e_j. Thus the list $\alpha_1, \ldots, \alpha_m$ is of the form

certain (e_i-e_j), possible e_k, corresponding (e_i+e_j).

The only problem that can occur is if $\alpha(\gamma)=e_k$ with γ long. Then we may assume $\gamma=e_k \pm e_{k'}$ is noncompact and $\alpha(\gamma)=e_k$. But $\alpha(e_k-e_{k'})=e_k$ implies $e_k-e_{k'}$ is strongly orthogonal to the e_i-e_j in our list and can be adjoined before e_k occurs in the list. This contradiction shows that $|\alpha(\gamma)| \geqq |\gamma|$ as required.

(4.5) **Proposition.** *Each system of simple roots $\varepsilon_1, \dots, \varepsilon_n$ for Δ admits at least one fundamental sequence of positive noncompact roots.*

Proof. We may assume $\mathfrak{g}$ is simple. In G_2 there are only three possibilities up to isomorphism, obtained by labeling some nonempty subset of simple roots as noncompact. If the long simple root is noncompact, choose it as α_1 and let α_2 be orthogonal; the result is a fundamental sequence with (4a) always valid. Otherwise choose α_1 to be the short simple root, and let α_2 be orthogonal; the result is a fundamental sequence that has (4b) holding for some γ. This handles G_2.

Since $\mathfrak{g}$ is simple and $\mathfrak{g}^{\mathbb{C}}=G_2$ has been considered, Lemmas 4.3 and 4.2 prove the proposition except when $\mathfrak{g}^{\mathbb{C}}=F_4$. Thus let $\mathfrak{g}^{\mathbb{C}}=F_4$. If all noncompact roots have the same length, the two lemmas again provide a fundamental sequence. Thus suppose there are noncompact roots of both lengths. Let the roots be $\pm e_i$, $\pm e_i \pm e_j$, $\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)$. Suppose, for definiteness, that e_1-e_2 or e_1+e_2 is noncompact. Then both are noncompact since

$$e_1+e_2=(e_1-e_2)+e_2+e_2,$$

and exactly one of $\frac{1}{2}(\pm(e_1-e_2)+e_3+e_4)$ is noncompact, since the difference is noncompact. But then the sum e_3+e_4 is noncompact, and e_3-e_4 must also be noncompact. Adjusting notation, we find as a result that either all the roots in the set

$$\{e_3-e_4, e_3+e_4, e_1-e_2, e_1+e_2\} \tag{4.2}$$

must be noncompact, or all the roots in the set

$$\{e_2-e_4, e_2+e_4, e_1-e_3, e_1+e_3\} \tag{4.3}$$

must be noncompact, or else all the roots in the set

$$\{e_2-e_3, e_2+e_3, e_1-e_4, e_1+e_4\} \tag{4.4}$$

must be noncompact. Now assume that the simple roots are

$$\tfrac{1}{2}(e_1-e_2-e_3-e_4),\ e_4,\ e_3-e_4,\ e_2-e_3.$$

If all the roots in (4.2) or in (4.4) are noncompact, easy computation shows that the sequence (4.2) or (4.4), respectively, satisfies (3) of Definition 4.1. Since it clearly satisfies (1) and (4), Lemma 4.2 shows it is a fundamental sequence. Finally our computations showed that the only remaining case has all roots in (4.2) and (4.4) compact and all roots in (4.3) noncompact. But this is impossible since then the noncompact e_2-e_4 would have to be the sum of the compact roots e_2-e_3 and e_3-e_4.

5. Iwasawa Decomposition in the Lie Algebra

Fix a fundamental sequence $\alpha_1, \dots, \alpha_m$ of positive noncompact roots, in the sense of Definition 4.1. Such a sequence exists, according to Proposition 4.5. We shall associate to this sequence a canonical Iwasawa decomposition of $\mathfrak{g}$, and we shall obtain explicit formulas for the projection operators. At the end of this section we shall prove a result about restricting representations of K to a subgroup M defined in terms of this Iwasawa decomposition.

Let $\mathfrak{a}$ be the maximal abelian subspace of $\mathfrak{p}$ given by

$$\mathfrak{a}=\sum_{j=1}^{m}\mathbf{R}(E_{\alpha_j}+E_{-\alpha_j}).$$

Form restricted roots with respect to $\mathfrak{a}$, and define an ordering on the restricted roots by means of the basis $E_{\alpha_1}+E_{-\alpha_1},\ldots,E_{\alpha_m}+E_{-\alpha_m}$. Let $\mathfrak{n}$ be the sum of the positive restricted-root spaces. Then $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{a}\oplus\mathfrak{n}$ is an Iwasawa decomposition of $\mathfrak{g}$, and we have a corresponding decomposition of the complexifications $\mathfrak{g}^{\mathbf{C}}=\mathfrak{k}^{\mathbf{C}}\oplus\mathfrak{a}^{\mathbf{C}}\oplus\mathfrak{n}^{\mathbf{C}}$. Let $P_{\mathfrak{k}}$ and $P_{\mathfrak{a}}$ be the projections of $\mathfrak{g}^{\mathbf{C}}$ on $\mathfrak{k}^{\mathbf{C}}$ and $\mathfrak{a}^{\mathbf{C}}$, respectively, defined by this decomposition.

Later we shall use the notation A and N for the analytic subgroups with Lie algebras $\mathfrak{a}$ and $\mathfrak{n}$, M for the centralizer of $\mathfrak{a}$ in K, and ρ^+ for half the sum of the positive restricted roots counted with multiplicities.

(5.1) **Lemma.** *If X in $\mathfrak{g}^{\mathbf{C}}$ has the property that*

$$[E_{\alpha_j}+E_{-\alpha_j},X]=\begin{cases}0 & \text{for } 1\leqq j\leqq k\\ cX & \text{for } j=k+1\end{cases}$$

with $c>0$, then X is in $\mathfrak{n}^{\mathbf{C}}$.

Proof. Let $\mathfrak{g}_\varphi$ be the complexified restricted root spaces. Write $X=\sum X_\varphi+X_0$ with $X_\varphi\in\mathfrak{g}_\varphi$ and $X_0\in\mathfrak{m}^{\mathbf{C}}\oplus\mathfrak{a}^{\mathbf{C}}$. Then

$$cX=[E_{\alpha_{k+1}}+E_{-\alpha_{k+1}},X]=\sum\varphi(E_{\alpha_{k+1}}+E_{-\alpha_{k+1}})X_\varphi.$$

Hence $X_0=0$ and $\varphi(E_{\alpha_{k+1}}+E_{-\alpha_{k+1}})=c$ whenever $X_\varphi\neq 0$. Similarly for $1\leqq j\leqq k$, $\varphi(E_{\alpha_j}+E_{-\alpha_j})=0$ whenever $X_\varphi\neq 0$. Thus X is in $\sum\mathfrak{g}_\varphi$ with the sum taken over those φ for which $\varphi(E_{\alpha_j}+E_{-\alpha_j})=0$, $1\leqq j\leqq k$, and $\varphi(E_{\alpha_{k+1}}+E_{-\alpha_{k+1}})=c$. Such restricted roots φ are positive, and thus X is in $\mathfrak{n}^{\mathbf{C}}$.

(5.2) **Proposition.** *Let the Iwasawa decomposition be defined by means of a fundamental sequence, and let β be a noncompact root. If $\beta=\pm\alpha(\beta)$, then*

$$P_{\mathfrak{a}}(E_\beta)=P_{\mathfrak{a}}(E_{-\beta})=\tfrac{1}{2}(E_\beta+E_{-\beta}),$$
$$P_{\mathfrak{k}}(E_\beta)=\tfrac{1}{2}H_\beta.$$

If $\beta\neq\pm\alpha(\beta)$, let $\alpha=\alpha(\beta)$ and let the α-string containing β be $\beta+n\alpha$, $-p\leqq n\leqq q$. There are two possibilities:

(i) *Every noncompact root γ with $\alpha(\gamma)=\alpha$ has $|\gamma|\leqq|\alpha|$. Then p and q are at most* 1 *and not both* 0, *and*

$$P_{\mathfrak{a}}(E_\beta)=P_{\mathfrak{a}}(E_{-\beta})=0,$$
$$P_{\mathfrak{k}}(E_\beta)=-\frac{1}{p+q}([E_{-\alpha},E_\beta]+[E_\alpha,E_\beta]).$$

(ii) *Some noncompact root $\gamma>0$ with $\alpha(\gamma)=\alpha$ has $|\gamma|>|\alpha|$, and $\gamma-3\alpha$ is a root.*

Then β *is one of* $\pm\gamma$ *or* $\pm(\gamma-2\alpha)$, *and*

$$P_{\mathfrak{a}}(E_\beta)=0$$

$$P_{\mathfrak{k}}(E_\beta)=\begin{cases} -\frac{1}{2}[E_\alpha, E_\beta]+\frac{1}{12}[E_\alpha,[E_\alpha,[E_\alpha,E_\beta]]] & \textit{if } \beta=-\gamma \\ -\frac{1}{2}[E_{-\alpha}, E_\beta]+\frac{1}{12}[E_{-\alpha},[E_{-\alpha},[E_{-\alpha},E_\beta]]] & \textit{if } \beta=\gamma \\ -\frac{1}{2}[E_\alpha, E_\beta]-\frac{1}{4}[E_{-\alpha},E_\beta] & \textit{if } \beta=-(\gamma-2\alpha) \\ -\frac{1}{2}[E_{-\alpha}, E_\beta]-\frac{1}{4}[E_{\alpha},E_\beta] & \textit{if } \beta=\gamma-2\alpha. \end{cases}$$

Proof. Since $P_{\mathfrak{a}}$ is an orthogonal projection, the expressions for $P_{\mathfrak{a}}(E_\beta)$ are immediate in all cases. However, $P_{\mathfrak{k}}$ is not an orthogonal projection. First suppose $\beta=\alpha(\beta)=\alpha_k$. This case can be handled by imbedding $\mathfrak{sl}(2,\mathbf{R})$ in $\mathfrak{g}$, but we shall do the computation directly to illustrate the general method. We have

$$[E_\beta+E_{-\beta}, E_\beta-\tfrac{1}{2}(E_\beta+E_{-\beta})-\tfrac{1}{2}H_\beta]=2(E_\beta-\tfrac{1}{2}(E_\beta+E_{-\beta})-\tfrac{1}{2}H_\beta) \tag{5.1}$$

and, for $1\leqq j\leqq k-1$,

$$[E_{\alpha_j}+E_{-\alpha_j}, E_\beta-\tfrac{1}{2}(E_\beta+E_{-\beta})-\tfrac{1}{2}H_\beta]=0. \tag{5.2}$$

By Lemma 5.1, $E_\beta-\frac{1}{2}(E_\beta+E_{-\beta})-\frac{1}{2}H_\beta$ is in $\mathfrak{n}^{\mathbf{C}}$. Hence

$$E_\beta=\{\tfrac{1}{2}H_\beta\}+\{\tfrac{1}{2}(E_\beta+E_{-\beta})\}+\{E_\beta-\tfrac{1}{2}(E_\beta+E_{-\beta})-\tfrac{1}{2}H_\beta\}$$

exhibits E_β as in $\mathfrak{k}^{\mathbf{C}}\oplus\mathfrak{a}^{\mathbf{C}}\oplus\mathfrak{n}^{\mathbf{C}}$ and must be the Iwasawa decomposition. That is, $P_{\mathfrak{k}}(E_\beta)=\frac{1}{2}H_\beta$.

Similarly if $\beta=-\alpha(\beta)=-\alpha_k$, equations (5.1) and (5.2) are still valid, and we again conclude $P_{\mathfrak{k}}(E_\beta)=\frac{1}{2}H_\beta$.

Now suppose $\alpha(\beta)\neq\pm\beta$. Write α for $\alpha(\beta)$, and suppose we are in case (i). Since $|\alpha(\beta)|\geqq|\beta|$, we have

$$p-q=\frac{2\langle\beta,\alpha\rangle}{|\alpha|^2}=-1 \text{ or } 0 \text{ or } 1.$$

Also $p+q\leqq 3$ in any case, and thus p and q are both $\leqq 1$ unless $p=1$ and $q=2$ or vice-versa. Say $p=1$ and $q=2$. Then $\beta+2\alpha$ is a noncompact root and $|\beta+2\alpha|>|\alpha|$. If we show that $\alpha(\beta+2\alpha)=\alpha$, then we have a contradiction to the assumption that we are in case (i). For previous α_j in the fundamental sequence, $\langle\beta+2\alpha,\alpha_j\rangle=0$. Thus $\alpha(\beta+2\alpha)=\alpha_j$ implies $\beta+2\alpha\pm\alpha_j$ are both roots. Then α, $\beta+2\alpha$, and $\beta+2\alpha+\alpha_j$ are roots of three distinct lengths, contradiction. Hence $\alpha(\beta+2\alpha)=\alpha$ and we conclude $p=1$ and $q=2$ is impossible. Similarly $p=2$ and $q=1$ is impossible. Thus p and q are both $\leqq 1$. They cannot both be 0 since $\alpha(\beta)=\alpha$ implies β and α are not strongly orthogonal.

To verify the formula for $P_{\mathfrak{k}}(E_\beta)$, we use the formula

$$[E_{-\alpha},[E_\alpha,E_\beta]]=q(p+1)E_\beta \tag{5.3a}$$

on p. 143 of [9] and its companion formula

$$[E_\alpha,[E_{-\alpha},E_\beta]]=p(q+1)E_\beta, \tag{5.3b}$$

both combined with the fact that p and q are $\leqq 1$. We simply compute

$$\left[E_\alpha+E_{-\alpha}, E_\beta+\frac{1}{p+q}([E_{-\alpha}, E_\beta]+[E_\alpha, E_\beta])\right]$$
$$=[E_{-\alpha}, E_\beta]+[E_\alpha, E_\beta]+\frac{1}{p+q}\{p(q+1)+q(p+1)\} E_\beta$$
$$=(p+q)\left(E_\beta+\frac{1}{p+q}([E_{-\alpha}, E_\beta]+[E_\alpha, E_\beta])\right) \tag{5.4}$$

since $p+2pq+q=p^2+2pq+q^2=(p+q)^2$ for our values of p and q. Moreover, if α_j precedes α in the fundamental sequence,

$$\left[E_{\alpha_j}+E_{-\alpha_j}, E_\beta+\frac{1}{p+q}([E_{-\alpha}, E_\beta]+[E_\alpha, E_\beta])\right]=0. \tag{5.5}$$

To show this, it is enough to show that α_j and $\beta\pm\alpha$ are strongly orthogonal. There are two cases. If $\beta\pm\alpha$ both are roots, then $\langle\beta, \alpha\rangle=0$. If, say, $\beta+\alpha+\alpha_j$ is a root, then $\langle\beta+\alpha+\alpha_j, \alpha\rangle=\langle\alpha, \alpha\rangle>0$ and $(\beta+\alpha+\alpha_j)-\alpha=\beta+\alpha_j$ is a root, contradicting $\alpha(\beta)=\alpha$. We can argue similarly for the other possibilities in this case. In the second case, only one of $\beta\pm\alpha$ is a root and $\langle\beta\pm\alpha, \alpha\rangle=-\langle\beta, \alpha\rangle\neq 0$. If, say, $\beta\pm\alpha+\alpha_j$ is a root, then $\langle\beta\pm\alpha+\alpha_j, \alpha\rangle=-\langle\beta, \alpha\rangle\neq 0$ and we find that $\beta+\alpha_j$ is a root, contradicting $\alpha(\beta)=\alpha$. We conclude that (5.5) is valid, and Lemma 5.1 then shows that

$$E_\beta+\frac{1}{p+q}([E_{-\alpha}, E_\beta]+[E_\alpha, E_\beta])$$

is in $\mathfrak{n}^{\mathbf{C}}$. Since $\beta\pm\alpha$ are compact roots, the correction term to E_β here is in $\mathfrak{k}^{\mathbf{C}}$ and we can argue as in the first half of the proof to complete the proof of the formula for $P_{\mathfrak{k}}(E_\beta)$.

