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Intertwining Operators and Small Unitary Representations

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In memory of Harish-Chandra

ABSTRACT. In several authors' attempts to classify the irreducible unitary representations of semisimple Lie groups, representations that are "small" play a pivotal role. The trouble is that there are too many small representations for the unitarity of all of them to be decided by direct calculations. This article proposes a technique for combining the use of intertwining operators and cohomological induction to reduce the investigation of all small representations to the investigation of just a few of them. It illustrates the technique by giving applications to analytic continuations of discrete series, both holomorphic and nonholomorphic. It includes a certain amount of expository background concerning discrete series, analytic continuations thereof, and cohomological induction.

Introduction

Harish-Chandra spent most of his mathematical life working on the harmonic analysis of semisimple groups. To him we owe an appreciation of the richness and importance of the subject, the idea of reducing harmonic analysis to questions in abelian Fourier analysis, the notion of a character of an infinite-dimensional representation, the analytic properties of characters, the classification of the discrete series, and many other things. These things are now taken by many researchers in the field as so fundamental that one often no longer even cites the relevant papers of Harish-Chandra, despite the hundreds of pages that Harish-Chandra may have required in their development. Such is the nature of Harish-Chandra's contribution to the field: it is so fundamental that it is not done justice by citations of papers.

In this article we investigate a technique combining three things—a continuation of Harish-Chandra's discrete series, the construction of representations by cohomological induction, and properties of intertwining operators—in order to identify certain irreducible representations as unitary. To put this construction in context, we begin with some background.

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The problem of classifying the irreducible unitary representations of a semisimple Lie group remains unsolved. Some of the partial results (e.g., [**BSK1**]), suggest that the statement of a complete classification may be so complicated that no reasonable theorem is possible. Instead of aiming for a complete classification, it may be more advisable to search for all the "important" irreducible unitary representations, whatever "important" may mean.

General techniques for proving that representations are unitary are few in number. The oldest is ordinary induction: induction carries unitary representations of a closed subgroup to unitary representations of the whole group, provided the induction process is suitably normalized. In practice the subgroup is often taken to be the Levi component of a parabolic subgroup, and then the induction process is called *parabolic induction* and also takes an irreducible representation to a representation with a finite composition series.

Another technique is continuity arguments—originally done in the context of parabolic induction. The idea is that positivity of an invariant Hermitian form is preserved when a representation varies through a continuous family if the form varies continuously through nonsingular Hermitian forms and if certain admissibility hypotheses are satisfied. The nonsingularity is often proved by establishing irreducibility for the representations in question.

Still another technique is cohomological induction, whose definition will be reviewed below. Cohomological induction carries a representation of a certain kind of subgroup to a representation of the whole group, and it preserves unitarity under a suitable positivity condition on the parameters. This positivity condition can be weakened when the inducing representation is "small" in a suitable sense, and this fact will be an important ingredient in the theory.

The work of several authors (e.g., [**B**], [**KS**], [**SRV**], [**BSK1**]) suggests that in order to find the "important" irreducible unitary representations of a semisimple Lie group, one should first look for those that are "small." The trouble is that there are too many "small" representations of most groups to handle by special arguments. The question that we therefore address is the building of most "small" representations from a few of them.

The technique will be to use intertwining operators between cohomologically induced representations. The organization is as follows:

When an irreducible representation is cohomologically induced with the positivity condition in place, the resulting representation is irreducible and should be regarded as unlikely to be small. To obtain small representations, we need to go outside the positivity range. There we can attempt to isolate irreducible subquotients of cohomologically induced representations by means of kernels and images of intertwining operators. The representations that we study in this paper will be subrepresentations of images of such operators, and our main theorem for providing these operators will be Theorem 5.1 below.

We shall illustrate matters first with a case where the unitarity is already completely known—the case of "analytic continuations of holomorphic discrete series." We may think of all these representations as small. Among them we distinguish "line-bundle cases" and "vector-bundle cases." The unitarity in the line-bundle cases was settled by Wallach [**W1**] essentially by a direct calculation. Although our technique yields the unitarity of some of these from others, let us regard all these representations as handled by special arguments. We use Theorem 5.1 to show how the unitarity of vector-bundle cases can in effect be deduced from that of the line-bundle cases, at least when the Dynkin diagram of the underlying group has only single lines. In this way we are building many small representations from a few of them.

It is a relevant question whether we are succeeding in dealing with a significant part of the problem of constructing small representations. As evidence that we are, we cite the example of $SU(N,2)$. The unitary dual for such a group is completely known ([**KS**] and [**BSK2**]). Among the important irreducible unitary representations are the Langlands quotients of representations induced from a minimal parabolic subgroup whose parameters are isolated in the set of parameters of all irreducible unitary representations. The unitarity of all these isolated representations reduces by cohomological induction to that of finitely many "basic cases," the number of them being $O(N^2)$. It turns out that all of the basic cases are line-bundle cases or vector-bundle cases of analytic continuations of holomorphic discrete series. On the other hand, the number of line-bundle cases of analytic continuations of holomorphic discrete series for $SU(N,2)$ is $O(N)$. Thus we achieve with our theorem a nontrivial reduction of the number of small representations that need special study.

The applications of Theorem 5.1 are by no means limited to analytic continuations of holomorphic discrete series, and we give in §7 some illustrations of what happens in another situation. The examples we treat are continuations of nonholomorphic discrete series.

This article is a mixture of an exposition and a development of new results. The material of §§1–4 is expository except for Theorem 2.8 and the description of the theme of this paper. Section 5 contains the main theorem, while §6 amounts to a new proof of an old theorem, proving unitarity for vector-bundle cases of analytic continuations of holomorphic discrete series by means of the theorem of §5. The point of the new proof is that it lends itself to a broad generalization. An illustration of this generalization for representations not previously investigated is carried out in §7.

1. Setting and approach

Let G be a connected reductive Lie group in the Harish-Chandra class. This means that G is locally the product of a connected abelian Lie group and a connected semisimple Lie group with finite center. Fix a maximal compact subgroup K of G. For simplicity of notation, we shall assume throughout that

$$\operatorname{rank} G = \operatorname{rank} K. \tag{1.1}$$

Let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be the Cartan decomposition of the Lie algebra of G corresponding to K, and let θ be the Cartan involution. We use subscripts 0 for real Lie algebras and real vector spaces, and we drop the subscripts for the complexifications, so that the decomposition of the complexification $\mathfrak{g}$ of $\mathfrak{g}_0$ is $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

Cohomological induction was introduced by Zuckerman in 1978 and was developed by Vogan in [**V2**] and later work. An exposition appears in [**KV**], and we give references within this source when possible.

Cohomological induction is defined relative to a θ stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ in $\mathfrak{g}$, $\mathfrak{u}$ being the nilpotent radical. Part of the definition of θ stable parabolic is that $\mathfrak{q}$ has a Levi factor $\mathfrak{l}$ that is the complexification of $\mathfrak{l}_0 = \mathfrak{l} \cap \mathfrak{g}_0$. If L denotes the corresponding analytic subgroup, then L is closed connected reductive in the Harish-Chandra class. Let $\bar{\mathfrak{q}} = \mathfrak{l} \oplus \bar{\mathfrak{u}}$ be the opposite parabolic, which is given in this situation by complex conjugation within $\mathfrak{g}$ relative to $\mathfrak{g}_0$. See [**KV**, §IV.6] for details.

We work with admissible representations of G and L, and it is known that it is equivalent, for many purposes including investigations of unitarity, to work instead with $(\mathfrak{g}, K)$ modules and $(\mathfrak{l}, L \cap K)$ modules. Cohomological induction is any of several functors for carrying $(\mathfrak{l}, L \cap K)$ modules into $(\mathfrak{g}, K)$ modules: Let Z be an $(\mathfrak{l}, L \cap K)$ module, and define

$$\mathcal{L}_j(Z) = (\mathcal{L}_{\mathfrak{l}}^{\mathfrak{g}})_j(Z) = (\Pi_{\mathfrak{g}, L\cap K}^{\mathfrak{g}, K})_j(\operatorname{ind}_{\bar{\mathfrak{q}}, L\cap K}^{\mathfrak{g}, L\cap K}(Z \otimes \textstyle\bigwedge^{\mathrm{top}} \mathfrak{u})).$$

In the middle expression we have dropped mention of the nilpotent part of $\mathfrak{q}$ to simplify the notation in later calculations; the nilpotent part does affect what the functor does, however. The right side is a composition of four operations: The first is that Z is effectively shifted in parameter by the sum of the roots contributing to $\mathfrak{u}$ (once we fix a Cartan subalgebra). The second is that the $(\mathfrak{l}, L \cap K)$ module $Z \otimes \bigwedge^{\mathrm{top}} \mathfrak{u}$ is changed to a $(\bar{\mathfrak{q}}, L \cap K)$ module by having $\bar{\mathfrak{u}}$ act as 0. The third, given by the ind functor, is akin to the familiar Verma module construction and is simply the tensor product over $U(\bar{\mathfrak{q}})$ with $U(\mathfrak{g})$. The fourth is the j^{th} derived functor of the "Bernstein functor" Π whose definition we shall recall in the next section. See [**KV**, §II.1] for details.