By property (4) of a fundamental sequence, the only alternative to case (i) is case (ii). In this case α and $\gamma-3\alpha$ are the simple roots of a split G_2 factor of $\mathfrak{g}$, and thus β has to lie on the α-string through γ or through $-\gamma$. Therefore β is one of $\pm\gamma$ or $\pm(\gamma-2\alpha)$. To consider the two root strings of length 4 simultaneously, let δ be the smallest root on the string; δ is $-\gamma$ or $\gamma-3\alpha$. The α-string through δ is then $\delta, \delta+\alpha, \delta+2\alpha, \delta+3\alpha$. Lemma 5.1 shows that

$$aE_\delta+b[E_\alpha, E_\delta]+c[E_\alpha,[E_\alpha, E_\delta]]+d[E_\alpha,[E_\alpha,[E_\alpha, E_\delta]]]$$

is in $\mathfrak{n}^{\mathbf{C}}$ if

$$(a, b, c, d)=(3, 3, \tfrac{3}{2}, \tfrac{1}{2}) \quad \text{or} \quad (3, 1, -\tfrac{1}{2}, -\tfrac{1}{2})$$

and hence is in $\mathfrak{n}^{\mathbf{C}}$ if

$$(a, b, c, d)=\begin{cases}(0, 1, 1, \tfrac{1}{2}) & \text{or}\\ (-3, 0, \tfrac{3}{2}, 1) & \text{or}\\ (1, \tfrac{1}{2}, 0, -\tfrac{1}{12}) & \text{or}\\ (\tfrac{3}{2}, 1, \tfrac{1}{4}, 0).\end{cases} \tag{5.6}$$

Returning to β, whose α-string is $\beta+n\alpha$ with $-p\leqq n\leqq q$, apply (5.6) with $\delta=\beta-p\alpha$ and use (5.3). Then we get the result of the proposition.

(5.3) **Lemma.** *$M = M_0 F$, where M_0 is the identity component of M and F is a finite subgroup of both the compact torus T and the center of M.*

Proof. If G is a matrix group, this result is well known; F can be taken as $G \cap \exp i\mathfrak{a}$. In the general case G is a finite cover of a matrix group G_1, and we obtain $M = M_0 F$ with F the complete inverse image under the covering homomorphism of the group F_1 in G_1. Then $\operatorname{Ad}(F) = \operatorname{Ad}(F_1)$ is trivial on the Lie algebra of M; hence F centralizes M_0. Moreover F is contained in T since T is the inverse image of the torus in G_1. Since F is contained in T, F is abelian. Thus F is in the center of M. This proves the lemma.

If α is a noncompact root, the standard Cayley transform relative to α is $\operatorname{Ad}(u_\alpha)$, where

$$u_\alpha = \exp \frac{\pi}{4} (E_\alpha - E_{-\alpha}). \tag{5.7}$$

(5.4) **Lemma.** *Let α and β be orthogonal roots.*

(i) *If α and β are strongly orthogonal,* $\operatorname{Ad}(u_\alpha) E_\beta = E_\beta$.

(ii) *If α and β are not strongly orthogonal,* $\operatorname{Ad}(u_\alpha) E_\beta = \frac{1}{2}([E_\alpha, E_\beta] - [E_{-\alpha}, E_\beta])$.

Proof. In (i), every term in the exponential series is 0 but the first. In (ii), we must have $|\beta| = |\alpha|$ since otherwise α, β, and $\alpha + \beta$ would be roots of three different lengths. Then the α-string containing β is $\beta - \alpha, \beta, \beta + \alpha$. Applying (5.3) with $p = q = 1$, we have

$$(\operatorname{ad} E_{-\alpha})(\operatorname{ad} E_\alpha) E_\beta = (\operatorname{ad} E_\alpha)(\operatorname{ad} E_{-\alpha}) E_\beta = 2 E_\beta.$$

Thus

$$\operatorname{ad}^2 (E_\alpha - E_{-\alpha}) E_\beta = -4 E_\beta.$$

Then

$$\operatorname{Ad}(u_\alpha) E_\beta = (\cos \tfrac{\pi}{2}) E_\beta + \tfrac{1}{2} (\sin \tfrac{\pi}{2}) ([E_\alpha, E_\beta] - [E_{-\alpha}, E_\beta]),$$

and the lemma follows.

Let τ_λ be an irreducible representation of K with highest weight λ, let V_λ be the representation space, and let ϕ_λ be a nonzero highest weight vector. We denote by σ_λ the restriction of $\tau_\lambda(M)$ to the M-cyclic subspace generated by ϕ_λ, and we let H_λ be the subspace of V_λ in which σ_λ operates.

(5.5) **Proposition.** *If τ_λ is an irreducible representation of K, then the representation σ_λ of M is irreducible. The highest weight of σ_λ on the Cartan subalgebra*

$$\mathfrak{h}^- = \mathfrak{t} \ominus \sum_{j=1}^{m} \mathbf{R} i H_{\alpha_j} \tag{5.8}$$

of $\mathfrak{m}$, with the relative ordering, is $\lambda|_{\mathfrak{h}^-}$, and ϕ_λ is a highest weight vector. The value of σ_λ on z in the central subgroup F is $\sigma_\lambda(z) = \xi_\lambda(z) I$, where ξ_λ is the character of T whose differential is λ.

Proof. The Lie algebra $\mathfrak{a}\oplus\mathfrak{h}^-$ is a Cartan subalgebra of $\mathfrak{g}$, and the Cayley transform $\mathrm{Ad}\,(u_{\alpha_1}\dots u_{\alpha_m})$ carries $\mathfrak{t}^{\mathbf{C}}$ to $(\mathfrak{a}\oplus\mathfrak{h}^-)^{\mathbf{C}}$. It carries roots vanishing on $\sum \mathbf{C} H_{\alpha_j}$ to roots vanishing on $\mathfrak{a}^{\mathbf{C}}$, and thus the roots for M with Cartan subalgebra $\mathfrak{h}^-$ are of the form $\gamma = \mathrm{Ad}\,(u_{\alpha_1}\dots u_{\alpha_m})\,\beta$ with β a root of $(\mathfrak{g}^{\mathbf{C}}, \mathfrak{t}^{\mathbf{C}})$ orthogonal to α_j for $1 \leqq j \leqq m$. The root vectors correspond similarly.

The Cayley transform acts trivially on $\mathfrak{h}^-$ and so preserves the ordering. However, it does not act trivially on the root vectors. To show that ϕ_λ is a highest weight vector of σ_λ, we are thus to show that if $\tilde{E}_\gamma = \mathrm{Ad}\,(u_{\alpha_1}\dots u_{\alpha_m})\,E_\beta$ with $\beta > 0$ and $\beta \perp \alpha_j$ for $1 \leqq j \leqq m$, then $\tau_\lambda(\tilde{E}_\gamma)\,\phi_\lambda = 0$.

First suppose β is compact. By Lemma 4.2 with $k = m$, β is strongly orthogonal to all α_j. Then Lemma 5.4 shows that $\tilde{E}_\gamma = E_\beta$ and we conclude that $\tau_\lambda(\tilde{E}_\gamma)\,\phi_\lambda = \tau_\lambda(E_\beta)\,\phi_\lambda = 0$ since ϕ_λ is a highest weight vector.

Next let β be noncompact. By maximality of $\{\alpha_1, \dots, \alpha_m\}$, β fails to be strongly orthogonal to some α_j. By Lemma 5.4,

$$\mathrm{Ad}\,(u_{\alpha_j})\,E_\beta = c_1 E_{\beta+\alpha_j} + c_2 E_{\beta-\alpha_j}.$$

For $i \neq j$, $\alpha_i \pm (\beta \pm \alpha_j)$ cannot be roots since we would have roots of three different lengths. By Lemma 5.4 all the other $\mathrm{Ad}(u_{\alpha_i})$ fix $\mathrm{Ad}(u_{\alpha_j})\,E_\beta$, and we obtain

$$\tilde{E}_\gamma = \mathrm{Ad}\,(u_{\alpha_1}\dots u_{\alpha_m})\,E_\beta = c_1 E_{\beta+\alpha_j} + c_2 E_{\beta-\alpha_j}.$$

In particular, this equation shows that j is unique and hence that $\alpha_j = \alpha(\beta)$. Since the sequence is fundamental, (4.1) says that $\beta - \alpha(\beta)$ is >0. That is, $\beta \pm \alpha_j$ are both positive roots. Hence $\tau_\lambda(\tilde{E}_\gamma)\,\phi_\lambda = 0$.

We have now proved that ϕ_λ is a highest weight vector for σ_λ. The proposition follows directly from Lemma 5.3.

6. Szegö Kernels

Fix an integral form λ on the compact Cartan subalgebra $\mathfrak{t}$, and introduce a system of positive roots such that λ is dominant with respect to the positive compact roots. With respect to this ordering, fix a fundamental sequence $\alpha_1, \dots, \alpha_m$ of positive noncompact roots (see Proposition 4.4), and form the corresponding Iwasawa decomposition of $\mathfrak{g}$ as in § 5. We shall write $G = ANK$ for the corresponding global decomposition of G and write $g = e^{H(g)}\,n\,\kappa(g)$ for an individual element of G.

Let τ_λ be an irreducible representation of K with highest weight λ, let V_λ be the representation space for τ_λ, and let ϕ_λ be a nonzero highest weight vector. Let ν be a real-valued linear functional on $\mathfrak{a}$. Then we define $S(x, l)$ to be the function on $G \times K$ given by

$$S(x, l) = e^{\nu H(lx^{-1})}\,\tau_\lambda(\kappa(lx^{-1}))^{-1} \tag{6.1}$$

$S(x, l)$ is the *Szegö kernel* with parameters λ and ν. It is clear that

$$S(kx, l) = \tau_\lambda(k)\,S(x, l) \tag{6.2}$$

for all k in K.

As in § 5 let H_λ be the M-cyclic subspace of V_λ generated by ϕ_λ, and let σ_λ be the representation of M given by τ_λ operating in H_λ. By Proposition 5.5, σ_λ is irreducible. Let $C^\infty(K, \sigma_\lambda)$ be the space of smooth functions $f: K \to H_\lambda$ satisfying

$$f(mk) = \sigma_\lambda(m) f(k) \quad \text{for } m \in M, k \in K. \tag{6.3}$$

Then we define the *Szegö mapping* with parameters λ and ν by

$$S(f)(x) = \int_K S(x, l) f(l)\, dl = \int_K e^{\nu H(lx^{-1})} \tau_\lambda(\kappa(lx^{-1}))^{-1} f(l)\, dl \tag{6.4}$$

for f in $C^\infty(K, \sigma_\lambda)$. The definition makes sense since f takes values in $H_\lambda \subseteq V_\lambda$. Equation (6.2) shows that the Szegö mapping carries $C^\infty(K, \sigma_\lambda)$ into $C^\infty(G, \tau_\lambda)$.

We can now state the main theorem of this section in its first form.

(6.1) **Theorem.** *The image of the Szegö mapping with parameters λ and ν is contained in the kernel of the operator $\mathscr{D}$, provided λ and ν are related by the formula*

$$\boxed{\nu(E_{\alpha_j} + E_{-\alpha_j}) = \frac{2\langle \lambda + n_j \alpha_j, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle}} \tag{6.5a}$$

where $1 \leqq j \leqq m$ *and*

$$n_j = |\{\gamma \in \Delta_n^+ \mid \alpha(\gamma) = \alpha_j \text{ and } \alpha(\gamma) + \gamma \in \Delta\}|. \tag{6.5b}$$

Remark. We use the notation $\nu = \nu(\lambda)$ later.

We shall give a second form of this theorem, cast in terms of the nonunitary principal series. The nonunitary principal series representation $W(\sigma_\lambda, \nu')$, where ν' is a linear functional on $\mathfrak{a}$, operates in the space of those smooth functions f from G to H_λ such that

$$f(manx) = e^{\nu' \log a} \sigma_\lambda(m) f(x) \tag{6.6}$$

with G-action given by

$$W(\sigma_\lambda, \nu', g) f(x) = f(xg). \tag{6.7}$$

The parameters are arranged so that $W(\sigma_\lambda, \nu')$ is unitary when $\nu' = \rho^+ + i\mu$ with μ real.

If we fix ν' and use (6.6) to extend members of $C^\infty(K, \sigma_\lambda)$ to be defined on all of G, then we can regard $C^\infty(K, \sigma_\lambda)$ as exactly the representation space for $W(\sigma_\lambda, \nu')$.

(6.2) **Lemma.** *If f is in $C^\infty(K, \sigma_\lambda)$ and is extended to G by means of ν' and if S is the Szegö mapping with parameters λ and ν, then*

$$S(f)(x) = \int_K \tau_\lambda(k)^{-1} f(kx)\, dk, \tag{6.8}$$

provided ν and ν' are related by $\nu' = 2\rho^+ - \nu$.

Proof. The change-of-variables formula

$$\int_K h(k)\, dk = \int_K h(\kappa(lx^{-1}))\, e^{2\rho^+ H(lx^{-1})} dl$$

is a consequence of the fact that $W(1, \rho^+, x^{-1})$ is unitary in $L^2(K)$. Applying this formula with

$$h(k) \equiv \tau(k)^{-1} f(k x) = e^{\nu' H(kx)} \tau(k)^{-1} f(\kappa(k x)),$$

we see that the right side of (6.8) equals

$$\int_K e^{\nu' H(\kappa(l x^{-1})x)} e^{2\rho^+ H(l x^{-1})} \tau(\kappa(l x^{-1}))^{-1} f(\kappa(\kappa(l x^{-1})x))\, dl.$$

It is easy to see that $\kappa(\kappa(l x^{-1})x) = l$ and $H(\kappa(l x^{-1})x) = -H(l x^{-1})$. Substituting and using the identity $\nu' = 2\rho^+ - \nu$, we obtain the lemma.

The lemma shows that the Szegö mapping with parameters λ and ν is a G-equivariant mapping from the nonunitary principal series $W(\sigma_\lambda, 2\rho^+ - \nu)$ to $C^\infty(G, \tau_\lambda)$. Thus we can restate Theorem 6.1 in the following form.