The $(\mathfrak{g}, K)$ module $\mathcal{L}_j(Z)$ is 0 if $j > S = \dim_{\mathbb{C}}(\mathfrak{u} \cap \mathfrak{k})$ and in favorable cases is 0 if $j < S$. The usual interest is in the case $j = S$.

MAIN DEVICE. If $\mathfrak{q}$ and $\mathfrak{q}'$ are θ stable parabolic subalgebras as above and if Z and Z' are $(\mathfrak{l}, L\cap K)$ and $(\mathfrak{l}', L'\cap K)$ modules, respectively, find interesting members of

$$\operatorname{Hom}_{\mathfrak{g},K}((\mathcal{L}_{\mathfrak{l}'}^{\mathfrak{g}})_{S'}(Z'), (\mathcal{L}_{\mathfrak{l}}^{\mathfrak{g}})_S(Z)). \tag{1.2}$$

Kobayashi [**Ko**] and Trapa [**T**] have done work related to this problem in a different context. Our work overlaps with theirs a little but not much.

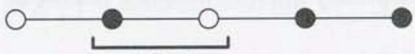
The following is the particular situation of interest. Think of $\mathcal{L}_{S'}(Z')$ as a $(\mathfrak{g}, K)$ module under study. Assume that Z and Z' are unitarizable and that Z is "small" for L in a suitable sense, so that $\mathcal{L}_S(Z)$ has a better-than-usual chance of being infinitesimally unitary. Also assume that some K type information is available that permits the conclusion that there is a nonzero member of the above Hom. Then the map gives a handle on the unitarity question for a particular irreducible subquotient of $\mathcal{L}_{S'}(Z')$. When the parameters of Z' are outside the usual range, this subquotient is likely to be small for G, and hence we get new information about the unitarity of a small $(\mathfrak{g}, K)$ module.

The main example of a "small" $(\mathfrak{l}, L \cap K)$ module for this paper is that Z is one-dimensional. In this case, let us write $Z = \mathbb{C}_\lambda$ with $\lambda \in \mathfrak{l}^*$ and $\lambda([\mathfrak{l}, \mathfrak{l}]) = 0$. Then $\mathcal{L}_S(\mathbb{C}_\lambda)$ is called a *Zuckerman module* and is often denoted $A_{\mathfrak{q}}(\lambda)$. The sense in which $A_{\mathfrak{q}}(\lambda)$ has a better-than-usual chance of being unitary will be discussed in the next section.

2. Properties of cohomological induction

In order to understand cohomological induction, it is helpful to understand how θ stable parabolic subalgebras are constructed operationally. Let $\mathfrak{t}_0$ be a maximal abelian subspace of $\mathfrak{k}_0$; under our equal-rank assumption, this is a Cartan subalgebra of $\mathfrak{g}_0$. Let $\Delta(\mathfrak{g},\mathfrak{t})$ be the set of roots; each root is said to be *compact* or *noncompact* according as its root space is in $\mathfrak{k}$ or is in $\mathfrak{p}$.

Fix a positive system $\Delta^+(\mathfrak{g},\mathfrak{t})$. Form the corresponding Dynkin diagram of $\mathfrak{g}$, and paint the noncompact simple roots. In [**K2**] the resulting diagram is called the *Vogan diagram* of $\mathfrak{g}_0$ relative to $\Delta^+(\mathfrak{g},\mathfrak{t})$. Label some of the simple roots as generating $\Delta(\mathfrak{l},\mathfrak{t})$. For example, the Vogan diagram of $\mathfrak{g}_0$ relative to $\Delta^+(\mathfrak{g},\mathfrak{t})$, with a labeled set of simple roots, might be the following:



$\Delta(\mathfrak{l},\mathfrak{t})$

Let $\Delta(\mathfrak{l},\mathfrak{t})$ be the root system generated by the marked simple roots, and let $\Delta(\mathfrak{u})$ be the set of positive roots not in $\Delta(\mathfrak{l},\mathfrak{t})$. Then put

$$\mathfrak{l} = \mathfrak{t} \oplus \left(\bigoplus_{\alpha\in\Delta(\mathfrak{l},\mathfrak{t})} \mathfrak{g}_\alpha \right)$$

and

$$\mathfrak{u} = \bigoplus_{\alpha\in\Delta(\mathfrak{u})} \mathfrak{g}_\alpha,$$

and the result is a θ stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$. For our $\mathfrak{g}_0$, all such $\mathfrak{q}$'s arise in this way.

We denote various sets of roots by $\Delta(\mathfrak{g},\mathfrak{t})$, $\Delta(\mathfrak{l},\mathfrak{t})$, $\Delta(\mathfrak{u})$, Δ_K, $\Delta_{L\cap K}$, and so on. The corresponding sets of positive roots are indicated by superscripts $+$. Let δ, δ_L, $\delta(\mathfrak{u})$, δ_K, and $\delta_{L\cap K}$ be the usual half sums of positive roots, and write $W(\mathfrak{g},\mathfrak{t})$ and $W(\mathfrak{l},\mathfrak{t})$ for the Weyl groups of $\Delta(\mathfrak{g},\mathfrak{t})$ and $\Delta(\mathfrak{l},\mathfrak{t})$.

An ingredient in the definition of cohomological induction is the "Bernstein functor" Π, whose definition we now review. The universal enveloping algebra of $\mathfrak{g}$ is a ring whose unital modules are exactly the representations of $\mathfrak{g}$. There is a corresponding ring $R(\mathfrak{g},K)$ for handling $(\mathfrak{g},K)$ modules, and its precise definition is given in Chapter I of [**KV**]. Then Π is given by a formula completely analogous to the formation of Verma modules:

$$\Pi^{\mathfrak{g},K}_{\mathfrak{g},L\cap K}(Z) = R(\mathfrak{g},K)\otimes_{R(\mathfrak{g},L\cap K)} Z.$$

The functor Π is covariant and right exact, and Π_j is defined to be the j^{th} right derived functor of Π. One has at present no geometric or analytic interpretation of Π_j, but a dual Hom functor, known as the "Zuckerman functor," has an interpretation in terms of sheaf-cohomology sections of vector bundles.

Fix a θ stable parabolic subalgebra $\mathfrak{q}$ as above, let $S = \dim(\mathfrak{u}\cap\mathfrak{k})$, and define $\mathcal{L}_j$ as in §1. The following theorems give some basic properties of cohomological induction. In each, Z is an $(\mathfrak{l}, L\cap K)$ module. Historically Theorems 2.1 through

2.7 are due to Zuckerman and Vogan. We give references to [**KV**] for proofs; more information about the history may be found in [**KV**].

THEOREM 2.1. (Trivial vanishing). *$\mathcal{L}_j(Z) = 0$ if $j > 2S$.*

REFERENCE. [**KV**, Corollary 2.125].

THEOREM 2.2. (Vanishing and admissibility). *If Z is admissible as an $L \cap K$ module, then $\mathcal{L}_j(Z)$ is admissible as a K module, and $\mathcal{L}_j(Z) = 0$ for $j > S$.*

REFERENCE. [**KV**, Theorem 5.35].

THEOREM 2.3. (Infinitesimal character). *If Z has infinitesimal character λ_0 relative to $(\mathfrak{l}, \mathfrak{t})$, then $\mathcal{L}_j(Z)$ has infinitesimal character $\lambda_0 + \delta(\mathfrak{u})$ relative to $(\mathfrak{g}, \mathfrak{t})$.*

REFERENCE. [**KV**, Corollary 5.25].

THEOREM 2.4. (Double induction). *Cohomological induction in stages is valid if the inside cohomologically induced modules vanish in all but one degree, and then the degrees add.*

REFERENCE. [**KV**, Corollary 11.86].

For the remaining theorems in this section, let $\mathfrak{t}_0$, $\mathfrak{t}$, $\Delta(\mathfrak{g}, \mathfrak{t})$, and $\Delta^+(\mathfrak{g}, \mathfrak{t})$ be as earlier in this section.

THEOREM 2.5. (Behavior in the *good range*). *If Z has infinitesimal character λ_0 relative to $(\mathfrak{l}, \mathfrak{t})$ and if $\langle \lambda_0 + \delta(\mathfrak{u}), \alpha \rangle > 0$ for all $\alpha \in \Delta(\mathfrak{u})$, then*
(a) *$\mathcal{L}_j(Z) = 0$ for $j < S$, and*
(b) *$\mathcal{L}_S(Z)$ is (nonzero) irreducible if Z is irreducible.*

REFERENCE [**KV**, Theorems 5.99 and 8.2].

THEOREM 2.6. (Behavior in the *weakly good range*). If Z has infinitesimal character λ_0 relative to $(\mathfrak{l}, \mathfrak{t})$ and if $\langle \lambda_0 + \delta(\mathfrak{u}), \alpha \rangle \geq 0$ for all $\alpha \in \Delta(\mathfrak{u})$, then
(a) $\mathcal{L}_j(Z) = 0$ for $j < S$, and
(b) $\mathcal{L}_S(Z)$ is irreducible or zero if Z is irreducible.

REFERENCE. [**KV**, Theorems 5.99 and 8.2].

THEOREM 2.7. (Bottom Layer Theorem). *If λ is the highest weight of an $L \cap K$ type in Z relative to $\Delta^+(\mathfrak{l} \cap \mathfrak{k}, \mathfrak{t})$, then $\lambda + 2\delta(\mathfrak{u} \cap \mathfrak{p}) = \Lambda$ is the highest weight of a K type in $\mathcal{L}_S(Z)$ relative to $\Delta^+(\mathfrak{k}, \mathfrak{t})$ if and only if Λ is Δ_K^+ dominant, and then the multiplicities are equal.*

REFERENCE. [**KV**, Theorem 5.80].

The K types Λ of $\mathcal{L}_S(Z)$ as in Theorem 2.7 are said to be in the *bottom layer*. For example, $A_\mathfrak{q}(\lambda)$ has the single $L \cap K$ highest weight λ, and the bottom layer is $\{\lambda + 2\delta(\mathfrak{u} \cap \mathfrak{p})\}$ if $\lambda + 2\delta(\mathfrak{u} \cap \mathfrak{p})$ is Δ_K^+ dominant and is empty otherwise.

Theorem 2.7 may be regarded as a theorem about objects in a certain category. Theorem 2.8 below is the corresponding result about morphisms; it reduces to Theorem 2.7 when applied to the identity map $Z \to Z$.

Suppose that Z' and Z are two $(\mathfrak{l}, L \cap K)$ modules and that λ is an $L \cap K$ highest weight. We say that an $(\mathfrak{l}, L \cap K)$ map $Z' \to Z$ has $L \cap K$ *rank n* on the $L \cap K$ type with highest weight λ if this $L \cap K$ type has multiplicity n in the image of Z' within Z. In analogous fashion, we can define K *rank* on a K type for a $(\mathfrak{g}, K)$ map.

THEOREM 2.8. (Bottom Layer Theorem for Maps). *Let $\mathfrak{q}' = \mathfrak{q}$, and suppose that there is an $(\mathfrak{l}, L\cap K)$ map $Z' \to Z$ that has $L\cap K$ rank n on the $L \cap K$ type with highest weight λ. If $\Lambda = \lambda + 2\delta(\mathfrak{u}\cap\mathfrak{p})$ is Δ_K^+ dominant, then the $(\mathfrak{g}, K)$ map $\mathcal{L}_S(Z' \to Z)$ is a member of $\operatorname{Hom}_{\mathfrak{g},K}(\mathcal{L}_S(Z'), \mathcal{L}_S(Z))$ that has K rank n on the K type with highest weight Λ.*

PROOF. Let us write π for the map $Z' \to Z$. Equation (5.74) of [**KV**] uses the inclusion $\mathfrak{k} \to \mathfrak{g}$ to produce a $(\mathfrak{k}, L\cap K)$ map

$$\beta_Z : \operatorname{ind}_{\bar{\mathfrak{q}}\cap\mathfrak{k}, L\cap K}^{\mathfrak{k}, L\cap K}(Z_{\bar{\mathfrak{q}}}^{\#}) \to \operatorname{ind}_{\bar{\mathfrak{q}}, L\cap K}^{\mathfrak{g}, L\cap K}(Z_{\bar{\mathfrak{q}}}^{\#}).$$

In the diagram of $(\mathfrak{k}, L\cap K)$ modules

$$\begin{array}{ccc} \operatorname{ind}_{\bar{\mathfrak{q}}\cap\mathfrak{k}, L\cap K}^{\mathfrak{k}, L\cap K}(Z'^{\#}_{\bar{\mathfrak{q}}}) & \xrightarrow{\operatorname{ind}_{\bar{\mathfrak{q}}\cap\mathfrak{k}, L\cap K}^{\mathfrak{k}, L\cap K}(\pi)} & \operatorname{ind}_{\bar{\mathfrak{q}}\cap\mathfrak{k}, L\cap K}^{\mathfrak{k}, L\cap K}(Z^{\#}_{\bar{\mathfrak{q}}}) \\ {\scriptstyle\beta_{Z'}}\big\downarrow & & \big\downarrow{\scriptstyle\beta_Z} \\ \operatorname{ind}_{\bar{\mathfrak{q}}, L\cap K}^{\mathfrak{g}, L\cap K}(Z'^{\#}_{\bar{\mathfrak{q}}}) & \xrightarrow{\operatorname{ind}_{\bar{\mathfrak{q}}, L\cap K}^{\mathfrak{g}, L\cap K}(\pi)} & \operatorname{ind}_{\bar{\mathfrak{q}}, L\cap K}^{\mathfrak{g}, L\cap K}(Z^{\#}_{\bar{\mathfrak{q}}}) \end{array} \tag{2.1}$$

the four modules are tensor products of $U(\mathfrak{k})$ or $U(\mathfrak{g})$ with $Z'^{\#}_{\bar{\mathfrak{q}}}$ or $Z^{\#}_{\bar{\mathfrak{q}}}$. The vertical maps act in the first member of the tensor product, and the horizontal maps act in the second member. Therefore (2.1) commutes. If we apply the functor $(\Pi_{\mathfrak{k}, L\cap K}^{\mathfrak{k}, K})_S$ to (2.1) and argue as in (5.75) and (5.76) of [**KV**], then we obtain a commutative diagram

$$\begin{array}{ccc} \mathcal{L}_S^K(Z') & \xrightarrow{\mathcal{L}_S^K(\pi)} & \mathcal{L}_S^K(Z) \\ {\scriptstyle\mathcal{B}}\big\downarrow & & \big\downarrow{\scriptstyle\mathcal{B}} \\ \mathcal{L}_S(Z') & \xrightarrow{\mathcal{L}_S(\pi)} & \mathcal{L}_S(Z) \end{array} \tag{2.2}$$

of K modules in which $\mathcal{L}^K$ is the $\mathcal{L}$ functor relative to K and in which $\mathcal{B}$ is the "bottom-layer map" for Z' or Z. The form of Theorem 2.7 in [**KV**] shows that each $\mathcal{B}$ is an isomorphism on the K type Λ. Consequently the commutativity of (2.2) implies that $\mathcal{L}_S^K(\pi)$ and $\mathcal{L}_S(\pi)$ have the same K rank on the K type Λ.

Meanwhile the functor $\mathcal{L}_S^K$ respects direct sums and carries an irreducible K module of type λ to an irreducible K module of type Λ. Thus the $L\cap K$ rank of π equals the K rank of $\mathcal{L}_S^K(\pi)$. Combining this equality with the equality of the K ranks of $\mathcal{L}_S^K(\pi)$ and $\mathcal{L}_S(\pi)$, we obtain the conclusion of the theorem.

3. Unitarizability and small representations

In this section we use the notation of §2, including the definitions of $\mathfrak{t}_0$, $\mathfrak{t}$, $\Delta(\mathfrak{g}, \mathfrak{t})$, and $\Delta^+(\mathfrak{g}, \mathfrak{t})$. Let Z be an $(\mathfrak{l}, L\cap K)$ module with infinitesimal character λ_0 relative to $(\mathfrak{l}, \mathfrak{t})$.

The main theorem giving a sufficient condition for the unitarizability of $\mathcal{L}_S(Z)$ is due to Vogan [**V3**]. Wallach [**W2**] obtained a simplified proof, which is what appears in [**KV**]. The theorem is often stated in the following form.

THEOREM 3.1. (Unitarizability Theorem) *Let Z be an infinitesimally unitary $(\mathfrak{l}, L \cap K)$ module with infinitesimal character λ_0 relative to $(\mathfrak{l}, \mathfrak{t})$. If λ_0 is in the weakly good range, i.e., $\langle \lambda_0 + \delta(\mathfrak{u}), \alpha \rangle \geq 0$ for all $\alpha \in \Delta(\mathfrak{u})$, then $\mathcal{L}_S(\mathfrak{u})$ is infinitesimally unitary.*

REFERENCE [**KV**, Theorem 9.1].

The unitarity of $\mathcal{L}_S(Z)$ is deduced by considering an explicit Hermitian form constructed on $\mathcal{L}_S(Z)$ from the given form on Z. This form on $\mathcal{L}_S(Z)$ is known as the *Shapovalov form*, and its positivity is deduced by means of a continuity argument.

Theorem 3.1 is not very useful for our current purposes. We are interested in the unitarity of some special irreducible subquotient $\mathcal{L}_{S'}(Z')$, and the technique of interest will be to pick out this irreducible subquotient by means of an intertwining operator. If Z' is irreducible and if its infinitesimal character is in the weakly good range, then $\mathcal{L}_{S'}(Z')$ will be irreducible or 0, by Theorem 2.6, and the irreducible subquotient has to be all of $\mathcal{L}_{S'}(Z')$. The intertwining operator is not needed to isolate the irreducible subquotient, and our present theory becomes uninteresting.