(6.1′) **Theorem.** *The Szegö mapping with parameters λ and ν is a G-equivariant map that carries the nonunitary principal series representation $W(\sigma_\lambda, 2\rho^+ - \nu)$ into the kernel of the operator $\mathcal{D}$ on $C^\infty(G, \tau_\lambda)$, provided λ and ν are related by* (6.5).

We shall prove the equivalent Theorems 6.1 and 6.1′ in § 7. Later we shall see that if the chosen system Δ^+ of positive roots makes $\lambda + \delta_k - \delta_n$ dominant and nonsingular with respect to all positive roots, then the image of the Szegö mapping in the theorem is substantially the discrete series representation with Harish-Chandra parameter $\lambda + \delta_k - \delta_n$. The lowest K-type in the sense of Schmid [19] is λ. If $\lambda + \delta_k - \delta_n$ is dominant with respect to all positive roots and nonsingular with respect to all compact roots, the image is a discrete series or "limit of discrete series."

In any event each discrete series representation is exhibited explicitly as a quotient of a nonunitary principal series representation. We can obtain an explicit subrepresentation theorem for discrete series by duality.

It is often true that the fundamental sequence is not unique. Generally, distinct choices for the fundamental sequence exhibit a discrete series representation as quotients of distinct nonunitary principal series representations. However, it is not necessarily true that all quotient maps are obtained by suitable choices of the fundamental sequence.

Before turning to the proof of Theorems 6.1 and 6.1′, it is appropriate to note that S is not the zero operator. To see this, let $P_{\sigma\lambda}$ be the orthogonal projection of V_λ on H_λ and define $f(k) = P_{\sigma\lambda} \tau_\lambda(k) \phi_\lambda$. Then f is in $C^\infty(K, \sigma_\lambda)$, and it is easy to see that

$$(Sf(1), \phi_\lambda) = \int_K |f(k)|^2\, dk.$$

The right side is not 0 since $f(1) = \phi_\lambda$.

7. Proof of Theorem 6.1

Let f be in $C^\infty(K, \sigma_\lambda)$ and let $F = S(f)$. Since by Lemma 6.2 the Szegö mapping is equivariant with respect to the actions of G on the right and since $\mathcal{D}$ is built from right-invariant derivatives, it is enough to prove that $\mathcal{D}F(1) = 0$.

By an argument similar to Proposition 4.1 of [1], we can choose a scalar-valued function $\tilde{f}$ in $C^\infty(K)$ such that

$$f(k)=\int_M \sigma_\lambda(m)^{-1}\tilde{f}(mk)\,\phi_\lambda\,dm.$$

Then

$$F(x)=\int_{K\times M} e^{\nu H(kx^{-1})}\tau_\lambda(\kappa(kx^{-1}))^{-1}\,\tau_\lambda(m)^{-1}\,\phi_\lambda\tilde{f}(mk)\,dm\,dk$$
$$=\int_K e^{\nu H(kx^{-1})}\,\tau_\lambda(\kappa(kx^{-1}))^{-1}\,\phi_\lambda\tilde{f}(k)\,dk$$

and

$$\mathscr{D}F(1)=\int_K \mathscr{D}\{S(x,k)\,\phi_\lambda\}_{x=1}\,\tilde{f}(k)\,dk.$$

Hence it is enough to show that

$$\mathscr{D}\{S(x,k)\,\phi_\lambda\}_{x=1}=0.$$

Next, let $X_1,\ldots,X_{2n}$ be an orthonormal basis of $\mathfrak{p}$. Then

$$\begin{aligned}\mathscr{D}\{S(x,k)\,\phi_\lambda\}_{x=1}&=\sum_{i=1}^{2n}P\left\{\frac{d}{dt}S(\exp(-tX_i),k)\,\phi_\lambda\otimes X_i\right\}_{t=0}\\&=\sum_{i=1}^{2n}P\left\{\frac{d}{dt}[e^{\nu H(k\exp tX_i)}\tau_\lambda(\kappa(k\exp tX_i))^{-1}\,\phi_\lambda]_{t=0}\otimes X_i\right\}\\&=\sum_{i=1}^{2n}P\Big\{\tau_\lambda\otimes\operatorname{Ad}(k^{-1})\frac{d}{dt}[e^{\nu H(\exp t\operatorname{Ad}(k)X_i)}\\&\qquad\cdot\tau_\lambda(\kappa(\exp t\operatorname{Ad}(k)X_i))^{-1}\,\phi_\lambda]_{t=0}\otimes\operatorname{Ad}(k)\,X_i\Big\}\\&=\tau_\lambda^{-}(k)^{-1}\,\mathscr{D}\{S(x,1)\,\phi_\lambda\}_{x=1}\end{aligned}$$

since $\{\operatorname{Ad}(k)\,X_i\}$ is another orthonormal basis of $\mathfrak{p}$. Hence it is enough to show that

$$\mathscr{D}\{S(x,1)\,\phi_\lambda\}_{x=1}=0.$$

To compute this expression, we use the formula (2.10) for $\mathscr{D}$ with the explicit orthonormal basis $(\frac{1}{2}|\beta|^2)^{1/2}\,E_\beta$ of $\mathfrak{p}^{\mathbf{C}}$. Let $E_\beta=X_\beta+iY_\beta$. We have

$$\begin{aligned}\mathscr{D}\{S(x,1)\,\phi_\lambda\}_{x=1}&=\sum_{\beta\in\Delta_n}\frac{1}{2}|\beta|^2\,P\left\{\frac{d}{dt}[e^{\nu H(\exp tX_\beta)}\tau_\lambda(\kappa(\exp tX_\beta))^{-1}\,\phi_\lambda]_{t=0}\otimes E_{-\beta}\right\}\\&\quad+\sum_{\beta\in\Delta_n}\frac{i}{2}|\beta|^2\,P\left\{\frac{d}{dt}[e^{\nu H(\exp tY_\beta)}\tau_\lambda(\kappa(\exp tY_\beta))^{-1}\,\phi_\lambda]_{t=0}\otimes E_{-\beta}\right\}\\&=\sum_{\beta\in\Delta_n}\tfrac{1}{2}|\beta|^2\,P\{\nu(P_{\mathfrak{a}}E_\beta)\phi_\lambda\otimes E_{-\beta}\}-\sum_{\beta\in\Delta_n}\tfrac{1}{2}|\beta|^2\,P\{\tau_\beta(P_{\mathfrak{k}}E_\beta)\phi_\lambda\otimes E_{-\beta}\}.\end{aligned}$$

We are ready to use the Iwasawa decomposition given in Proposition 5.2. The fact that we shall use about P is that it annihilates all weight vectors that are not of the form $\lambda-\mu$ with μ a sum of elements of Δ^+. Temporarily let us assume that case (ii) of Proposition 5.2 does not arise. Then

$$\mathscr{D}\{S(x,1)\phi_\lambda\}_{x=1}=\sum_{j=1}^{m}\tfrac{1}{4}|\alpha_j|^2\,\nu(E_{\alpha_j}+E_{-\alpha_j})\,P\{\phi_\lambda\otimes E_{-\alpha_j}\}$$
$$-\sum_{j=1}^{m}\tfrac{1}{4}|\alpha_j|^2\,\lambda(H_{\alpha_j})\,P\{\phi_\lambda\otimes E_{-\alpha_j}\}$$
$$+\sum_{\beta\neq\pm\alpha_j}\frac{1}{2(p+q)}|\beta|^2\,P\{\tau_\lambda[E_{-\alpha},E_\beta]\,\phi_\lambda\otimes E_{-\beta}\}. \qquad (7.1)$$

In the third sum, the terms corresponding to $\beta>0$ are 0 by (4.1) since ϕ_λ is a highest weight vector. For the terms with $\beta<0$, we have

$$\begin{aligned}0&=\tau_\lambda^{-}[E_{-\alpha},E_\beta]\,P(\phi_\lambda\otimes E_{-\beta})\\ &=P(\tau_\lambda\otimes\mathrm{ad}\,[E_{-\alpha},E_\beta]\,(\phi_\lambda\otimes E_{-\beta}))\\ &=P(\tau_\lambda[E_{-\alpha},E_\beta]\,\phi_\lambda\otimes E_{-\beta})+P(\phi_\lambda\otimes[[E_{-\alpha},E_\beta],E_{-\beta}]).\end{aligned}$$

Introduce normalized vectors $X_\alpha=(\tfrac{1}{2}|\alpha|^2)^{1/2}E_\alpha$ and $X_\beta=(\tfrac{1}{2}|\beta|^2)^{1/2}E_\beta$. With notation $N_{\alpha\beta}$ as on page 146 of [9], we have

$$\begin{aligned}[[X_{-\alpha},X_\beta],X_{-\beta}]&=N_{-\alpha,\beta}[X_{-\alpha+\beta},X_{-\beta}]=N_{-\alpha,\beta}N_{-\alpha+\beta,-\beta}X_{-\alpha}\\ &=N_{-\alpha,\beta}N_{\alpha,-\alpha+\beta}X_{-\alpha}.\end{aligned}$$

But by (5.3)

$$\tfrac{1}{2}|\alpha|^2p(q+1)X_\beta=[X_\alpha,[X_{-\alpha},X_\beta]]=N_{-\alpha,\beta}N_{\alpha,-\alpha+\beta}X_\beta.$$

Thus

$$[[E_{-\alpha},E_\beta],E_{-\beta}]=\frac{|\alpha|^2}{|\beta|^2}\,p(q+1)\,E_{-\alpha},$$

and we conclude, for $\beta<0$, that

$$P(\tau_\lambda[E_{-\alpha},E_\beta]\,\phi_\lambda\otimes E_{-\beta})=-\frac{p(q+1)\,|\alpha|^2}{|\beta|^2}\;P(\phi_\lambda\otimes E_{-\alpha}).$$

Substituting in (7.1), we obtain

$$\mathscr{D}\{S(x,1)\phi_\lambda\}_{x=1}=\tfrac{1}{4}\sum_{j=1}^{m}|\alpha_j|^2\left\{\nu(E_{\alpha_j}+E_{-\alpha_j})-\lambda(H_{\alpha_j})-\sum_{\substack{\beta<0\\ \alpha(\beta)=\alpha_j}}\frac{2p(q+1)}{(p+q)}\right\}$$
$$\cdot P(\phi_\lambda\otimes E_{-\alpha_j}).$$

To evaluate the sum over β, let us replace β by $-\beta$, interchanging p and q. A term is

$$\frac{2q(p+1)}{(p+q)}.$$

If $q=0$, this is 0. Otherwise $q=1$ and $p+1=p+q$. Hence the term counts 2 if $\beta+\alpha(\beta)$ is a root, 0 if not. The sum is $2n_j$, with n_j defined by (6.5b). Consequently

$$\mathscr{D}\{S(x,1)\phi_\lambda\}_{x=1}=\tfrac{1}{4}\sum_{j=1}^{m}|\alpha_j|^2\left\{\nu(E_{\alpha_j}+E_{-\alpha_j})-\frac{2\langle\lambda+n_j\alpha_j,\alpha_j\rangle}{\langle\alpha_j,\alpha_j\rangle}\right\}P(\phi_\lambda\otimes E_{-\alpha}). \qquad (7.2)$$

By (6.5a) the expression in braces vanishes. Thus the left side is 0 as required.

Finally we indicate what modifications are necessary if case (ii) of Proposition 5.2 arises. In the third sum of (7.1) fix $\alpha=\alpha_j$ so that case (ii) applies and sum over β with $\alpha(\beta)=\alpha_j$. A typical term is a coefficient times $P\{\tau_\lambda(E_{\beta+n\alpha})\phi_\lambda\otimes E_{-\beta}\}$. Here we may take $n<0$ since P annihilates the other terms. If $\beta>0$, (4.1) shows the term is 0. Thus we can take $\beta<0$ and n negative and odd. In the notation or Proposition 5.2, we have simple roots α and $\gamma-3\alpha$ for a split G_2 factor, and β must be $-\gamma$ or $-\gamma+2\alpha$ to be negative and noncompact. Since $\beta+n\alpha$ is to be a root for some odd $n<0$, the only possible choice is $\beta=-\gamma+2\alpha$ and $n=1$. Tracking down the coefficient by means of Proposition 5.2, we see that the relevant term in the third sum of (7.1) is

$$+\tfrac{1}{8}|\beta|^2 P\{\tau_\lambda[E_{-\alpha}, E_\beta]\phi_\lambda\otimes E_{-\beta}\}. \tag{7.3}$$

Since $p+q=3$, this is $\tfrac{3}{4}$ of

$$\frac{1}{2(p+q)}|\beta|^2 P\{\tau_\lambda[E_{-\alpha}, E_\beta]\phi_\lambda\otimes E_{-\beta}\}.$$

Hence we can repeat the argument that computed the relevant term in the third sum of (7.1) in the previous case to see that (7.3) equals

$$\tfrac{3}{4}\cdot\tfrac{1}{4}|\alpha_j|^2\left(-\frac{2q(p+1)}{p+q}\right)P(\phi_\lambda\otimes E_{-\alpha_j}).$$

Since $n_j=1$, $p=1$, and $q=2$ for this exceptional α_j, formula (7.2) is still valid. The proof is complete.

8. Infinitesimal Character of $W(\sigma_\lambda, \nu'(\lambda))$

The main result of this section, Proposition 8.2, relates λ to the parameters σ_λ and ν of Theorem 6.1 by means of the standard Cayley transform built from the roots in a fundamental sequence. This result will be applied in § 10. An immediate consequence of the result is a simple formula in terms of λ for the infinitesimal character of the image of the Szegö kernel.

We continue with notation as in § 2. Let $G=ANK$ be the Iwasawa decomposition of G constructed as in § 5 from a fundamental sequence $\{\alpha_1, \ldots, \alpha_m\}$ of positive noncompact roots. With $\mathfrak{h}^-$ defined as in (5.8), let $\mathfrak{h}=\mathfrak{h}^-\oplus\mathfrak{a}$. Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, and the corresponding Cartan subgroup is FT_0A, where $M=M_0F$ as in Lemma 5.3 and T_0 is the analytic subgroup corresponding to $\mathfrak{h}^-$. T_0 is the identity component of $M\cap T$.

Let Ψ denote the root system of $(\mathfrak{g}^{\mathbf{C}}, \mathfrak{h}^{\mathbf{C}})$, and let $\Psi_m\subseteq\Psi$ be the root system of M. Let Ψ^+ be the system of positive roots of Ψ obtained by requiring that $\mathfrak{a}$ comes before $\mathfrak{h}^-$. The corresponding restricted positive roots of $(\mathfrak{g}, \mathfrak{a})$ are as in § 5, and the positive roots of Ψ_m are as in Proposition 5.5. Put $\Psi_m^+=\Psi_m\cap\Psi^+$.

Now $M=M_0F$ as in Lemma 5.3. Let T_0 be the identity component of $M\cap T$. Then FT_0A is a Cartan subgroup of G. If (σ, H) is an irreducible representation of M and if ν is a complex-valued real-linear form on $\mathfrak{a}$, let $\sigma\otimes\nu$ denote the representation of MAN on H given by $(\sigma\otimes\nu)(man)=\sigma(m)e^{\nu(\log a)}$. Let $\lambda(\sigma, \nu)$

denote the representation of FT_0A on the *lowest* weight space of $(\sigma\otimes\nu, H)$ relative to Ψ_m^+. Clearly $\lambda(\sigma,\nu)$ determines $\sigma\otimes\nu$.