Examination of either continuity argument, Vogan's or Wallach's, used to prove Theorem 3.1 reveals a better theorem, however.

THEOREM 3.2. *Let Z be a $(\mathfrak{l}, L \cap K)$ module with infinitesimal character λ_0 relative to $(\mathfrak{l}, \mathfrak{t})$, and suppose that Z carries a positive definite invariant Hermitian form. Suppose further that the $U(\mathfrak{g})$ module $\operatorname{ind}_{\mathfrak{q}}^{\mathfrak{g}}((Z \otimes_{\mathbb{C}} \mathbb{C}_{t\delta(\mathfrak{u})})_{\mathfrak{q}}^{\#})$ is irreducible for all $t \geq 0$. Then the corresponding Shapovalov form on $\mathcal{L}_S(Z)$ is positive definite.*

REFERENCE. [**KV**, Theorem 9.68].

Theorem 3.2 is the general result, and in part we shall make use of it directly. We shall make use also of a corollary of it that assumes that Z is "small" in a suitable sense. The corollary is by no means best possible, as we shall see in one of our applications later in this paper.

Define $\mathfrak{t}_{ss} = \mathfrak{t} \cap [\mathfrak{g}, \mathfrak{g}]$. A finite-length $(\mathfrak{g}, K)$ module V with an infinitesimal character λ is *weakly unipotent* if

(i) the restriction of λ to $\mathfrak{t}_{ss}$ is in the $\mathbb{R}$ linear span of the roots
(ii) whenever F is a finite-dimensional $U(\mathfrak{g})$ module and μ is an infinitesimal character appearing in $V \otimes_{\mathbb{C}} F$, then $\left|\lambda|_{\mathfrak{t}_{ss}}\right| \leq \left|\mu|_{\mathfrak{t}_{ss}}\right|$.

EXAMPLE 3.3. If V is one-dimensional, we show that V is weakly unipotent. In fact, let λ' be the unique weight of V, so that V has infinitesimal character $\lambda = \lambda' + \delta_L$. The inclusion $\mathfrak{t}_0 \subseteq \mathfrak{k}_0$ forces λ to be imaginary on $\mathfrak{t}_0$, and (i) holds. In (ii), we may assume that F is irreducible with highest weight ν, and what needs to be checked is that $\left|\lambda|_{\mathfrak{t}_{ss}}\right| \leq \left|(\lambda+\nu)|_{\mathfrak{t}_{ss}}\right|$. But the left side is $\left|\delta_L|_{\mathfrak{t}_{ss}}\right|$ since λ' vanishes on $\mathfrak{t}_{ss}$, and the right side is $\left|(\nu + \delta_L)|_{\mathfrak{t}_{ss}}\right|$. The inequality follows by expanding the square of the right side since $\langle \nu, \delta_L \rangle \geq 0$.

COROLLARY 3.4. *Let Z be an infinitesimally unitary irreducible $(\mathfrak{l}, L \cap K)$ module with infinitesimal character λ_0 relative to $(\mathfrak{l}, \mathfrak{t})$, and suppose that Z is weakly unipotent relative to $\mathfrak{l}$. If λ_0 is in the weakly fair range, i.e., if*

$$\langle \lambda_0 + \delta(\mathfrak{u}), \alpha|_{\mathfrak{z}} \rangle \geq 0$$

for all $\alpha \in \Delta(\mathfrak{u})$ when $\mathfrak{z}$ is the center of $\mathfrak{l}$, then $\mathcal{L}_S(\mathfrak{u})$ is infinitesimally unitary.

REFERENCE. [**KV**, Theorem 12.4] except that only the barest sketch of a proof is given.

PROOF. We run through the first part of the proof of Lemma 5.100 of [**KV**] up through equation (5.104) except that we drop $L \cap K$ from the notation and we replace Z by $Z \otimes \mathbb{C}_{t\delta(\mathfrak{u})}$. Thus $M = \operatorname{ind}_{\mathfrak{q}}^{\mathfrak{g}}((Z \otimes \mathbb{C}_{t\delta(\mathfrak{u})})_{\mathfrak{q}}^{\#})$. To be able to apply Theorem 3.2, we want to prove that M is irreducible. Arguing by contradiction, suppose N is a nontrivial $U(\mathfrak{g})$ submodule of M. We claim that N meets $1 \otimes (Z \otimes \mathbb{C}_{t\delta(\mathfrak{u})})^{\#}$.

Assume the contrary. Then the argument in [**KV**] for (5.104) shows that there exists a finite-dimensional $U(\mathfrak{l})$ submodule F of $U(\mathfrak{u}) \ominus \mathbb{C}$, i.e., the members of $U(\mathfrak{u})$ with 0 constant term, such that $F \otimes (Z \otimes \mathbb{C}_{t\delta(\mathfrak{u})})^{\#}$ has a nonzero $U(\mathfrak{l})$ submodule with infinitesimal character ν satisfying

$$\nu - \delta(\mathfrak{u}) = s(\lambda_0 + (t+1)\delta(\mathfrak{u})) \tag{3.1}$$

for some $s \in W(\mathfrak{g}, \mathfrak{t})$.

Let us decompose the norm squared on members of $(i\mathfrak{t}_0)^*$ according to the formula $|\cdot|^2 = |\cdot|_{\mathfrak{z}}^2 + |\cdot|_{\mathrm{ss}}^2$, the subscripts indicating the parts carried on $\mathfrak{z}$ and $\mathfrak{t} \cap [\mathfrak{l}, \mathfrak{l}]$, respectively. Tensoring $F \otimes (Z \otimes \mathbb{C}_{t\delta(\mathfrak{u})})^{\#}$ with $\mathbb{C}_{-(t+2)\delta(\mathfrak{u})}$, we see that $\nu - (t+2)\delta(\mathfrak{u})$ is an infinitesimal character in $F \otimes Z$. Since Z is weakly unipotent, we obtain

$$|\lambda_0|_{\mathrm{ss}}^2 \le |\nu - (t+2)\delta(\mathfrak{u})|_{\mathrm{ss}}^2 = |\nu|_{\mathrm{ss}}^2. \tag{3.2}$$

Taking norms in (3.1) gives

$$|\nu - \delta(\mathfrak{u})|^2 = |\lambda_0 + (t+1)\delta(\mathfrak{u})|^2,$$

and the ss parts satisfy

$$|\nu - \delta(\mathfrak{u})|_{\mathrm{ss}}^2 = |\nu|_{\mathrm{ss}}^2 \ge |\lambda_0|_{\mathrm{ss}}^2 = |\lambda_0 + (t+1)\delta(\mathfrak{u})|_{\mathrm{ss}}^2$$

by (3.2). Therefore the $\mathfrak{z}$ parts satisfy

$$|\nu - \delta(\mathfrak{u})|_{\mathfrak{z}}^2 \le |\lambda_0 + (t+1)\delta(\mathfrak{u})|_{\mathfrak{z}}^2. \tag{3.3}$$

Since ν is an infinitesimal character in $F \otimes (Z \otimes \mathbb{C}_{t\delta(\mathfrak{u})})^{\#}$, $\nu - \delta(\mathfrak{u})$ is an infinitesimal character in $F \otimes (Z \otimes \mathbb{C}_{(t+1)\delta(\mathfrak{u})})$. Therefore a theorem of Kostant [**KV**, Theorem 7.133] shows that some $W(\mathfrak{l}, \mathfrak{t})$ transform of $\nu - \delta(\mathfrak{u})$ is of the form

$$\text{weight of } F + \text{infinitesimal character of } Z \otimes \mathbb{C}_{(t+1)\delta(\mathfrak{u})},$$

hence of the form

$$\sum_{\alpha \in \Delta(\mathfrak{u})} n_\alpha \alpha + \lambda_0 + (t+1)\delta(\mathfrak{u})$$

with all $n_\alpha \ge 0$. Since the action of $W(\mathfrak{l}, \mathfrak{t})$ preserves the $\mathfrak{z}$ part of the norm, we obtain

$$\begin{aligned}|\nu - \delta(\mathfrak{u})|_{\mathfrak{z}}^2 &= \Big|\sum_{\alpha \in \Delta(\mathfrak{u})} n_\alpha \alpha + \lambda_0 + (t+1)\delta(\mathfrak{u})\Big|_{\mathfrak{z}}^2 \\ &= \Big|\sum_{\alpha \in \Delta(\mathfrak{u})} n_\alpha \alpha\Big|_{\mathfrak{z}}^2 + \sum_{\alpha \in \Delta(\mathfrak{u})} 2n_\alpha \langle \lambda_0 + (t+1)\delta(\mathfrak{u}), \alpha|_{\mathfrak{z}} \rangle + |\lambda_0 + (t+1)\delta(\mathfrak{u})|_{\mathfrak{z}}^2.\end{aligned}$$

The middle term on the right side is ≥ 0 since λ_0 is in the weakly fair range and since $\langle \delta(\mathfrak{u}), \alpha|_{\mathfrak{z}} \rangle > 0$ [**KV**, Corollary 4.69]. Comparison with (3.3) thus yields

$\left|\sum_{\alpha\in\Delta(\mathfrak{u})} n_\alpha \alpha\right|_{\mathfrak{z}}^2 \leq 0$. It follows that the weight of F in question is 0, and this contradicts the fact that F lies in $U(\mathfrak{u})\ominus\mathbb{C}$. Therefore N meets $1\otimes(Z\otimes\mathbb{C}_{t\delta(\mathfrak{u})})^{\#}$.

The proof is completed in the same way that Corollary 5.105 of [**KV**] is proved: Let $1\otimes z$ be a nonzero element in the intersection of N with $1\otimes(Z\otimes\mathbb{C}_{t\delta(\mathfrak{u})})^{\#}$. Then $U(\mathfrak{l})(1\otimes z) = 1\otimes(Z\otimes\mathbb{C}_{t\delta(\mathfrak{u})})^{\#}$ since $(Z\otimes\mathbb{C}_{t\delta(\mathfrak{u})})^{\#}$ is irreducible under $\mathfrak{l}$. Hence $N\supseteq 1\otimes(Z\otimes\mathbb{C}_{t\delta(\mathfrak{u})})^{\#}$. Applying $U(\mathfrak{g})$, we see that $N\supseteq M$. Hence $N=M$.

EXAMPLE 3.5. If $\mathbb{C}_\lambda$ is a one-dimensional $(\mathfrak{l}, L\cap K)$ module, we show that its infinitesimal character is in the weakly fair range if

$$\langle\lambda+\delta(\mathfrak{u}),\alpha\rangle\geq 0 \qquad \text{for all } \alpha\in\Delta(\mathfrak{u}); \tag{3.4}$$

consequently $A_{\mathfrak{q}}(\lambda)$ is infinitesimally unitary. In fact, $\mathbb{C}_\lambda$ is weakly unipotent by Example 3.3 since it is one-dimensional. Thus the unitarity will follow from Corollary 3.4 if we show that the infinitesimal character is in the weakly fair range, i.e., that

$$\langle\lambda+\delta_L+\delta(\mathfrak{u}),\alpha|_{\mathfrak{z}}\rangle\geq 0 \qquad \text{for all } \alpha\in\Delta(\mathfrak{u}), \tag{3.5}$$

since $\lambda_0=\lambda+\delta_L$. In (3.5), δ_L vanishes on $\mathfrak{z}$ and may be dropped. The remaining terms in the left member vanish on $\mathfrak{t}_{ss}$ since $\mathbb{C}_\lambda$ is one-dimensional; thus we may drop the subscript $\mathfrak{z}$ in the right member. Hence (3.5) is equivalent with (3.4), and our assertion follows. This result appears explicitly in [**KV**] as Corollary 9.70.

For future reference let us isolate from the proof of Corollary 3.4 the sufficient condition for unitarity that was being deduced from Theorem 3.2.

COROLLARY 3.6. *Let Z be an infinitesimally unitary irreducible $(\mathfrak{l}, L\cap K)$ module with infinitesimal character λ_0 relative to $(\mathfrak{l},\mathfrak{t})$. In order for $\mathcal{L}_S(Z)$ to fail to be infinitesimally unitary, it is necessary that there exist a number $t\geq 0$ and an $\mathfrak{l}$ weight μ of $U(\mathfrak{u})\ominus\mathbb{C}$ such that*

$$\mu+\lambda_0+(t+1)\delta(\mathfrak{u}) = s(\lambda_0+(t+1)\delta(\mathfrak{u})) \tag{3.6}$$

for some $s\in W(\mathfrak{g},\mathfrak{t})$. If Z is one-dimensional and λ_0 is $\Delta^+(\mathfrak{l},\mathfrak{t})$ dominant, then μ may be assumed $\Delta^+(\mathfrak{l},\mathfrak{t})$ dominant.

PROOF. The condition from (3.1) is that there exist an $\mathfrak{l}$ infinitesimal character ν in $(U(\mathfrak{u})\ominus\mathbb{C})\otimes(Z\otimes\mathbb{C}_{(t+2)\delta(\mathfrak{u})})$ such that

$$\nu-\delta(\mathfrak{u}) = s(\lambda_0+(t+1)\delta(\mathfrak{u})).$$

Then $\nu-\delta(\mathfrak{u})$ is an infinitesimal character in $(U(\mathfrak{u})\ominus\mathbb{C})\otimes(Z\otimes\mathbb{C}_{(t+1)\delta(\mathfrak{u})})$. The theorem of Kostant mentioned after (3.3) says that $\nu-\delta(\mathfrak{u})$ is, up to a member of the Weyl group $W(\mathfrak{l},\mathfrak{t})$, the sum of a weight μ of $U(\mathfrak{u})\ominus\mathbb{C}$ and the infinitesimal character $\lambda_0+(t+1)\delta(\mathfrak{u})$ of $Z\otimes\mathbb{C}_{(t+1)\delta(\mathfrak{u})}$. Then (3.6) follows.

If Z is one-dimensional, we may calculate the infinitesimal character of $(U(\mathfrak{u})\ominus\mathbb{C})\otimes(Z\otimes\mathbb{C}_{(t+2)\delta(\mathfrak{u})})$ in terms of highest weights. From this observation, the last statement of the corollary follows.

Now we can indicate more precisely how we shall pursue the theme of this paper. In the setting of the Bottom Layer Theorem for Maps (Theorem 2.8), we consider an $(\mathfrak{l}, L\cap K)$ map $Z'\to Z$. As noted in the discussion in §1, we are interested in the case that Z is small, and in fact we shall consider exclusively the case that Z is one-dimensional in this article, say $Z=\mathbb{C}_\lambda$. Put $\Lambda=\lambda+2\delta(\mathfrak{u}\cap\mathfrak{p})$. To be able to use Theorem 2.8, we need to have an $(\mathfrak{l}, L\cap K)$ map $Z'\to\mathbb{C}_\lambda$ that has

$L \cap K$ rank 1 on the $L \cap K$ type λ. Theorem 2.8 then tells us that $\mathcal{L}_S(Z' \to \mathbb{C}_\lambda)$ has K rank 1 on the K type Λ. This intertwining operator is the interesting member of (1.2) that we have in mind.

The simplest case to analyze is that the K type Λ has multiplicity 1 in $\mathcal{L}_S(Z')$. Let us assume that we are in this situation. Then there is a unique irreducible subquotient X of $\mathcal{L}_S(Z')$ containing the K type Λ, and Theorem 2.8 is telling us that a certain $(\mathfrak{g}, K)$ map $X \to A_\mathfrak{q}(\lambda)$ is nonzero. If we can show that $A_\mathfrak{q}(\lambda)$ is infinitesimally unitary (for example by applying Theorem 3.2 or Example 3.5), then we can conclude that the irreducible subquotient X of $\mathcal{L}_S(Z')$ is infinitesimally unitary.

The existence of an $(\mathfrak{l}, L \cap K)$ map $Z' \to \mathbb{C}_\lambda$ says that Z' has a one-dimensional quotient. There are various natural $(\mathfrak{l}, L \cap K)$ modules Z' with this property. For example, the spherical principal series representation of L with parameter half the sum of positive restricted roots has the trivial representation of L as its Langlands quotient (in the sense of the Langlands classification); changing the principal series on the center of L allows for other one-dimensional representations of L as Langlands quotient. However, no matter what $Z' \to \mathbb{C}_\lambda$ we use, we always have to analyze $\mathcal{L}_S(Z')$. To analyze this particular $\mathcal{L}_S(Z')$, we would have to simplify a composition involving a cohomological induction and a parabolic induction. This step is related to finding the Langlands parameters of a cohomologically induced representation when one knows the Langlands parameters of the inducing representation. A result in this direction when the parameter of Z' is in the weakly good range is Theorem 11.225 of [**KV**]; some results when the parameter is outside the weakly good range appear in [**F**]. At any rate, this is quite a complicated matter, and we shall not discuss this case further in this paper.

Instead let us consider the case that $Z' \to \mathbb{C}_\lambda$ exhibits $\mathbb{C}_\lambda$ as the quotient of a generalized Verma module. Since Z' must be also an $(\mathfrak{l}, L \cap K)$ module, the possibilities are fairly limited. We shall be quickly led to studying "holomorphic discrete series" and their "analytic continuations."

Actually generalized Verma modules are highest weight modules, and it will be more convenient to work with lowest weight modules. Thus we shall have to make some minor adjustments in the traditional formulation of the theory.

4. Analytic continuations of holomorphic discrete series

The discrete series consists of the irreducible unitary representations of G that have a nonzero square-integrable matrix coefficient. The holomorphic discrete series consists of those with a highest weight vector. Harish-Chandra constructed and classified the holomorphic discrete series in a sequence of three papers [**HC1**], and he parametrized the full discrete series in a later sequence of three papers [**HC2**]. One of the main theorems of [**HC2**] is that the discrete series is nonempty if and only if $\operatorname{rank} G = \operatorname{rank} K$, and we are assuming this equality in this paper. See [**K1**] for an exposition of holomorphic and general discrete series.