Let $\Lambda(\sigma,\nu)$ denote the complex-linear extension of the differential of $\lambda(\sigma,\nu)$ to $\mathfrak{h}^{\mathbf{C}}$, Set

$$\mathfrak{n}^+=\sum_{\alpha\in\Psi^+}\mathfrak{g}_\alpha,\quad \mathfrak{n}^-=\sum_{\alpha\in\Psi^+}\mathfrak{g}_{-\alpha}.$$

The universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}^{\mathbf{C}}$ decomposes as

$$U(\mathfrak{g})=U(\mathfrak{h})\oplus(\mathfrak{n}^+U(\mathfrak{g})+U(\mathfrak{g})\mathfrak{n}^-),$$

and we let η: $U(\mathfrak{g})\to U(\mathfrak{h})$ be the corresponding projection. For z in the center $\mathscr{Z}$ of $U(\mathfrak{g})$ and for Λ a complex-linear form on $\mathfrak{h}^{\mathbf{C}}$, define

$$\chi_\Lambda(z)=\Lambda(\eta(z)).$$

We say that χ_Λ is the infinitesimal character corresponding to Λ. According, for example, to [4, p. 87], the infinitesimal character of the nonunitary principal series is given as follows.

(8.1) **Lemma.** *For z in $\mathscr{Z}$, $W(\sigma,\nu,z)=\chi_{\Lambda(\sigma,\nu)}(z)I$.*

The main result of this section is a formula that yields the infinitesimal character of the nonunitary principal series representation appearing in Theorem (6.1). With $(\tau_\lambda, V_\lambda)$ as in § 6, let σ_λ be defined as in Proposition (5.5) and let $\nu=\nu(\lambda)$ be as in (6.5). Set $\nu'(\lambda)=2\rho^+-\nu(\lambda)$.

To state the result, we introduce Cayley transforms as in (5.7). Let. $u_i=\exp\left(\frac{\pi}{4}(E_{\alpha_i}-E_{-\alpha_i})\right)$ for $i=1,\dots,m$. Set $u=u_1\cdot\dots\cdot u_m$. Then $\mathrm{Ad}(u)\mathfrak{t}^{\mathbf{C}}=\mathfrak{h}^{\mathbf{C}}$. Also, the Weyl group $W(\Psi_m)$ of Ψ_m canomically imbeds in the Weyl group of Ψ, and we let s_0 be the element of $W(\Psi_m)$ such that $s_0\Psi_m^+=-\Psi_m^+$.

(8.2) **Proposition.**

$$\boxed{\Lambda(\sigma_\lambda,\nu'(\lambda))=s_0(\Lambda\circ\mathrm{Ad}(u)^{-1})+\rho,}$$

where $\rho=\frac{1}{2}\sum_{\alpha\in\Psi^+}\alpha$ *and* $\Lambda=\lambda+\delta_k-\delta_n$.

The proof of this proposition takes some preparation. To simplify the notation, we write u^{-1} for $\mathrm{Ad}(u)^{-1}$.

(8.3) **Lemma.** *If β is in Δ_n, if $\beta\neq\pm\alpha_j$ for $j=1,\dots,m$, and if $\alpha(\beta)=\alpha_i$ but $\langle\beta,\alpha_i\rangle=0$, then $\langle\beta,\alpha_j\rangle=0$ for $j=1,\dots,m$.*

Proof. If $\langle\beta,\alpha_j\rangle\neq0$, then $j>i$. We may suppose by changing β to $-\beta$ that $\langle\beta,\alpha_j\rangle>0$. But $\beta+\alpha_i$ is in Δ since $\alpha(\beta)=\alpha_i$ and $\langle\beta,\alpha_i\rangle=0$. Since $\langle\beta+\alpha_i,\alpha_j\rangle=\langle\beta,\alpha_j\rangle$, we see that $\beta+\alpha_i-\alpha_j$ is a root, clearly noncompact. If $\alpha(\beta+\alpha_i-\alpha_j)=\alpha_r$ for some $r<i$, then $|\beta+\alpha_i-\alpha_j|^2=|\beta-\alpha_j|^2+|\alpha_i|^2$, and $\beta+\alpha_i-\alpha_j\pm\alpha_r$ are roots with $|\beta+\alpha_i-\alpha_j\pm\alpha_r|^2=|\beta-\alpha_j|^2+|\alpha_i|^2+|\alpha_r|^2$. Thus we would have roots of three different lengths. Hence $\alpha(\beta+\alpha_i-\alpha_j)=\alpha_i$. But $|\alpha_i|^2<|\beta+\alpha_i-\alpha_j|^2=|\beta-\alpha_j|^2+|\alpha_i|^2$. Thus (4a) fails in Definition 4.1 for $\gamma=\beta+\alpha_i-\alpha_j$, and (4b) must hold. But then α_i cannot be orthogonal to β without being strongly orthogonal, since orthogonality implies strong orthogonality in G_2.

(8.4) **Lemma.** *Let*

$$\Delta^+_{n,j,1}=\{\beta\in\Delta^+_n \mid \alpha(\beta)=\alpha_j \textit{ and } \beta-\alpha_j\in\Delta\},$$
$$\Delta^+_{n,j,2}=\{\beta\in\Delta^+_n \mid \alpha(\beta)=\alpha_j \textit{ and } \beta+\alpha_j\in\Delta\},$$
$$\Delta^+_{k,j,1}=\{\beta\in\Delta^+_k \mid \langle\beta,\alpha_i\rangle=0 \textit{ for } i<j \textit{ and } \langle\beta,\alpha_j\rangle>0\},$$
$$\Delta^+_{k,j,2}=\{\beta\in\Delta^+_k \mid \langle\beta,\alpha_i\rangle=0 \textit{ for } i<j \textit{ and } \langle\beta,\alpha_j\rangle<0\}.$$

If each β in Δ^+_n with $\alpha(\beta)=\alpha_j$ has $|\beta|\leqq|\alpha_j|$, then the mappings

$$\Delta^+_{n,j,1}\to\Delta^+_{k,j,2} \quad \textit{given by } \beta\to\beta-\alpha_j$$

and

$$\Delta^+_{n,j,2}\to\Delta^+_{k,j,1} \quad \textit{given by } \beta\to\beta+\alpha_j$$

are bijective.

Proof. If β is in $\Delta^+_{n,j,1}$, then $\beta-\alpha_j$ is in Δ^+_k by (3) of Definition 4.1. Since $|\alpha_j|\geqq|\beta|$, the Schwarz inequality implies $\langle\beta-\alpha_j,\alpha_j\rangle<0$. Hence $\beta-\alpha_j$ is in $\Delta^+_{k,j,2}$. Next, if β is in $\Delta^+_{k,j,2}$, then $\beta+\alpha_j$ is in Δ^+_n. Then $\alpha(\beta+\alpha_j)=\alpha_j$. [In fact, if $\alpha(\beta+\alpha_j)=\alpha_i$ with $i<j$, let γ be the longer of β and α_j. Then $\langle\beta+\alpha_j+\alpha_i-\gamma,\gamma\rangle$ is >0, and we conclude that either $\beta+\alpha_i$ or $\alpha_j+\alpha_i$ is a root. But $\beta+\alpha_i$ cannot be a root by Lemma 4.2, and $\alpha_j+\alpha_i$ cannot be a root by strong orthogonality of the fundamental sequence. Thus $\alpha(\beta+\alpha_j)=\alpha_j$.] Then $\beta+\alpha_j$ is in $\Delta^+_{n,j,1}$ and the map $\Delta^+_{n,j,1}$ to $\Delta^+_{k,j,2}$ is bijective.

If β is in $\Delta^+_{n,j,2}$, then $\beta+\alpha_j$ is in Δ^+_k and the Schwarz inequality implies $\langle\beta+\alpha_j,\alpha_j\rangle>0$ since $|\alpha_j|\geqq|\beta|$. Hence $\beta+\alpha_j$ is in $\Delta^+_{k,j,1}$. Finally, if β is in $\Delta^+_{k,j,1}$, then $\beta-\alpha_j$ is a noncompact root. Lemma 4.2 shows that β is strongly orthogonal to $\alpha_1,\ldots,\alpha_{j-1}$. Hence $\beta-\alpha_j$ is positive by (3) of Definition 4.1. Arguing as in the previous paragraph, we see that $\alpha(\beta-\alpha_j)=\alpha_j$. Thus $\beta-\alpha_j$ is in $\Delta^+_{n,j,2}$. The proof of the lemma is now complete.

(8.5) **Lemma.** *Let $m_j=|\Delta^+_{n,j,1}|$ and $n_j=|\Delta^+_{n,j,2}|$. If each β in Δ^+_n has $|\beta|\leqq|\alpha(\beta)|$, then*

(a) $\rho^+(E_{\alpha_j}+E_{-\alpha_j})=1+m_j+n_j$ *and*

(b) $\dfrac{2\langle\delta_k-\delta_n,\alpha_j\rangle}{\langle\alpha_j,\alpha_j\rangle}=-1-m_j+n_j.$

Remark. The integer n_j coincides with the integer in (6.5b) of Theorem 6.1.

Proof. $\rho^+(E_{\alpha_j}+E_{-\alpha_j})=\frac{1}{2}\sum_{\tilde\alpha\in\Psi^+}\tilde\alpha(E_{\alpha_j}+E_{-\alpha_j})$. Now $\mathrm{Ad}(u)H_{\alpha_j}=-E_{\alpha_j}-E_{-\alpha_j}$. Thus $\Psi^+=\{\alpha\circ u^{-1}\mid\alpha\in R\subseteq\Delta\}$, where

$$R=\Psi^+_m\cup\{\alpha\in\Delta \mid \text{for some } j,\ \langle\alpha,\alpha_j\rangle<0 \text{ and } \langle\alpha,\alpha_i\rangle=0 \text{ when } i<j\}.$$

Since $\alpha_j(H_{\alpha_j})=2$, we obtain

$$\rho^+(E_{\alpha_j}+E_{-\alpha_j})=-\sum_{\alpha\in R}\frac{\langle\alpha,\alpha_j\rangle}{\langle\alpha_j,\alpha_j\rangle}.$$

We note that if β is in $\Delta^+_{n,j,1}\cap\Delta^+_{n,j,2}$, then $\langle\beta,\alpha_r\rangle=0$ for all r by Lemma 8.3.

Hence

$$\begin{aligned}\rho^+(E_{\alpha_j}+E_{-\alpha_j}) &= 1+\sum_{i\leqq j}\sum_{\beta\in\Delta^+_{n,i,1}}\frac{\langle\beta,\alpha_j\rangle}{\langle\alpha_j,\alpha_j\rangle} \\ &\quad -\sum_{i\leqq j}\sum_{\beta\in\Delta^+_{n,i,2}}\frac{\langle\beta,\alpha_j\rangle}{\langle\alpha_j,\alpha_j\rangle} \\ &\quad +\sum_{i\leqq j}\sum_{\beta\in\Delta^+_{k,i,1}}\frac{\langle\beta,\alpha_j\rangle}{\langle\alpha_j,\alpha_j\rangle}-\sum_{i\leqq j}\sum_{\beta\in\Delta^+_{k,i,2}}\frac{\langle\beta,\alpha_j\rangle}{\langle\alpha_j,\alpha_j\rangle} \\ &= 1+\sum_{i\leqq j}\sum_{\beta\in\Delta^+_{n,i,1}}\frac{\langle\beta,\alpha_j\rangle}{\langle\alpha_j,\alpha_j\rangle}-\sum_{i\leqq j}\sum_{\beta\in\Delta^+_{n,i,2}}\frac{\langle\beta,\alpha_j\rangle}{\langle\alpha_j,\alpha_j\rangle} \\ &\quad +\sum_{i\leqq j}\sum_{\beta\in\Delta^+_{n,i,2}}\frac{\langle\beta+\alpha_i,\alpha_j\rangle}{\langle\alpha_j,\alpha_j\rangle}-\sum_{i\leqq j}\sum_{\beta\in\Delta^+_{n,i,1}}\frac{\langle\beta-\alpha_i,\alpha_j\rangle}{\langle\alpha_j,\alpha_j\rangle}\end{aligned}$$

by Lemma 8.4. If $i<j$, $\langle\alpha_i,\alpha_j\rangle=0$. Hence we find

$$\begin{aligned}\rho^+(E_{\alpha_j}+E_{-\alpha_j}) &= 1+\sum_{\beta\in\Delta^+_{n,j,1}}\frac{\langle\beta,\alpha_j\rangle}{\langle\alpha_j,\alpha_j\rangle}-\sum_{\beta\in\Delta^+_{n,j,2}}\frac{\langle\beta,\alpha_j\rangle}{\langle\alpha_j,\alpha_j\rangle} \\ &\quad +\sum_{\beta\in\Delta^+_{n,j,2}}\frac{\langle\beta+\alpha_j,\alpha_j\rangle}{\langle\alpha_j,\alpha_j\rangle}-\sum_{\beta\in\Delta^+_{n,j,1}}\frac{\langle\beta-\alpha_j,\alpha_j\rangle}{\langle\alpha_j,\alpha_j\rangle} \\ &= 1+m_j+n_j\end{aligned}$$

as asserted. Conclusion (b) is proved in the same way.