It will be more convenient for us to work with $(\mathfrak{g}, K)$ modules, and then we can describe the discrete series by using cohomological induction. In fact, finding such a description was one of Zuckerman's original goals in introducing cohomological induction. In [**KV**] the discrete series representations are *defined* as follows: Fix the Cartan subalgebras $\mathfrak{t}_0$ and $\mathfrak{t}$ as earlier, and let $\mathfrak{q} = \mathfrak{t} \oplus \mathfrak{u}$ be any θ stable parabolic subalgebra whose Levi factor is $\mathfrak{t}$. Then the discrete series consists of all

$$(\mathcal{L}_{\mathfrak{t}}^{\mathfrak{g}})_S(\mathbb{C}_{\lambda'}) = (\Pi_{\mathfrak{g},T}^{\mathfrak{g},K})_S(\mathrm{ind}_{\mathfrak{q},T}^{\mathfrak{g},T}(\mathbb{C}_{\lambda'} \otimes \textstyle\bigwedge^{\mathrm{top}} \mathfrak{u}))$$

whose parameters are in the good range, provided all choices of $\mathfrak{u}$ are allowed.

To make the correspondence with Harish-Chandra's parametrization, one uses the results of §2 above to recognize these $(\mathfrak{g}, K)$ modules as irreducible and infinitesimally unitary, and then it is possible to give a direct proof of their square integrability (cf. [**W3**]). Sorting out parameters then shows that these representations exhaust the discrete series, and it is easy to identify the equivalences among them. When the parameters are outside the good range but in the weakly good range, the nonzero $(\mathfrak{g}, K)$ modules that are obtained are exactly the limits of discrete series.

It is not hard, using double induction and the results of §IV.11 of [**KV**], to see that $(\mathcal{L}_{\mathfrak{l}}^{\mathfrak{g}})_S(Z)$ is in the discrete series whenever $\mathfrak{l}_0$ is compact and Z is irreducible with parameter in the good range. It is a limit of discrete series if it is nonzero and the parameter is merely in the weakly good range.

The condition on G to have a nonempty holomorphic discrete series is that the centralizer in $\mathfrak{g}_0$ of the center of $\mathfrak{k}_0$ is $\mathfrak{k}_0$. This is equivalent with the condition that G/K has an invariant complex structure (i.e., G/K is *Hermitian symmetric*), and in this case the holomorphic discrete series representations can be realized in spaces of holomorphic sections of equivariant holomorphic vector bundles over G/K. When G is noncompact simple, G/K has an invariant complex structure if and only if the center of $\mathfrak{k}_0$ is nonzero, and in this case it is one-dimensional. For background one may consult [**K2**].

To relate holomorphic discrete series to cohomological induction, it is more convenient to work with lowest-weight representations than with highest-weight representations. First let us suppose that G is noncompact simple. Then the holomorphic discrete series are the discrete series $(\mathcal{L}_{\mathfrak{t}}^{\mathfrak{g}})_S(\mathbb{C}_{\lambda'})$ for which the positive system $\Delta^+(\mathfrak{g}, \mathfrak{t})$ defined by $\mathfrak{u}$ has one noncompact simple root, say α, and α occurs just once in the largest root. In terms of the Vogan diagram of $\mathfrak{g}_0$, exactly one simple root α is painted, and α occurs just once when the largest root is expanded in terms of simple roots. When G is not necessarily noncompact or simple, the condition on the Vogan diagram of $\mathfrak{g}_0$ is that each simple component of the diagram have at most one painted root and any painted root occurs just once when the largest root for that component is expanded in terms of simple roots.

Under these conditions, $\mathfrak{k}$ is the $\mathfrak{l}$ built from the simple roots that are compact, and the noncompact roots involve a painted root. Since a painted root cannot occur twice in any root, the sum of two noncompact positive roots cannot be a root. We denote the sum of the root spaces for the positive noncompact roots by $\mathfrak{p}^+$ and the sum of the root spaces for the negative noncompact roots by $\mathfrak{p}^-$. Then $\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$, $\mathfrak{p}^+$ and $\mathfrak{p}^-$ are abelian subspaces of $\mathfrak{p}$ normalized by $\mathfrak{k}$, and the sum $\mathfrak{k} \oplus \mathfrak{p}^+$ is a θ stable parabolic subalgebra of $\mathfrak{g}$. By double induction and the results of §IV.11 of [**KV**], we can write

$$(\mathcal{L}_{\mathfrak{t}}^{\mathfrak{g}})_S(\mathbb{C}_{\lambda'}) = (\mathcal{L}_{\mathfrak{k}}^{\mathfrak{g}})_0(\mathcal{L}_{\mathfrak{t}}^{\mathfrak{k}})_S(\mathbb{C}_{\lambda'}) = (\mathcal{L}_{\mathfrak{k}}^{\mathfrak{g}})_0(V_{\lambda'}) = \mathrm{ind}_{\mathfrak{k}+\mathfrak{p}^-,K}^{\mathfrak{g},K}((V_{\lambda'})^{\#}),$$

where $V_{\lambda'}$ is the representation of K with highest weight λ'. This equation shows that the $(\mathfrak{g}, K)$ module in question is just a generalized Verma module, except that the parabolic subalgebra $\mathfrak{k} \oplus \mathfrak{p}^+$ that would produce highest weights has been replaced by $\mathfrak{k} \oplus \mathfrak{p}^-$ and the representations have lowest weights. A global version on G

of the representation in question occurs in sections of an associated holomorphic vector bundle over G/K built from the representation of K on $V_{\lambda'}^{\#}$. Accordingly when $\dim V_{\lambda'} = 1$, we refer to the representation as a *line-bundle case*; when $\dim V_{\lambda'} > 1$, we refer to it as a *vector-bundle case*.

In a 1979 paper [**W1**] Wallach classified the irreducible infinitesimally unitary $(\mathfrak{g}, K)$ modules that have a highest weight vector and a one-dimensional K type. For all practical purposes we may assume that G/K is Hermitian symmetric, and we do assume that. The $(\mathfrak{g}, K)$ modules of interest are necessarily quotients of generalized Verma modules having a one-dimensional K type. To describe the situation, let us specialize to the case that G is noncompact simple.

Since G/K is Hermitian symmetric and G is noncompact simple, we have $\mathfrak{k}_0 = [\mathfrak{k}_0, \mathfrak{k}_0] \oplus \mathfrak{z}_{\mathfrak{k}_0}$ with the center $\mathfrak{z}_{\mathfrak{k}_0}$ one dimensional. A one-dimensional skew-Hermitian representation of $\mathfrak{k}_0$ is 0 on $[\mathfrak{k}_0, \mathfrak{k}_0]$ and therefore lies in a one-parameter family according to the behavior on $\mathfrak{z}_{\mathfrak{k}_0}$. Wallach thought in terms of working with the universal cover of G, so that there was no integrability condition on $\mathfrak{z}_{\mathfrak{k}_0}$. We shall work with a linear G having a simply-connected complexification, so that the parameter corresponding to $\mathfrak{z}_{\mathfrak{k}_0}$ is subject to an integrability condition. At any rate, when this parameter is sufficiently large in one direction, holomorphic discrete series representations result, so that the general case can be considered as a kind of analytic continuation of the line-bundle case of holomorphic discrete series. Wallach was able to settle the continued line-bundle cases by an explicit calculation of the Shapovalov form, but the corresponding calculation could not be carried out in the vector-bundle cases.

In 1983, Enright-Howe-Wallach [**EHW**] and Jakobsen [**J**] were able to settle completely the vector-bundle cases largely without making explicit calculations. The proofs of unitarity involved a number of *ad hoc* arguments that we seek to avoid. Let us state the result, adjusting the notation of [**EHW**] to reflect the assumption that the representations have lowest-weight vectors instead of highest-weight vectors. We confine ourselves to the case that G is noncompact simple. The result when the Vogan diagram of $\mathfrak{g}_0$ has a double line is more complicated to state than the result when the diagram has only single lines, and we shall stick to the single-line cases.

Thus the situation is that $\mathfrak{g}_0$ is noncompact simple, there is one noncompact simple root α, and α has coefficient 1 when the largest root is expanded in terms of simple roots. We suppose that G is a linear group with a simply connected complexification, so that a linear form μ on $\mathfrak{t}$ is (analytically) integral if and only if $2\langle \mu, \beta\rangle/|\beta|^2$ is an integer for all simple roots β. We assume further that the underlying Dynkin diagram of the Vogan diagram of $\mathfrak{g}_0$ has only single lines, i.e., is of type A, D, or E. This last assumption excludes exactly $\mathfrak{g}_0 = \mathfrak{so}(2, 2n-1)$ and $\mathfrak{g}_0 = \mathfrak{sp}(n, \mathbb{R})$.

Define a parabolic subalgebra of $\mathfrak{g}$ by

$$\mathfrak{q}' = \mathfrak{l}' \oplus \mathfrak{u}' = \mathfrak{k} \oplus \mathfrak{p}^+.$$

Let $\lambda' \in \mathfrak{t}^*$ be integral and K dominant, and let $V_{\lambda'}$ be an irreducible representation of K with highest weight λ'. Relative to this parabolic subalgebra $\mathfrak{q}'$, we have $\dim(\mathfrak{u}' \cap \mathfrak{k}) = 0$, and thus $S' = 0$. The $(\mathfrak{g}, K)$ module to study is

$$\mathcal{L}_0(V_{\lambda'}) = \mathrm{ind}_{\mathfrak{k}+\mathfrak{p}^-}^{\mathfrak{g}}(V_{\lambda'} \otimes \textstyle\bigwedge^{\mathrm{top}} \mathfrak{p}^+).$$

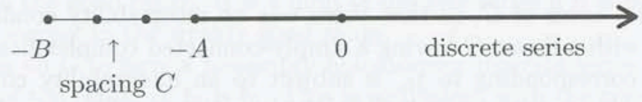
We want to know whether the irreducible quotient of $\mathcal{L}_0(V_{\lambda'})$ containing the K type $\lambda' + 2\delta(\mathfrak{p}^+)$ is infinitesimally unitary.

To orient ourselves, it is helpful to consider $V_{\lambda'}$ as a member of a one-parameter family of representations of the universal covering group $\widetilde{K}$ of K. Let $\Lambda_\mathfrak{z}$ be the fundamental weight of $\mathfrak{g}_0$ defined by the properties that $2\langle \Lambda_\mathfrak{z}, \alpha\rangle/|\alpha|^2 = 1$ and $2\langle \Lambda_\mathfrak{z}, \beta\rangle/|\beta|^2 = 0$ for all other simple β; $\Lambda_\mathfrak{z}$ is a nonzero multiple of $2\delta(\mathfrak{p}^+)$. The one-parameter family is

$$V_{\lambda' + (z - \frac{2\langle\lambda'+\delta,\alpha\rangle}{|\alpha|^2})\Lambda_\mathfrak{z}}.$$

The *Wallach set* is defined to be the set of real z for which the irreducible quotient of the corresponding $\mathfrak{g}$ module is infinitesimally unitary. The $\mathfrak{g}$ module is a $(\mathfrak{g}, K)$ module if and only if z is an integer.

Qualitatively the picture of the Wallach set is the set of bold points in the following diagram, the numbers A, B, and C being defined in Theorem 4.1 below.



The good range is $\{z > 0\}$ and gives discrete series. A limit of discrete series occurs at 0. To the left of 0 is a continuous range (in the universal cover) and then a finite discrete set, with equal spacing.

The point with

$$z = \frac{2\langle\lambda' + \delta, \alpha\rangle}{|\alpha|^2} \tag{4.1}$$

in the above diagram tells about unitarity for the original $(\mathfrak{g}, K)$ module $\mathcal{L}_0(V_{\lambda'})$.

THEOREM 4.1. *The irreducible quotient of $\mathcal{L}_0(V_{\lambda'})$ containing the K type $\lambda' + 2\delta(\mathfrak{p}^+)$ is infinitesimally unitary if and only if the number z in (4.1) is in the Wallach set defined by numbers A, B, and C as follows: Let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ be the θ stable parabolic subalgebra defined by taking*

$\mathfrak{l}$ to be built from α and all simple roots γ that are compact and are orthogonal to λ'

$\mathfrak{u}$ to be built from the positive roots not contributing to $\mathfrak{l}$.

Then

$$B = 1 - \frac{2\langle 2\delta_{L\cap K}, \alpha\rangle}{|\alpha|^2}$$

$$A = B - ((\text{real rank of } \mathfrak{l}_0) - 1)C$$

$$C = \begin{cases} 1 & \text{for } \mathfrak{g}_0 = \mathfrak{su}(m,n) \\ n-2 & \text{for } \mathfrak{g}_0 = \mathfrak{so}(2, 2n-2) \\ 2 & \text{for } \mathfrak{g}_0 = \mathfrak{so}^*(2n) \\ 3 & \text{for } \mathfrak{g}_0 \text{ of type E III with } \mathfrak{g} = \mathrm{E}_6 \\ 4 & \text{for } \mathfrak{g}_0 \text{ of type E VII with } \mathfrak{g} = \mathrm{E}_7. \end{cases}$$

REFERENCE [**EHW**] or [**J**].

REMARKS. The largest possible $\mathfrak{l}$ is $\mathfrak{l} = \mathfrak{g}$, which occurs exactly when λ' is orthogonal to all simple roots that are compact. This is the line-bundle case. The following table describes the Wallach set in this situation:

$\mathfrak{g}_0$	Dynkin diagram	real rank	α	A	B	C
$\mathfrak{su}(m,n)$, $m \leq n$	A_{m+n-1}	$m-1$	$e_m - e_{m+1}$	n	$m+n-1$	1
$\mathfrak{so}(2,2n-2)$, $n \geq 4$	D_n	2	$e_1 - e_2$	$n-1$	$2n-3$	$n-2$
$\mathfrak{so}^*(2n)$, $n \geq 4$	D_n	$[\frac{n}{2}]$	$e_{n-1}+e_n$	$2[\frac{n}{2}]-1$	$2n-3$	2
E III	E_6	2	end of long branch	8	11	3
E VII	E_7	3	end of long branch	9	17	4

5. Main theorem

We now return to the case of an $(\mathfrak{l}, L \cap K)$ map $Z' \to \mathbb{C}_\lambda$ with Z' a lowest weight module. We suppose that $\mathfrak{g}_0$, $\mathfrak{t}_0$, $\mathfrak{k}_0$, and $\Delta^+(\mathfrak{g}, \mathfrak{t})$ are fixed throughout.

THEOREM 5.1. *Suppose that $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ is a θ stable parabolic of $\mathfrak{g}$ and that $L/(L \cap K)$ is Hermitian symmetric in a fashion compatible with the inherited positive system $\Delta^+(\mathfrak{l}, \mathfrak{t})$ for $\mathfrak{l}$. Let $\lambda \in \mathfrak{t}^*$ be an integral parameter orthogonal to the roots of $\mathfrak{l}$, and suppose that $\Lambda = \lambda + 2\delta(\mathfrak{u} \cap \mathfrak{p})$ is dominant for Δ_K^+. Let $\mathfrak{q}' = \mathfrak{l}' \oplus \mathfrak{u}'$ be any θ stable parabolic subalgebra of $\mathfrak{g}$ compatible with $\Delta^+(\mathfrak{g}, \mathfrak{t})$ such that $\mathfrak{l}' \supseteq \mathfrak{l} \cap \mathfrak{k}$ and $\mathfrak{l}'$ is compact, and define*

$$\lambda' = \lambda - 2\delta(\mathfrak{l} \cap \mathfrak{p}^+),$$

where $\mathfrak{p}^+$ denotes the sum of the root spaces for the positive noncompact roots. Put $S' = \dim(\mathfrak{u}' \cap \mathfrak{k})$. Then

(a) *λ' is $\Delta_{L'}^+$ dominant, so that there exists an irreducible representation $V_{\lambda'}$ of the compact group L' with highest weight λ',*
(b) *the K type Λ has multiplicity one in both $(\mathcal{L}_{\mathfrak{l}'}^{\mathfrak{g}})_{S'}(V_{\lambda'})$ and $A_\mathfrak{q}(\lambda)$, and*
(c) *there exists a $(\mathfrak{g}, K)$ map*

$$(\mathcal{L}_{\mathfrak{l}'}^{\mathfrak{g}})_{S'}(V_{\lambda'}) \to A_\mathfrak{q}(\lambda)$$

of K rank one on the K type Λ.

Consequently if $A_\mathfrak{q}(\lambda)$ is infinitesimally unitary, then the unique irreducible subquotient of $(\mathcal{L}_{\mathfrak{l}'}^{\mathfrak{g}})_{S'}(V_{\lambda'})$ with K type Λ is infinitesimally unitary.

REMARK. A prototype for this theorem may be found in [**KV**, pp. 589–592 and 632]. In that example, $L = G = Sp(2, \mathbb{R})$ and $\mathfrak{l}' = \mathfrak{k}$. Most of that discussion is intended to show that for a suitable λ, $A_\mathfrak{q}(\lambda)$ is infinitesimally unitary but the map in (c) is not onto.

PROOF. For (a) we write

$$\lambda' = \lambda - 2\delta(\mathfrak{l} \cap \mathfrak{p}^+) = \Lambda - 2\delta(\mathfrak{p}^+). \tag{5.1}$$

The simple roots of $\mathfrak{l}'$ are simple for $\mathfrak{g}$ and are compact. If β is one of them, the reflection s_β permutes the positive noncompact roots of $\mathfrak{g}$, and hence $\langle 2\delta(\mathfrak{p}^+), \beta\rangle = 0$. Combining this equality with (5.1) and the Δ_K^+ dominance of Λ, we see that $\langle \lambda', \beta\rangle \geq 0$.