Proof of Proposition 8.2. We may assume that G is simple. First suppose that $|\beta|\leqq|\alpha(\beta)|$ for all β in Δ^+_n. By (6.5a) and Lemma 8.5a,

$$\begin{aligned}\Lambda(\sigma_\lambda,\nu'(\lambda))(E_{\alpha_j}+E_{-\alpha_j}) &= 2\rho^+(E_{\alpha_j}+E_{-\alpha_j})-\frac{2\langle\lambda+n_j\alpha_j,\alpha_j\rangle}{\langle\alpha_j,\alpha_j\rangle} \\ &= 2+2m_j+2n_j-\frac{2\langle\lambda,\alpha_j\rangle}{\langle\alpha_j,\alpha_j\rangle}-2n_j \\ &= 2+2m_j-\frac{2\langle\lambda,\alpha_j\rangle}{\langle\alpha_j,\alpha_j\rangle}.\end{aligned}$$

By Lemma 8.5b

$$\begin{aligned}(s_0(\Lambda\circ u^{-1})+\rho)(E_{\alpha_j}+E_{-\alpha_j}) &= -\frac{2\langle\Lambda,\alpha_j\rangle}{\langle\alpha_j,\alpha_j\rangle}+\rho^+(E_{\alpha_j}{}^+E_{-\alpha_j}) \\ &= -\frac{2\langle\lambda,\alpha_j\rangle}{\langle\alpha_j,\alpha_j\rangle}-\frac{2\langle\delta_k-\delta_n,\alpha_j\rangle}{\langle\alpha_j,\alpha_j\rangle}+1+m_j+n_j \\ &= -\frac{2\langle\lambda,\alpha_j\rangle}{\langle\alpha_j,\alpha_j\rangle}+1+m_j-n_j+1+m_j+n_j \\ &= -\frac{2\langle\lambda,\alpha_j\rangle}{\langle\alpha_j,\alpha_j\rangle}+2+2m_j.\end{aligned}$$

Hence $\Lambda(\sigma_\lambda, \nu'(\lambda))|_{\mathfrak{a}} = (s_0(\Lambda \circ u^{-1}) + \rho)|_{\mathfrak{a}}$. Now $\Lambda(\sigma_\lambda, \nu'(\lambda))|_{\mathfrak{h}^-} = s_0 \lambda|_{\mathfrak{h}^-}$. Thus we must show $s_0 \Lambda|_{\mathfrak{h}^-} + \rho|_{\mathfrak{h}^-} = s_0 \lambda|_{\mathfrak{h}^-}$. That is, we want $s_0(\delta_k - \delta_n)|_{\mathfrak{h}^-} + \rho|_{\mathfrak{h}^-} = 0$. Now

$$\Delta_k^+ = \{\beta \in \Delta_k^+ \mid \langle \beta, \alpha_j \rangle = 0 \quad \text{for } j = 1, \ldots, m\} \cup \bigcup_{j=1}^{m} (\Delta_{k,j,1}^+ \cup \Delta_{k,j,2}^+)$$

$$\Delta_n^+ = \{\alpha_1, \ldots, \alpha_m\} \cup \bigcup_{j=1}^{m} (\Delta_{n,j,1}^+ \cup \Delta_{n,j,2}^+).$$

The decomposition of Δ_k^+ is disjoint, but the decomposition for Δ_n^+ is not. In fact, Lemma 8.3 gives

$$\bigcup_{j=1}^{m} (\Delta_{n,j,1}^+ \cap \Delta_{n,j,2}^+) = \{\beta \in \Delta_n^+ \mid \langle \beta, \alpha_j \rangle = 0 \text{ for } j = 1, \ldots, m\}.$$

By Lemma 8.4,

$$\Delta_{n,j,1}^+|_{\mathfrak{h}^-} = \Delta_{k,j,2}^+|_{\mathfrak{h}^-} \quad \text{and} \quad \Delta_{n,j,2}^+|_{\mathfrak{h}^-} = \Delta_{k,j,1}^+|_{\mathfrak{h}^-}.$$

Hence

$$\begin{aligned}(\delta_k - \delta_n)|_{\mathfrak{h}^-} &= \tfrac{1}{2} \sum_{\substack{\beta \in \Delta_k^+ \\ \langle \beta, \alpha_j \rangle = 0 \\ \text{for all } j}} \beta + \tfrac{1}{2} \sum_{\substack{\beta \in \Delta_n^+ \\ \langle \beta, \alpha_j \rangle = 0 \\ \text{for all } j}} \beta \\ &= \tfrac{1}{2} \sum_{\beta \in \Psi_m^+} \beta = \rho|_{\mathfrak{h}^-}. \end{aligned} \tag{8.1}$$

This proves the proposition in the case that $|\beta| \leqq |\alpha(\beta)|$ for all β in Δ_n^+.

Since we are assuming that G is simple, the only case in which $|\beta| > |\alpha(\beta)|$ can occur for some β in Δ_n^+, i.e., in which (4b) of Definition 4.1 can occur, has $G = G_2$ and $m = 2$. As in the proof of Proposition 4.5 we see that if $\varepsilon_1, \varepsilon_2$ are the simple roots of Δ^+, then $|\varepsilon_2|^2 = 3|\varepsilon_1|^2$. We find $\Delta_k^+ = \{\varepsilon_2, 2\varepsilon_1 + \varepsilon_2\}$, $\Delta_n^+ = \{\varepsilon_1, \varepsilon_1 + \varepsilon_2, 3\varepsilon_1 + \varepsilon_2, 3\varepsilon_1 + 2\varepsilon_2\}$, $\alpha_1 = \varepsilon_1$, $\alpha_2 = 3\varepsilon_1 + 2\varepsilon_2$, $n_1 = 1$, $n_2 = 0$. We have

$$\frac{\langle \alpha_2, \beta \rangle}{\langle \alpha_2, \alpha_2 \rangle} = \frac{1}{2}$$

if β is in Δ^+ and β is not α_1 or α_2. Also

$$\frac{\langle \varepsilon_2, \alpha_1 \rangle}{\langle \alpha_1, \alpha_1 \rangle} = -\frac{3}{2}, \qquad \frac{\langle 2\varepsilon_1 + \varepsilon_2, \alpha_1 \rangle}{\langle \alpha_1, \alpha_1 \rangle} = \frac{1}{2},$$

$$\frac{\langle \varepsilon_1 + \varepsilon_2, \alpha_1 \rangle}{\langle \alpha_1, \alpha_1 \rangle} = -\frac{1}{2}, \qquad \frac{\langle 3\varepsilon_1 + \varepsilon_2, \alpha_1 \rangle}{\langle \alpha_1, \alpha_1 \rangle} = \frac{3}{2}.$$

Hence we find

(a) $\rho^+(E_{\alpha_1} + E_{-\alpha_1}) = 5$,

(b) $\dfrac{2\langle \delta_k - \delta_n, \alpha_1 \rangle}{\langle \alpha_1, \alpha_1 \rangle} = -3$,

(c) $\rho^+(E_{\alpha_2} + E_{-\alpha_2}) = 1$,

(d) $\dfrac{2\langle \delta_k - \delta_n, \alpha_2 \rangle}{\langle \alpha_2, \alpha_2 \rangle} = -1$.

Now

$$\Lambda(\sigma_\lambda, v'(\lambda))(E_{\alpha_1}+E_{-\alpha_1})=2\rho^+(E_{\alpha_1}+E_{-\alpha_1})-\frac{2\langle\lambda,\alpha_1\rangle}{\langle\alpha_1,\alpha_1\rangle}-2=8-\frac{2\langle\lambda,\alpha_1\rangle}{\langle\alpha_1,\alpha_1\rangle},$$

$$(s_0(\Lambda\circ u^{-1})+\rho)(E_{\alpha_1}+E_{-\alpha_1})=5-\frac{2\langle\Lambda,\alpha_1\rangle}{\langle\alpha_1,\alpha_1\rangle}=5-\frac{2\langle\lambda,\alpha_1\rangle}{\langle\alpha_1,\alpha_1\rangle}+3=8-\frac{2\langle\lambda,\alpha_1\rangle}{\langle\alpha_1,\alpha_1\rangle},$$

$$\Lambda(\sigma_\lambda, v'(\lambda))(E_{\alpha_2}+E_{-\alpha_2})=2\rho^+(E_{\alpha_2}+E_{-\alpha_2})-\frac{2\langle\lambda,\alpha_2\rangle}{\langle\alpha_2,\alpha_2\rangle}=2-\frac{2\langle\lambda,\alpha_2\rangle}{\langle\alpha_2,\alpha_2\rangle},$$

$$(s_0(\Lambda\circ u^{-1})+\rho)(E_{\alpha_2}+E_{-\alpha_2})=1-\frac{2\langle\Lambda,\alpha_2\rangle}{\langle\alpha_2,\alpha_2\rangle}=1-\frac{2\langle\lambda,\alpha_2\rangle}{\langle\alpha_2,\alpha_2\rangle}+1=2-\frac{2\langle\lambda,\alpha_2\rangle}{\langle\alpha_2,\alpha_2\rangle}.$$

Comparison of these formulas completes the proof of Proposition 8.2.

9. Image of S "Far from the Walls"

With λ integral and dominant with respect to Δ_k^+, let $S_\lambda(x,l)$ be the Szegö kernel defined by (6.1) with parameters λ and $v=v(\lambda)$, with $v(\lambda)$ given by (6.5). The Szegö mapping, which is also denoted S_λ and is defined by (6.4), carries $C^\infty(K,\sigma_\lambda)$ into the kernel of $\mathscr{D}$ in $C^\infty(G,\tau_\lambda)$. Let Q_λ be the right regular representation of G on $C^\infty(G,\tau_\lambda)$ and $\mathscr{H}_\lambda$ be the subspace of K-finite vectors in the kernel of $\mathscr{D}$ in $C^\infty(G,\tau_\lambda)$. Then $(Q_\lambda, \mathscr{H}^\lambda)$ is a representation of $\mathfrak{g}$.

(9.1) *Definition.* The integral parameter λ is said to be *far from the walls* if $\lambda-\langle Q\rangle$ is Δ_k^+-dominant for all $Q\subseteq\Delta_n^+$. (Here $\langle Q\rangle=\sum_{\alpha\in Q}\alpha$.)

We now quote more precisely the three results mentioned in § 1 that we need. If $\Lambda=\lambda-\delta_k+\delta_n$ is regular and Δ^+-dominant, we let (π_Λ, H^Λ) be the discrete series representation with Harish-Chandra parameter Λ, as in § 1.

(9.2) **Theorem** (Schmid [19], Hotta-Parthasarathy [10, p. 156]). *If λ is far from the walls, then the representation $(Q_\lambda|_K, \mathscr{H}^\lambda)$ splits into a direct sum $\Sigma\oplus m_\lambda(\mu)\tau_\mu$ with*

$$m_\lambda(\mu)\leqq b_\lambda(\mu)=\sum_{s\in W_K}\det(s)\,Q(s(\mu+\delta_k)-(\lambda+\delta_k)),$$

where $Q(\xi)$ is the number of distinct ways that ξ can be written as a sum of elements of Δ_n^+.

Remarks. In Hotta-Parthasarathy [10] the unnecessary condition that G be linear is imposed. The proof in [10] goes through without this condition since $\mathscr{D}$ is then the first term of the elliptic complex $\mathbf{E}'_\mu$ of [10, p. 153]. This complex is defined for all λ far from the walls. The proofs in § 5 of [10] go through without change.

(9.3) **Theorem** (Hecht-Schmid [8], Enright [3]). *Suppose that the integral parameter λ is far from the walls and that $\Lambda=\lambda-\delta_k+\delta_n$ is regular and Δ^+-dominant. Then the representation $\pi_\Lambda|_K$ of K splits as*

$$\pi_\Lambda|_K=\sum_\mu b_\lambda(\mu)\,\tau_\mu,$$

with $b_\lambda(\mu)$ as in Theorem 9.2.

(9.4) **Theorem** (Schmid [19], Hotta-Parthasarathy [10, p. 164]). *If the integral parameter λ is far from the walls and if $\Lambda = \lambda - \delta_k + \delta_n$ is regular and Δ^+-dominant, then the subspace of square-integrable elements of $(Q_\lambda, \mathscr{H}^\lambda)$ is equivalent as a representation of $\mathfrak{g}$ with the representation of $\mathfrak{g}$ on the space of K-finite vectors of (π_Λ, H^Λ).*

(9.5) **Corollary.** *If the integral parameter λ is far from the walls and if $\Lambda = \lambda - \delta_k + \delta_n$ is regular and Δ^+-dominant, then $(Q_\lambda, \mathscr{H}^\lambda)$ is equivalent as a representation of $\mathfrak{g}$ with the representation of $\mathfrak{g}$ on the space of K-finite vectors of (π_Λ, H^Λ).*

Proof. The point is to show that every member of $\mathscr{H}^\lambda$ is square-integrable. Theorems 9.3 and 9.4 show that τ_μ occurs in the space of square-integrable elements with multiplicity at least $b_\lambda(\mu)$, and Theorem 9.2 shows that τ_μ occurs in all of $\mathscr{H}^\lambda$ with multiplicity at most $b_\lambda(\mu)$. Since all elements of $\mathscr{H}^\lambda$ are K-finite, the result follows.

(9.6) **Corollary.** *If the integral parameter λ is far from the walls and if $\Lambda = \lambda - \delta_k + \delta_n$ is regular and Δ^+-dominant, then the image under the Szegö mapping S_λ of the space of K-finite vectors in $C^\infty(K, \sigma_\lambda)$ is the full K-finite kernel $\mathscr{H}^\lambda$ of $\mathscr{D}$ in $C^\infty(G, \tau_\lambda)$. Therefore S_λ is a $\mathfrak{g}$-intertwining operator from the K-finite subspace of $W(\sigma_\lambda, \nu'(\lambda))$, where $\nu'(\lambda) = 2\rho^+ - \nu(\lambda)$, onto $(Q_\lambda, \mathscr{H}^\lambda)$ and exhibits the K-finite vectors of $(\pi_\Lambda, \mathscr{H}^\Lambda)$ as a $\mathfrak{g}$-equivariant quotient space of the K-finite subspace of $W(\sigma_\lambda, \nu'(\lambda))$.*

This corollary is immediate from Corollary 9.5 and Theorem 6.1 since we know that S_λ is not the zero operator. We have thus proved Theorem 1.1 in a particularly sharp form far from the walls.

10. Tensoring with Finite-Dimensional Representations

Theorem 1.1 was proved in § 9 when λ is far from the walls. In oversimplified form the idea in the general case is that everything can be shifted compatibly from a parameter $\lambda + \mu$ far from the walls to a general parameter λ by forming suitable projections of tensor products with finite-dimensional representations. This technique was introduced by Zuckerman [25] in a general context, and a specific result of his concerning discrete series will be used in Theorem 10.8. The other machinery concerning tensor products will be developed in this section, and Theorem 1.1 will then be proved at the end.

Let λ be an integral form that is Δ_k^+-dominant. We retain the notation of §§ 8–9. Fix μ to be the highest weight of an irreducible finite-dimensional representation of G. Let (π, U) denote a finite-dimensional representation of G with *lowest* weight $-\mu$. We make repeated use of the following lemma.

(10.1) **Lemma.** *$\tau_{\lambda+\mu} \otimes \pi|_K$ contains the K-type τ_λ with multiplicity 1, and its other K-types have highest weights $\lambda + \gamma$ with γ a sum of elements of Δ^+. If $\phi_{\lambda+\mu}$ is a nonzero highest weight vector for $\tau_{\lambda+\mu}$ and v_0 is a nonzero lowest weight vector for π, then the τ_λ projection of $\phi_{\lambda+\mu} \otimes v_0$ is a nonzero highest weight vector for τ_λ.*

Remarks. All but the last assertion of the lemma is well known, but we reproduce part of one standard proof of the remainder in order to obtain the last assertion.

Proof. Let $\mu_0, \ldots, \mu_d$ be the weights of $\pi|_K$ arranged in increasing order and repeated according to their multiplicities, and let $v_0, \ldots, v_d$ be a corresponding basis of weight vectors. Let M_i be the cyclic space of $\phi_{\lambda+\mu}\otimes v_i$ in the tensor product, and let $V_i = \sum_{j\geq i} M_j$. Then $V_0 \supseteq \cdots \supseteq V_d \supseteq V_{d+1} = 0$, and it is clear that the root vectors corresponding to Δ_k^+ carry $\phi_{\lambda+\mu}\otimes v_i$ into V_{i+1}. It follows readily that

$$V_i/V_{i+1} = 0 \quad \text{or} \quad \tau_{\lambda+\mu+\mu_i} \tag{10.1}$$

and correspondingly that

$$\tau_{\lambda+\mu}\otimes\pi|_K = \sum n_i \tau_{\lambda+\mu+\mu_i} \tag{10.2}$$

with each $n_i = 0$ or 1. An elementary argument with characters shows that $n_0 = 1$. Hence $V_0/V_1 = \tau_\lambda \neq 0$. Therefore $\phi_{\lambda+\mu}\otimes v_0$ has nonzero projection on the τ_λ space. Since its weight is λ, it projects to a nonzero highest weight vector. This proves the lemma.

For the remainder of this section we shall assume that μ is chosen so large that $\lambda+\mu-\delta_k+\delta_n$ is regular and Δ^+-dominant and that $\lambda+\mu$ is far from the walls in the sense of Definition 9.1. For now, we do not require that $\lambda-\delta_k+\delta_n$ itself be Δ^+-dominant.

We shall use Lemma 10.1 to introduce the shift that carries $\mathscr{H}^{\lambda+\mu}$ to $\mathscr{H}^\lambda$. Let

$$Q_1 : C^\infty(G, \tau_{\lambda+\mu})\otimes U \to C^\infty(G, \tau_{\lambda+\mu}\otimes\pi|_K) \tag{10.3}$$

be the G-equivariant map defined by

$$Q_1(f\otimes v)(g) = f(g)\otimes\pi(g)\, v$$

for f in $C^\infty(G, \tau_{\lambda+\mu})$ and v in U. Let $P_\lambda : V_{\lambda+\mu}\otimes U \to V_\lambda$ be a nonzero K-intertwining operator; P_λ exists and is unique up to a scalar factor by Lemma 10.1. Define

$$\tilde{P}_\lambda(f\otimes v)(g) = P_\lambda(Q_1(f\otimes v)(g)) \tag{10.4}$$

for f in $C^\infty(G, \tau_{\lambda+\mu})$ and v in U, and let

$$\mathscr{P}^\lambda = \tilde{P}_\lambda(\mathscr{H}^{\lambda+\mu}\otimes U). \tag{10.5}$$

(See § 9 for the definition of $\mathscr{H}^{\lambda+\mu}$.)

(10.2) **Proposition.** (a) *$\tilde{P}_\lambda$ carries $\mathscr{H}^{\lambda+\mu}\otimes U$ into $\mathscr{H}^\lambda$ and is a $\mathfrak{g}$-intertwining operator from $(Q_{\lambda+\mu}\otimes\pi, \mathscr{H}^{\lambda+\mu}\otimes U)$ to $(Q_\lambda, \mathscr{H}^\lambda)$.*

(b) *The image $\mathscr{P}^\lambda$ of $\mathscr{H}^{\lambda+\mu}\otimes U$ under $\tilde{P}_\lambda$ contains the K-type τ_λ with multiplicity* 1, *and its other K-types have highest weights $\lambda+\gamma$ with γ a sum of elements in Δ^+.*

(c) *$\mathscr{P}^\lambda$ has an infinitesimal charcter.*

Remark. In Theorem 10.8 we shall see that $\mathscr{P}^\lambda$ is independent of μ if λ satisfies certain properties.

Proof. (a) It is enough to show that $\mathscr{D}(\tilde{P}_\lambda(f\otimes v)) = 0$ whenever f is in $\mathscr{H}^{\lambda+\mu}$ and v is in U. Since $\tilde{P}_\lambda$ is G-equivariant from $C^\infty(G, \tau_{\lambda+\mu})\otimes U$ to $C^\infty(G, \tau_\lambda)$, it is enough to show that $\mathscr{D}(\tilde{P}_\lambda(f\otimes v))(1) = 0$. This expression will be 0 because of the map P

in the defining formula (2.9) for $\mathscr{D}$. In fact, the map $\mathscr{D} \circ \tilde{P}_\lambda$ followed by evaluation at 1, as a map from $\mathscr{H}^{\lambda+\mu} \otimes U$ to the representation space for τ_λ^- (see (2.6)), is a K-intertwining operator. Every K-type of $\mathscr{H}^{\lambda+\mu}$ is of the form τ_ξ with $\xi = \lambda + \mu + \gamma$ and with γ a sum of elements of Δ^+, by Theorem 9.2, and $\tau_{\lambda+\mu}$ has multiplicity 1. By Lemma 10.1 applied to $\lambda + \gamma$, we obtain a decomposition

$$\mathscr{H}^{\lambda+\mu} \otimes U = \tau_\lambda + \sum \tau_{\lambda+\gamma'} \tag{10.6}$$

with each γ' a nonempty sum of members of Δ^+. Since, by (2.6), every K-type of τ_λ^- is of the form $\tau_{\lambda-\beta}$ with β in Δ_n^+, we see that $\mathscr{D}(\tilde{P}_\lambda(f \otimes v))(1) = 0$.

(b) Equation (10.6) proves the assertion about K-types, provided we show $\tilde{P}_\lambda(\tau_\lambda \text{ space}) \neq 0$. We saw in § 6 that there is some f_0 in $\mathscr{H}^{\lambda+\mu}$ with $f_0(1) \neq 0$. It then follows from the transformation property for $C^\infty(G, \tau_{\lambda+\mu})$ that evaluation at 1 carries $\mathscr{H}^{\lambda+\mu}$ onto $V_{\lambda+\mu}$ and hence carries $\mathscr{H}^{\lambda+\mu} \otimes U$ onto $V_{\lambda+\mu} \otimes U$. Composing with P_λ, we see that the map $F \to \tilde{P}_\lambda F(1)$ is a K-intertwining operator carrying $\mathscr{H}^{\lambda+\mu} \otimes U$ onto V_λ. By (10.5), this map must annihilate all but the τ_λ space. Hence it must carry the τ_λ space onto V_λ. Therefore $\tilde{P}_\lambda(\tau_\lambda \text{ space}) \neq 0$, and τ_λ does appear in $\mathscr{P}^\lambda$.

(c) The K-finite dual space $(\mathscr{P}^\lambda)^*$ of $\mathscr{P}^\lambda$ is a $\mathfrak{g}$-module. For v in V_λ^* let $\delta_v(f) = \langle f(1), v \rangle$. Then δ_v is in $(\mathscr{P}^\lambda)^*$ and we obtain a K-homomorphism of V_λ^* into $(\mathscr{P}^\lambda)^*$. The image is not 0 by the argument for (b) above. If f is in $\mathscr{P}^\lambda$ and $U(\mathfrak{g})\, V_\lambda^*$ maps f to 0, then $f = 0$ by real analyticity of f. Hence V_λ^* is $U(\mathfrak{g})$-cyclic in $(\mathscr{P}^\lambda)^*$. Since V_λ^* appears with multiplicity 1, it follows that $(\mathscr{P}^\lambda)^*$ has an infinitesimal character: $z \cdot u = \chi(z) u$ for all u in $(\mathscr{P}^\lambda)^*$ and z in the center $\mathscr{Z}$ of $U(\mathfrak{g})$. If u is in $(\mathscr{P}^\lambda)^*$ and f is in $\mathscr{P}^\lambda$, we have

$$\langle z \cdot f, u \rangle = \langle f, {}^t z \cdot u \rangle = \langle \chi({}^t z) f, u \rangle$$

and

$$\langle z \cdot f - \chi({}^t z) f, u \rangle = 0.$$

Since u is arbitrary, $zf = \chi({}^t z) f$. The proof is complete.

The next step is to study tensor products of nonunitary principal series representations with finite-dimensional representations. To describe matters we introduce a class of representations wider than the nonunitary principal series. If (ξ, H^ξ) is a finite-dimensional representation of MAN, let Y^ξ be the space of all C^∞ functions $f\colon G \to H^\xi$ such that

(i) $f(bg) = \xi(b) f(g)$ for g in G and b in MAN

(ii) f is right K-finite,

and define

$$(W(\xi, X) f)(g) = \frac{d}{dt} f(g \exp tX)|_{t=0}$$

for X in $\mathfrak{g}$ and f in Y^ξ. Then $(W(\xi), Y^\xi)$ is a representation of $\mathfrak{g}$. For a special case, let σ be an irreducible representation of M, let ν be a complex-valued real-linear form on $\mathfrak{a}$, and put $(\sigma \otimes \nu)(man) = e^{\nu \log a} \sigma(m)$. The representations $\sigma \otimes \nu$ describe all irreducible representations of MAN, by [1, p. 186]. Moreover, $W(\sigma \otimes \nu)$ is the infinitesimal version of the nonunitary principal series representation $W(\sigma, \nu)$.

(10.3) **Lemma.** *Let (ξ, H^ξ) be a finite-dimensional representation of MAN, let $H_1 \subseteq H^\xi$ be an MAN-invariant subspace, and let ξ_1 and ξ_2 be the associated representations of MAN on H_1 and H/H_1, respectively. Let $p\colon H \to H/H_1$ be the quotient map, and regard Y^{ξ_1} as a subspace of Y^ξ. For f in Y^ξ, define*

$$p(f)(g) = p(f(g)).$$

Then p is a $\mathfrak{g}$-intertwining operator of Y^ξ onto Y^{ξ_2} with kernel Y^{ξ_1}.

Proof. It is enough to prove that $p(Y^\xi) = Y^{\xi_2}$, the rest being obvious. On the level of representations of M, we can regard $H_2 = H/H_1$ as a subspace of H, writing $H = H_1 \oplus H_2$ and $\xi|_M = \xi_1|_M \oplus \xi_2|_M$. If f is in Y^{ξ_2}, we look upon $f|_K$ as a function from K into $H_2 \subseteq H$ satisfying $f(mk) = \xi_2(m) f(k)$. Define $g(bk) = \xi(b) f(k)$. Then g is in Y^ξ and $p(g) = f$. Hence p maps onto Y^{ξ_2}.

(10.4) **Lemma.** *Let (ξ, H^ξ) be a finite-dimensional representation of MAN. Let*

$$H^\xi = H_1 \supset H_2 \supset \cdots \supset H_d \supset \{0\}$$

be a composition series for (ξ, H^ξ) with $\sigma_i \otimes \nu_i$ the corresponding irreducible representation of MAN on H_i/H_{i+1}. If ξ_i denotes the restriction of ξ to H_i, then there corresponds a chain

$$Y^\xi = Y^{\xi_1} \supset Y^{\xi_2} \supset \cdots \supset Y^{\xi_d} \supset \{0\},$$

and the representation of $\mathfrak{g}$ on $Y^{\xi_i}/Y^{\xi_{i+1}}$ is infinitesimally equivalent with $W(\sigma_i, \nu_i)$.

Remark. We follow [11] in the terminology "chain" and "composition series."

Proof. Iterate Lemma 10.3.

(10.5) **Lemma.** *Let (ξ, H^ξ) be a finite-dimensional representation of MAN, and let (π, U) be a finite-dimensional representation of G. Set $\beta = \pi|_{MAN}$. For f in Y^ξ and v in U define*

$$Q_2(f \otimes v)(g) = f(g) \otimes \pi(g)\, v. \tag{10.7}$$

Then $Q_2\colon Y^\xi \otimes U \to Y^{\xi \otimes \beta}$ defines a one-one $\mathfrak{g}$-intertwining operator of $W(\xi) \otimes \pi$ onto $W(\xi \otimes \beta)$.

Proof. An obvious computation shows that Q_2 carries $Y^\xi \otimes U$ into $Y^{\xi \otimes \beta}$ and is a $\mathfrak{g}$-intertwining operator. For f in $Y^{\xi \otimes \beta}$, let

$$(Q_3 f)(g) = (I \otimes \pi(g)^{-1})(f(g)).$$

Then $Q_3 \circ Q_2$ is the identity on $Y^\xi \otimes U$. Moreover, if f is in $Y^{\xi \otimes \beta}$, let $\{v_i\}$ be a basis of U and define functions f_i by

$$f(g) = \sum f_i(g) \otimes \pi(g)\, v_i.$$

Then each f_i is in Y^ξ and it follows readily that $Q_2 \circ Q_3$ is the identity on $Y^{\xi \otimes \beta}$.

(10.6) **Proposition.** *Let $\xi = \sigma \otimes \nu$ be an irreducible finite-dimensional representation of MAN on H, and let (π, U) be a finite-dimensional representation of G. If $\xi \otimes \pi|_{MAN}$ has a composition series*

$$H \otimes U = H_1 \supset H_2 \supset \cdots \supset H_d \supset \{0\}$$

with irreducible quotient representations $\sigma_i \otimes \nu_i$ on H_i/H_{i+1}, then $W(\sigma,\nu)\otimes\pi$ has a corresponding chain, and the respective quotients are infinitesimally equivalent with the $W(\sigma_i,\nu_i)$ for $1 \leqq i \leqq d$.

Proof. Combine Lemmas 10.4 and 10.5.

We have studied tensor products of finite-dimensional representations with the representation on the space $\mathscr{H}^\lambda$ and with nonunitary principal series. Next we study the effect of tensoring on the Szegö kernel.

(10.7) **Proposition.** *If the integral parameter λ is such that $\Lambda=\lambda+\delta_k-\delta_n$ is Δ^+-dominant and $\Lambda-\delta_k=\lambda-\delta_n$ is Δ_k^+-dominant, then the Szegö kernel S_λ carries $W(\sigma_\lambda, \nu'(\lambda))$ into $\mathscr{P}^\lambda$.*

Remark. So far, we have not proved that $\mathscr{P}^\lambda$ is independent of μ. We do not prove this until Theorem 10.8.

Proof. Again introduce μ and (π, U). Let ξ and β be the representations $\sigma_{\lambda+\mu}\otimes\nu'(\lambda+\mu)$ and $\pi|_{MAN}$, respectively, of MAN. By Corollary 9.6, $S_{\lambda+\mu}$ maps Y^ξ onto $\mathscr{H}^{\lambda+\mu}$. By (10.5), $\tilde{P}_\lambda(S_{\lambda+\mu}\otimes I)$ maps $Y^\xi\otimes U$ onto $\mathscr{P}^\lambda$. On the other hand, if f is in Y^ξ and v is in U, (10.4) and (10.7) give

$$\begin{aligned}\tilde{P}_\lambda(S_{\lambda+\mu} f\otimes v)(g) &= P_\lambda\Big(\int_K \tau_{\lambda+\mu}(k) f(k^{-1}g)\,dk\otimes\pi(g)\,v\Big)\\ &= P_\lambda\Big(\int_K \tau_{\lambda+\mu}(k)\otimes\pi(k)\Big)\big(f(k^{-1}g)\otimes\pi(k^{-1}g)\,v\big)\,dk\\ &= \int_K \tau_\lambda(k)\,P_\lambda(f(k^{-1}g)\otimes\pi(k^{-1}g)\,v)\,dk\\ &= \int_K \tau_\lambda(k)\,P_\lambda Q_2(f\otimes v)(k^{-1}g)\,dk.\end{aligned}$$

Here $Q_2(f\otimes v)$ is in $Y^{\xi\otimes\beta}$ by Lemma 10.5, and we conclude that the operator Z defined on $Y^{\xi\otimes\beta}$ by

$$Z(f)(g)=\int_K \tau_\lambda(k)\,P_\lambda(f(k^{-1}g))\,dk \tag{10.8}$$

carries $Y^{\xi\otimes\beta}$ onto $\mathscr{P}^\lambda$.

Let v_0 be a nonzero lowest weight vector for π, taken relative to the Cartan subalgebra $\mathfrak{t}$. Let $u=u_{\alpha_1}\cdot\dots\cdot u_{\alpha_m}$ be the Cayley transform with u_{α_j} defined by (5.7). A computation in $SL(2,\mathbb{C})$ gives

$$\pi(u_{\alpha_j})=\exp\pi(E_{\alpha_j})\exp\pi(\log(\sqrt{2})\,H_{\alpha_j})\exp\pi(-E_{-\alpha_j}). \tag{10.9}$$

Since v_0 is a lowest weight vector, we can apply (10.9) to see that

$$\pi(u)\,v_0=c\exp\pi(E_{\alpha_1}+\dots+E_{\alpha_m})\,v_0=c\,v_0+\text{higher weight vectors}, \tag{10.10}$$

where c is the nonzero constant

$$c=2^{-\Sigma\langle\mu,\alpha_j\rangle/\langle\alpha_j,\alpha_j\rangle}.$$

In view of (10.10) and Lemma 10.1, we see that

$$P_\lambda(\phi_{\lambda+\mu}\otimes\pi(u)\,v_0)=cP_\lambda(\phi_{\lambda+\mu}\otimes v_0)=c'\,\phi_\lambda, \tag{10.11}$$

where $c'\neq 0$ and ϕ_λ is a unit highest weight vector for the τ_λ subspace of $\tau_{\lambda+\mu}\otimes\pi|_K$.

We return to the operator Z, given by (10.8). For f in $Y^{\sigma_\lambda \otimes \nu'(\lambda)}$, define

$$\tilde{f}(k) = \int_M (\xi(m) \otimes \beta(m)) \langle f(m^{-1} k), \phi_\lambda \rangle (\phi_{\lambda+\mu} \otimes \pi(u) v_0)\, dm. \tag{10.12}$$

A change of variables shows that

$$\tilde{f}(mk) = (\xi(m) \otimes \beta(m)) \tilde{f}(k),$$

so that we can extend $\tilde{f}$ to G and obtain a member of $Y^{\xi \otimes \beta}$ by defining

$$\tilde{f}(bk) = \xi \otimes \beta(b)) \tilde{f}(k)$$

for b in MAN and k in K. Thus Z is defined on $\tilde{f}$. Under the hypotheses of the proposition, we shall prove that

$$S_\lambda f(g) = c''^{-1} Z \tilde{f}(g) \tag{10.13}$$

with $c'' \neq 0$, and this equality will finish the proof since the image of Z is $\mathscr{P}^\lambda$. For now, let us observe that (10.13) holds at $g = 1$. In fact, first we note that

$$P_\lambda(\tilde{f}(k)) = c'' f(k) \tag{10.14}$$

because (10.12) and (10.11) give

$$\begin{aligned} P_\lambda(\tilde{f}(k)) &= \int_M \langle f(m^{-1} k), \phi_\lambda \rangle P_\lambda((\tau_{\lambda+\mu}(m) \otimes \pi(m)) (\phi_{\lambda+\mu} \otimes \pi(u) v_0))\, dm \\ &= \int_M \langle f(k), \sigma_\lambda(m) \phi_\lambda \rangle \tau_\lambda(m) P_\lambda(\phi_{\lambda+\mu} \otimes \pi(u) v_0)\, dm \\ &= c' \int_M \langle f(k), \sigma_\lambda(m) \phi_\lambda \rangle \tau_\lambda(m) \phi_\lambda\, dm \\ &= c' \int_M \langle f(k), \sigma_\lambda(m) \phi_\lambda \rangle \sigma_\lambda(m) \phi_\lambda\, dm \\ &= c'' f(k) \end{aligned}$$

by Schur orthogonality. Combining (10.14) and (10.8), we see that (10.13) holds for $g = 1$. Since S_λ is a $\mathfrak{g}$-intertwining operator; the proof will be complete if we show that $f \to Z\tilde{f}$ is a $\mathfrak{g}$-intertwining operator. This step will require some preparation.

The Cayley transform u carries $\mathfrak{t}^{\mathbf{C}}$ to the Cartan subalgebra $\mathfrak{h}^{\mathbf{C}}$ defined in § 8, and hence $\pi(u) v_0$ is a weight vector of π relative to $\mathfrak{h}^{\mathbf{C}}$. Its weight is $-\mu \circ u^{-1}$. Let $\mu_1, \ldots, \mu_d$ be the weights of (π, U) relative to $\mathfrak{h}^{\mathbf{C}}$, repeated according to their multiplicities and arranged in increasing order relative to Ψ^+. Let $v_1, \ldots, v_d$ be corresponding nonzero weight vectors; we can arrange that $v_i = \pi(u) v_0$ is a member of this list. Let $w_j = \phi_{\lambda+\mu} \otimes v_j$, let W_j be the MAN cyclic space for w_j within the space $H_{\lambda+\mu} \otimes U$ of $\xi \otimes \beta$, and let $U_k = \sum_{j \geq k} W_j$. Then we have

$$I_1 \supseteq U_2 \supseteq \cdots \supseteq U_d \supseteq \{0\}.$$

The Lie algebra $\mathfrak{n}$ operates trivially on $\phi_{\lambda+\mu}$ and increases the weight of v_j relative to Ψ^+; thus $\xi \otimes \beta(\mathfrak{n})\, U_j \subseteq U_{j+1}$. Also root vectors corresponding to Ψ_m^+ carry U_j into U_{j+1}. It follows that MAN operates irreducibly on U_j/U_{j+1} if U_j/U_{j+1} is not 0. That is,

$$U_j/U_{j+1} = 0 \quad \text{or} \quad \sigma_j \otimes \nu_j.$$

Setting $Y_j = \{f \in Y^{\xi \otimes \beta} | f(g) \in U_j \text{ for all } g\}$, we see from Lemma 10.4 that

$$Y_j / Y_{j+1} = 0 \quad \text{or} \quad Y^{\sigma_j \otimes \nu_j}$$

as $\mathfrak{g}$-modules.

If i is the index with $v_i = \phi_{\lambda+\mu} \otimes \pi(u) v_0$ and if f is in $Y^{\sigma_\lambda \otimes \nu'(\lambda)}$, then $\tilde{f}$ takes its values in $W_i \subseteq U_i$, and hence $\tilde{f}$ is in Y_i. We shall identify the parameters (σ_i, ν_i). In fact, for any j, our arguments above give the highest weight of $\sigma_j \otimes \nu_j$ as

$$s_0 \Lambda(\sigma_j, \nu_j) = s_0 \Lambda(\sigma_{\lambda+\mu}, \nu'(\lambda+\mu)) + (\text{weight of } \pi), \tag{10.15}$$

and the weight of π is $-\mu \circ u^{-1} + Q \circ u^{-1}$ with Q a sum of members of Δ^+. Moreover, Q is 0 if and only if $j = i$. Applying Proposition 8.2, we see that the right side of (10.15) is

$$= (\Lambda + \mu) \circ u^{-1} + s_0 \rho - \mu \circ u^{-1} + Q \circ u^{-1}$$

$$= \Lambda \circ u^{-1} + s_0 \rho + Q \circ u^{-1} \tag{10.16}$$

$$= s_0 \Lambda(\sigma_\lambda, \nu'(\lambda)) + Q \circ u^{-1}. \tag{10.17}$$

Taking $j = i$, we see from (10.17) that $\sigma_i|_{M_0} \cong \sigma_\lambda|_{M_0}$ and

$$\nu_i = \nu'(\lambda). \tag{10.18}$$

With this much information, we can see that $f \to \tilde{f} + Y_{i+1}$ is a $\mathfrak{g}$-intertwining operator. In fact, it is easier to work on the G-level and differentiate afterward. Denoting the operation of G by subscripts, we are to compare $(f_g)^\sim$ with $(\tilde{f})_g$. With the Iwasawa decomposition $G = ANK$ as $g = e^{H(g)} n \kappa(g)$, we have

$$\begin{aligned}
(f_g)^\sim(k) &= \int_M \xi \otimes \beta(m) \langle f_g(k), \sigma_\lambda(m) \phi_\lambda \rangle (\phi_{\lambda+\mu} \otimes \pi(u) v_0)\, dm \\
&= \int_M \xi \otimes \beta(m) \langle f(kg), \sigma_\lambda(m) \phi_\lambda \rangle (\phi_{\lambda+\mu} \otimes \pi(u) v_0)\, dm \\
&= \int_M \xi \otimes \beta(m)\, e^{\nu'(\lambda) H(kg)} \langle f(\kappa(kg)), \sigma_\lambda(m) \phi_\lambda \rangle (\phi_{\lambda+\mu} \otimes \pi(u) v_0)\, dm \\
&= e^{\nu'(\lambda) H(kg)} \tilde{f}(\kappa(kg))
\end{aligned}$$

and

$$\begin{aligned}
(\tilde{f})_g(k) &= \tilde{f}(kg) = \xi \otimes \beta(e^{H(kg)} n)\, \tilde{f}(\kappa(kg)) \\
&\equiv e^{\nu_i H(kg)} \tilde{f}(\kappa(kg)) \bmod U_{i+1}.
\end{aligned}$$

By (10.18), we therefore have

$$(\tilde{f})_g(k) \equiv (f_g)^\sim(k) \bmod U_{i+1}.$$

Differentiating in g and applying Lemma 10.3, we conclude that the map

$$f \to \tilde{f} + Y_{i+1}$$

of $Y^{\sigma_\lambda \otimes \nu'(\lambda)}$ into Y_i / Y_{i+1} is a $\mathfrak{g}$-intertwining operator. Now Z is a $\mathfrak{g}$-intertwining operator, and hence $f \to Z\tilde{f} + Z(Y_{i+1})$ is a $\mathfrak{g}$-intertwining operator from $Y^{\sigma_\lambda \otimes \nu'(\lambda)}$ to $\mathscr{P}^\lambda / Z(Y_{i+1})$.

To complete the proof, we shall show that $Z(Y_{i+1}) = 0$. It is enough to show that the value of the Casimir operator Ω on $\mathscr{P}^\lambda$ (Proposition 10.2c and Corollary 3.2) is different from the value of Ω on $W(\sigma_j, v_j)$ if $j \geqq i+1$. Corollary 3.2 gives the value on $\mathscr{P}^\lambda$ as $|\Lambda|^2 - |\delta|^2$, since Proposition 10.2a says $\mathscr{P}^\lambda \subseteq \mathscr{H}^\lambda$. The value on $W(\sigma_j, v_j)$ is

$$|\Lambda(\sigma_j, v_j) - \rho|^2 - |\rho|^2,$$

which by (10.16) is

$$= |s_0(\Lambda \circ u^{-1}) + \rho + s_0(Q \circ u^{-1}) - \rho|^2 - |\rho|^2$$
$$= |\Lambda + Q|^2 - |\rho|^2.$$

Since Λ is Δ^+-dominant, $\langle \Lambda, Q \rangle$ is $\geqq 0$. Therefore

$$|\Lambda + Q|^2 - |\rho|^2 > |\Lambda|^2 - |\rho|^2$$

as soon as Q is not 0. Therefore the values of Ω do not match and we have $Z(Y_{i+1}) = 0$. The proof is complete.

The final step is to bring in tensor products with discrete series. We obtain the theorem below as a sharp form of Theorem 1.1, in view of Theorem 6.1′.

(10.8) **Theorem.** *Suppose the integral parameter λ is such that $\Lambda = \lambda + \delta_k - \delta_n$ is Δ^+-dominant and $\Lambda - \delta_k$ is Δ_k^+-dominant. Then*

(i) *$\mathscr{P}^\lambda$ is canonically defined, is independent of μ, and is irreducible,*

(ii) *S_λ carries $W(\sigma_\lambda, v'(\lambda))$ onto $\mathscr{P}^\lambda$, and*

(iii) *$\mathscr{P}^\lambda$ contains τ_λ with multiplicity 1, and τ_λ is the lowest K-type in $\mathscr{P}^\lambda$.*

Moreover, if $\langle \Lambda, \alpha \rangle > 0$ for all α in Δ^+, then $\mathscr{P}^\lambda$ is infinitesimally equivalent with the discrete series representation (π_Λ, H^Λ).

Proof. By Corollary 9.6, $\mathscr{H}^{\lambda+\mu}$ is irreducible and is equivalent with $(\pi_{\Lambda+\mu}, H^{\Lambda+\mu})$ if μ is sufficiently large. Form (π, U) with lowest weight $-\mu$. Zuckerman [25] has proved that the image of the projection of $H^{\Lambda+\mu} \otimes U$ according to the infinitesimal character corresponding to $s_0(\Lambda \circ u^{-1}) + \rho$ is irreducible if Λ is Δ^+-dominant and is infinitesimally equivalent with π_λ if Λ is also nonsingular. Now $H^{\Lambda+\mu} \otimes U$ is isomorphic with $\mathscr{H}^{\lambda+\mu} \otimes U$, and the image of the Szegö kernel also has infinitesimal character corresponding to $s_0(\Lambda \circ u^{-1}) + \rho$, by Proposition 8.2. By Proposition 10.7, the image of the Szegö kernel is contained in $\mathscr{P}^\lambda$, and by Proposition 10.2c $\mathscr{P}^\lambda$ has an infinitesimal character. Hence $\mathscr{P}^\lambda$ has infinitesimal character corresponding to $s_0(\Lambda \circ u^{-1}) + \rho$ and is contained in the image of the projection according to this infinitesimal character. By Zuckerman's irreducibility result, we must have equality. Then $\mathscr{P}^\lambda$ must be irreducible, and S_λ must map onto it. That is, $\mathscr{P}^\lambda$ is canonical. Result (iii) is by Proposition 10.2b. The final statement of the theorem is now clear.

Remark. The hypothesis in both Proposition 10.7 and Theorem 10.8 that $\Lambda - \delta_k = \lambda - \delta_n$ be Δ_k^+-dominant was made in order to guarantee that the eigenvalue of the Casimir operator on $\mathscr{P}^\lambda$ be $|\Lambda|^2 - |\delta|^2$. By means of Corollary 3.9 of Wallach [23], it is easily seen that this fact persists for all Δ_k^+-dominant λ. The proofs of Proposition 10.7 and Theorem 10.8 go through unchanged for the wider class of λ if this observation is made. Therefore Proposition 10.7 and Theorem 10.8 are valid without the additional assumption that $\lambda-\delta_n$ is Δ_k^+-dominant.

11. Analog of Harish-Chandra's ψ_Λ-Function

In this section we describe an alternate, and as yet unsuccessful, approach to the proof of Theorem 1.1. We start with a generalization of Harish-Chandra's ψ_Λ function for holomorphic discrete series. With the parameter λ chosen so that $\Lambda=\lambda+\delta_k-\delta_n$ is Δ^+-dominant and nonsingular, introduce the member f_λ of $C^\infty(K,\sigma_\lambda)$ defined by

$$f_\lambda(k)=\int_M \tau_\lambda(m)^{-1}\langle\tau_\lambda(mk)\,\phi_\lambda,\phi_\lambda\rangle\,\phi_\lambda\,dm.$$

Then f_λ is K-finite. Extend it to a function on G as in § 6.

We compute a particular K-finite matrix coefficient for the representation Q_λ in the image of the Szegö kernel S_λ. The coefficient is

$$\psi_\lambda(g)=\langle Q_\lambda(g)\,S_\lambda f,\delta\rangle,$$

where f is f_λ and δ is the K-finite linear functional on the image of S_λ given as the composition of evaluation at 1 followed by inner product with ϕ_λ. If ϕ_λ is a unit vector, easy computation gives

$$\psi_\lambda(g)=\int_K e^{\nu H(kg)}\langle\tau_\lambda(\kappa(kg))\,\phi_\lambda,\phi_\lambda\rangle\,\overline{\langle\tau_\lambda(k)\,\phi_\lambda,\phi_\lambda\rangle}\,dk. \tag{11.1}$$

Suppose that one could prove

(i) ψ_λ is square-integrable on G and

(ii) the image of S_λ is irreducible.

Since ψ_λ is $\mathscr{Z}$-finite and right-and-left K-finite, it would follow from Harish-Chandra's theory that the image of S_λ is infinitesimally equivalent with the cyclic space for ψ_λ, which in turn would be an irreducible subspace of $L^2(G)$. Then it would have been proved that the image of S_λ is an irreducible discrete series representation. Proposition 8.2 would show its infinitesimal character matches that of (π_Λ, H^Λ), and we would know it contained the K-type τ_λ, since $S_\lambda f_\lambda$ has that K-type. These facts would substantially prove Theorem 1.1.

Partial results in this direction have been proved by the authors in some special cases. For $SU(2,1)$, see [24].

12. Groups of Real-Rank One

We assume in this section that rank $G=$ rank K, that G has a simply-connected complexification $G^{\mathbb{C}}$, and that G has real-rank one. That is, $\dim A=1$ in any Iwasawa decomposition $G=ANK$. We shall show how irreducible images of Szegö kernels account for all of the reducibility of the unitary principal series of G.

Temporarily fix a system Δ^+ of positive roots. We begin with four easy results that give interpretations of the real-rank one data.

(12.1) **Lemma.** *All the elements of Δ_n^+ have the same length.*

Proof. If α and β are in Δ_n^+, then $\mathbf{R}(E_\alpha+E_{-\alpha})$ and $\mathbf{R}(E_\beta+E_{-\beta})$ are maximal abelian subspaces of $\mathfrak{p}$. Consequently there is an element k in K with $\mathrm{Ad}(k)(E_\alpha+E_{-\alpha})=c(E_\beta+E_{-\beta})$. Possibly by interchanging α and β, we may assume that $|c|\leqq 1$. By Lemma 3 of [12], the eigenvalues of $\mathrm{ad}(E_\alpha+E_{-\alpha})$ on $\mathfrak{g}$ are bounded by 2 in absolute value, and 2 is achieved, say on a vector X. Similarly the eigenvalues of $\mathrm{ad}(E_\beta+E_{-\beta})$ are bounded by 2. But we readily compute that

$$[E_\beta+E_{-\beta}, \mathrm{Ad}(k)X]=2c^{-1}\,\mathrm{Ad}(k)X,$$

and we conclude that $|c|=1$. Then (2.1) shows that $|\alpha|=|\beta|$.

(12.2) **Corollary.** *If α in Δ_n^+ is simple, then α is a fundamental sequence of positive noncompact roots, in the sense of Definition* 4.1.

Fix α as in the corollary, and construct the corresponding Iwasawa decomposition as in § 5.

(12.3) **Lemma.** *$M=M_0Z$, where Z is the center of G.*

Proof. We may assume G is simple. By Proposition 5 of [12], either M is connected or $G\cong SL(2,\mathbf{R})$. The lemma is clear in both cases.

(12.4) **Lemma.** *Let λ be integral and Δ_k^+ dominant, and let $\Lambda=\lambda+\delta_k-\delta_n$. Then $W(\sigma_\lambda, \nu'(\lambda))$ is in the unitary principal series if and only if $\langle\Lambda,\alpha\rangle=0$.*

Proof. By (6.5) and Lemma 8.5,

$$\begin{aligned}\nu'(\lambda)(E_\alpha+E_{-\alpha})&=2\rho^+(E_\alpha+E_{-\alpha})-\frac{2\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle}-2n_1\\&=\rho^+(E_\alpha+E_{-\alpha})-\frac{2\langle\lambda+\delta_k-\delta_n,\alpha\rangle}{\langle\alpha,\alpha\rangle}\\&=\rho^+(E_\alpha+E_{-\alpha})-\frac{2\langle\Lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle}.\end{aligned}$$

Since λ is integral, $\nu'=\rho^++i\mu$ with μ real on $\mathfrak{a}$ if and only if $\langle\Lambda,\alpha\rangle=0$.

To proceed, we shall adopt in succession two points of view—first that we want to imbed a class of "limits of discrete series" in unitary principal series, and second that we want to decompose all reducible unitary principal series by means of Szegö kernels as sums of "limits of discrete series." For a special case of these results, see § 8 of [14].

For the first point of view we start with Λ integral such that Λ is orthogonal to each member of a nonempty set of noncompact roots and to no compact roots. Introduce Δ^+ so that Λ is dominant with respect to Δ^+. The data (Λ,Δ^+) determine one representation that we shall imbed.

(12.5) **Lemma.** *If the integral parameter Λ is Δ^+-dominant and is orthogonal to at least one noncompact root and to no compact roots, then there is exactly one α in Δ^+ such that $\langle\Lambda,\alpha\rangle=0$, and α is simple.*

Proof. Since Λ is dominant, the set of roots in Δ^+ orthogonal to Λ is generated by the simple roots in the set. All of these must be noncompact. If there is more than one, two such cannot be orthogonal since they would be strongly orthogonal and would exhibit the real rank of G as greater than one. If they are nonorthogonal, their sum is a compact root, and Λ is orthogonal to it, contradiction. This proves the lemma.

With α as in Lemma 12.5, let

$$\Delta^{+\prime} = P_\alpha \Delta^+ = (\Delta^+ - \{\alpha\}) \cup \{-\alpha\}. \tag{12.1}$$

Then $\Delta^{+\prime}$ is a new system of positive roots, $\Delta^{+\prime}$ has the same positive compact roots as Δ^+, and Λ is dominant with respect to $\Delta^{+\prime}$. The data $(\Lambda, \Delta^{+\prime})$ determine a second representation that we shall imbed. Let δ_n' be half the sum of the positive noncompact roots for $\Delta^{+\prime}$.

Since α is simple in Δ^+ and $-\alpha$ is simple in $\Delta^{+\prime}$, Corollary 12.2 allows us to apply our theory to (Δ^+, α) and to $(\Delta^{+\prime}, -\alpha)$. If $G = ANK$ is the Iwasawa decomposition associated to (Δ^+, α), then $G = ANK$ is also the Iwasawa decomposition associated to $(\Delta^{+\prime}, -\alpha)$, since $E_{-\alpha} + E_\alpha = E_\alpha + E_{-\alpha}$.

(12.6) **Theorem.** *Suppose the integral parameter Λ is Δ^+-dominant and satisfies $\langle \Lambda, \alpha \rangle = 0$ for a noncompact simple root α and $\langle \Lambda, \beta \rangle \neq 0$ for all other positive roots. Define $\Delta^{+\prime}$ by* (12.1) *and let*

$$\lambda = \Lambda - \delta_k + \delta_n$$

$$\lambda' = \Lambda - \delta_k + \delta_n' = \lambda - \alpha.$$

Then λ and λ' are integral and Δ_k^+-dominant and σ_λ is equivalent with $\sigma_{\lambda'}$. Moreover, the unitary principal series representation $W(\sigma_\lambda, \rho^+)$ is infinitesimally equivalent with the direct sum of the K-finite images of $S_{(\lambda, \Delta^+)}$ and $S_{(\lambda', \Delta^{+\prime})}$.

Proof. Since $G^{\mathbf{C}}$ is simply-connected, δ is integral. Thus $-\delta_k + \delta_n = -2\delta_k + \delta$ is integral and λ is integral. The form λ is Δ_k^+-dominant since δ_n is and since $\Lambda - \delta_k$ is (Λ being Δ_k^+ nonsingular and G being linear). Similarly λ' is integral and Δ_k^+-dominant. Now $\lambda' = \lambda - \alpha$ shows $\lambda|_{\mathfrak{h}^-} = \lambda'|_{\mathfrak{h}^-}$, where $\mathfrak{h}^-$ is the Cartan subalgebra of $\mathfrak{m}$ given by (5.8). Since λ' and λ differ by a root, their associated characters are equal on the center Z of G. Then Lemma 12.3 and Proposition 5.5 show that σ_λ and $\sigma_{\lambda'}$ are equivalent. The domain of $S_{(\lambda, \Delta^+)}$ is $W(\sigma_\lambda, \rho^+)$ by the proof of Lemma 12.4. Since $W(\sigma_\lambda, \rho^+)$ is unitary, the K-finite image of $S_{(\lambda, \Delta^+)}$ imbeds in $W(\sigma_\lambda, \rho^+)$. Now ρ^+ is insensitive to the change from Δ^+ to $\Delta^{+\prime}$, and thus $S_{(\lambda', \Delta^{+\prime})}$ has domain $W(\sigma_{\lambda'}, \rho^+) \cong W(\sigma_\lambda, \rho^+)$. Then the K-finite image of $S_{(\lambda', \Delta^{+\prime})}$ imbeds in $W(\sigma_\lambda, \rho^+)$.

In any ordering compatible with Δ^+, Theorem 10.8 shows that τ_λ is the lowest K-type in the image of $S_{(\lambda, \Delta^+)}$. In particular, τ_λ occurs and $\tau_{\lambda'} = \tau_{\lambda - \alpha}$ does not. Symmetrically $\tau_{\lambda'}$ occurs in the image of $S_{(\lambda', \Delta^{+\prime})}$ and τ_λ does not. Since by [1] $W(\sigma_\lambda, \rho^+)$ decomposes into at most two irreducible pieces, we obtain the theorem.

Turning to the completeness theorem, we note that the case $\Lambda = 0$ when $G = SL(2, \mathbf{R})$ in Theorem 12.6 accounts for the only reducibility of the principal series of $SL(2, \mathbf{R})$. Thus we may assume that M is connected, by Proposition 5 of [12]. For this theorem we regard MAN as fixed, obtained in the standard way from a noncompact root $\pm\alpha$. Let $\mathfrak{h}^-$ be the Cartan subalgebra of $\mathfrak{m}$ given by

(5.8), and fix a system Ψ_m^+ of positive roots of $\mathfrak{m}$. Let ρ^- be half the sum of the positive roots of $\mathfrak{m}$.

(12.7) **Theorem.** *Suppose M is connected. Let σ be an irreducible representation of M with highest weight Λ^-, and suppose that the unitary principal series representation $W(\sigma, \rho^+ + i\mu)$ is reducible. Then $\mu=0$. Moreover if Λ denotes the extension of $\Lambda^- + \rho^-$ to $\mathfrak{t}^{\mathbf{C}}$ by 0 on $\mathbf{R}H_\alpha$ and if Δ^+ is chosen to make Λ Δ^+-dominant and to make α, say, positive, then*

(i) *α is simple*

(ii) *$\langle \Lambda, \alpha \rangle = 0$, and $\langle \Lambda, \beta \rangle \neq 0$ for all other positive roots*

(iii) *Λ is integral*

(iv) *$\lambda = \Lambda - \delta_k + \delta_n$ has the property that σ_λ is equivalent with σ.*

Consequently the reducibility of $W(\sigma, \rho^+)$ is accounted for by Theorem 12.6.

Proof. By Theorem 5 of [15] and the remarks after it, the reducibility implies that $\mu=0$, that $\langle \Lambda^- + \rho^-, \beta \rangle = 0$ only for $\beta = \pm\alpha$, and that $\exp\{(\Lambda^- + \rho^-)(H)\} = +1$ for any element H in $\mathfrak{h}^-$ with $\exp H = \gamma \equiv \exp \pi i (E_\alpha + E_{-\alpha})$.

Since $\langle \Lambda, \beta \rangle = \langle \Lambda^- + \rho^-, \beta \rangle$ for all β, (ii) is immediate and (i) follows from Lemma 12.5. To prove (iv) we are to show that $\lambda|_{\mathfrak{h}_-} = \Lambda^-$. That is, we want $\rho^- = (-\delta_k + \delta_n)|_{\mathfrak{h}_-}$. But this is simply formula (8.1), which is valid since (i) shows that M has been constructed from a fundamental sequence.

We are left with (iii). A computation in $SL(2, \mathbf{R})$ shows that the element γ is also given by $\gamma = \exp \pi i H_\alpha$. For any root β let $\gamma_\beta = \exp \pi i H_\beta$ and $\bar{\beta} = -p_\alpha \beta$. Suppose $\langle \beta, \alpha \rangle > 0$ and $\beta \neq \alpha$. Then Lemmas 1 and 3 of [12] show that either $\frac{2\langle \beta, \bar{\beta} \rangle}{|\beta|^2} = -1$ and $\beta + \bar{\beta} = \alpha$ or $\frac{2\langle \beta, \bar{\beta} \rangle}{|\beta|^2} = 0$ and $\frac{1}{2}(\beta + \bar{\beta}) = \alpha$. In either case it follows that $\gamma_\beta \gamma_{\bar{\beta}} = \gamma$. A computation in $SL(2, \mathbf{C})$ shows that $\gamma_{\bar{\beta}}^2 = 1$, and so $\gamma = \gamma_\beta \gamma_{\bar{\beta}}^{-1} = \exp \pi i (H_\beta - H_{\bar{\beta}})$. The element $\pi i (H_\beta - H_{\bar{\beta}})$ is in $\mathfrak{h}^-$, and therefore

$$(\Lambda^- + \rho^-)(\pi i (H_\beta - H_{\bar{\beta}})) \text{ is in } 2\pi i \mathbf{Z}.$$

That is, $\frac{2\langle \Lambda^- + \rho^-, \beta \rangle}{|\beta|^2}$ is an integer. For the remaining roots β, we note that $\langle \Lambda^- + \rho^-, \alpha \rangle = 0$ and that $\frac{\langle \Lambda^- + \rho^-, \beta \rangle}{|\beta|^2}$ is an integer if $\langle \beta, \alpha \rangle = 0$ since β is then a root of $\mathfrak{m}$. Consequently $\Lambda = \Lambda^- + \rho^-$ is integral in the algebraic sense. Since $G^{\mathbf{C}}$ is simply-connected, Λ is integral. The proof is complete.

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Notes Added in Proof. (1) J. Carmona has discovered a proof of Proposition 4.5 that avoids the case-by-case check for B_n, C_n, and F_4, but not for G_2.

(2) The second named author has recently carried out the program of § 11 in the case of real-rank one.