For (b) and (c) we note that $\mathbb{C}_{\lambda'}$ is a well defined $L \cap K$ module since λ' is integral and is orthogonal to the roots of $\mathfrak{l} \cap \mathfrak{k}$. We begin by constructing a nonzero $(\mathfrak{l}, L \cap K)$ map

$$(\mathcal{L}^{\mathfrak{l}}_{\mathfrak{l}\cap\mathfrak{k}})_0(\mathbb{C}_{\lambda'}) \to \mathbb{C}_\lambda. \tag{5.2}$$

Defining $\mathfrak{p}^-$ in analogy with $\mathfrak{p}^+$, we write

$$(\mathcal{L}^{\mathfrak{l}}_{\mathfrak{l}\cap\mathfrak{k}})_0(\mathbb{C}_{\lambda'}) = \operatorname{ind}_{(\mathfrak{l}\cap\mathfrak{k})+(\mathfrak{l}\cap\mathfrak{p}^-)}(\mathbb{C}^{\#}_{\lambda'}) = U(\mathfrak{l}) \otimes_{U((\mathfrak{l}\cap\mathfrak{k})+(\mathfrak{l}\cap\mathfrak{p}^-))} \mathbb{C}_\lambda.$$

Call the right side M, and let $\mathfrak{z}$ be the center of $\mathfrak{l} \cap \mathfrak{k}$. All weights of M, when restricted to $\mathfrak{z}$, are of the form

$$\lambda_{\mathfrak{z}} + \sum_{\substack{\beta\in\Delta^+(\mathfrak{l},\mathfrak{t}),\\ \beta \text{ noncompact}}} n_\beta \beta|_{\mathfrak{z}}$$

with all $n_\beta \geq 0$. Let N be the sum of the weight spaces for which not all n_β are 0. We show that N is an $\mathfrak{l}$ submodule of M. Since M/N is then evidently isomorphic to $\mathbb{C}_\lambda$, we will be able to conclude the existence of (5.2).

Without loss of generality, we may assume that $\mathfrak{g}_0$ is simple noncompact, so that all $\beta|_{\mathfrak{z}}$ are equal, say to ε, for β noncompact in $\Delta^+(\mathfrak{l}, \mathfrak{t})$. The fact that N is an $\mathfrak{l}$ submodule of M will thus follow if we show that each root vector $E_{-\beta}$, for β noncompact in $\Delta^+(\mathfrak{l}, \mathfrak{t})$, annihilates any vector in N of weight $\lambda|_{\mathfrak{z}} + \varepsilon$. Since $U(\mathfrak{l} \cap \mathfrak{p}^+)\mathbb{C}_\lambda = M$, it is enough to treat a vector in N of the form $E_{\beta'}v_0$ with β' noncompact in $\Delta^+(\mathfrak{l}, \mathfrak{t})$ and v_0 in $\mathbb{C}_\lambda$. We have

$$E_{-\beta}E_{\beta'}v_0 = E_{\beta'}E_{-\beta}v_0 + [E_{-\beta}, E_{\beta'}]v_0. \tag{5.3}$$

The first term on the right side of (5.3) is 0 since the restriction to $\mathfrak{z}$ of the weight of $E_{-\beta}v_0$ is $\lambda|_{\mathfrak{z}} - \varepsilon$, which is not allowed. The second term on the right side of (5.3) is 0 since $[E_{-\beta}, E_{\beta'}]$ is in $[\mathfrak{l}, \mathfrak{l}]$ and $[\mathfrak{l}, \mathfrak{l}]v_0 = 0$. Thus (5.3) is 0, and the construction of the map (5.2) is complete.

In the map (5.2), we note that the $L \cap K$ type $\lambda = \lambda' + 2\delta(\mathfrak{l} \cap \mathfrak{p})$ is in the bottom layer for the domain module and thus (5.2) has $L \cap K$ rank 1 on the $L \cap K$ type λ. By the Bottom Layer Theorem for Maps (Theorem 2.8), the map

$$(\mathcal{L}^{\mathfrak{g}}_{\mathfrak{l}})_S((\mathcal{L}^{\mathfrak{l}}_{\mathfrak{l}\cap\mathfrak{k}})_0(\mathbb{C}_{\lambda'}) \to \mathbb{C}_\lambda), \qquad S = \dim(\mathfrak{u} \cap \mathfrak{k}), \tag{5.4}$$

has K rank 1 on the K type $\Lambda = \lambda + 2\delta(\mathfrak{u} \cap \mathfrak{p})$. By Theorem 2.4 and the results of §IV.11 of [**KV**],

$$(\mathcal{L}^{\mathfrak{g}}_{\mathfrak{l}})_S(\mathcal{L}^{\mathfrak{l}}_{\mathfrak{l}\cap\mathfrak{k}})_0(\mathbb{C}_{\lambda'}) = (\mathcal{L}^{\mathfrak{g}}_{\mathfrak{l}\cap\mathfrak{k}})_S(\mathbb{C}_{\lambda'}) = (\mathcal{L}^{\mathfrak{g}}_{\mathfrak{l}'})_{S'}(\mathcal{L}^{\mathfrak{l}'}_{\mathfrak{l}\cap\mathfrak{k}})_{S-S'}(\mathbb{C}_{\lambda'}) = (\mathcal{L}^{\mathfrak{g}}_{\mathfrak{l}'})_{S'}(V_{\lambda'}). \tag{5.5}$$

Since also

$$A_{\mathfrak{q}}(\lambda) = (\mathcal{L}^{\mathfrak{g}}_{\mathfrak{l}})_S(\mathbb{C}_\lambda), \tag{5.6}$$

the map (5.4) is the required map for (c). We have seen that the K type Λ has multiplicity one in both $(\mathcal{L}^{\mathfrak{g}}_{\mathfrak{l}})_S(\mathcal{L}^{\mathfrak{l}}_{\mathfrak{l}\cap\mathfrak{k}})_0(\mathbb{C}_{\lambda'})$ and $(\mathcal{L}^{\mathfrak{g}}_{\mathfrak{l}})_S(\mathbb{C}_\lambda)$. Thus the equalities (5.5) and (5.6) complete the proof of (b).

6. Application to single-line holomorphic cases

Our objective in this section is to obtain by continuity arguments a proof of all the unitarity of the vector-bundle cases of analytic continuations of holomorphic discrete series for linear groups whose Dynkin diagrams have only single lines. The line of reasoning will be a model for proofs in §7 of unitarity in new nonholomorphic cases.

Theorem 5.1 is phrased in such a way that $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ and λ are given first and $\mathfrak{q}' = \mathfrak{l}' \oplus \mathfrak{u}'$ and λ' are constructed from them. In applications the construction goes in the reverse direction; we look, successfully or unsuccessfully, for $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ and λ satisfying the hypotheses that lead to a given pair $\mathfrak{q}' = \mathfrak{l}' \oplus \mathfrak{u}'$ and λ'.

In this section we study the situation in which G/K is Hermitian symmetric and $\Delta^+(\mathfrak{g}, \mathfrak{t})$ is chosen compatibly with this hypothesis. We assume that $\mathfrak{g}_0$ is noncompact simple, that the Dynkin diagram of $\mathfrak{g}_0$ has only single lines, and that G is linear with a simply connected complexification. This is the setting in which the description in §4 of the Wallach set is applicable.

In this setting we choose $\mathfrak{l}' = \mathfrak{k}$ and $\mathfrak{u}' = \mathfrak{p}^+$, and we assume that λ' is integral and Δ_K^+ dominant. Then $S' = 0$, and the $(\mathfrak{g}, K)$ module $(\mathcal{L}_{\mathfrak{l}'}^{\mathfrak{g}})_{S'}(V_{\lambda'})$ is equal to

$$\operatorname{ind}_{\mathfrak{k}+\mathfrak{p}^-}^{\mathfrak{g}}(V_{\lambda'} \otimes \wedge^{\mathrm{top}}\mathfrak{p}^+) = U(\mathfrak{g}) \otimes_{U(\mathfrak{k}+\mathfrak{p}^-)} V_{\lambda'+2\delta(\mathfrak{p}^+)}. \tag{6.1}$$

The question concerns the unitarity of the irreducible subquotient containing the K type $\Lambda = \lambda' + 2\delta(\mathfrak{p}^+)$. Specifically if z is the integer corresponding to λ' via (4.1) and if z is in the Wallach set, we want to understand the unitarity of (6.1) by using a $(\mathfrak{g}, K)$ map of (6.1) into some $A_\mathfrak{q}(\lambda)$ with $\mathfrak{q}$ and λ as in Theorem 5.1.

Here is what happens: In the vector-bundle cases (i.e., $\dim V_{\lambda'} > 1$), we can always choose $\mathfrak{q}$ and an associated λ so that the map into $A_\mathfrak{q}(\lambda)$ exists and $A_\mathfrak{q}(\lambda)$ is infinitesimally unitary. In all these cases, $A_\mathfrak{q}(\lambda)$ is infinitesimally unitary as a consequence of a continuity argument, namely Theorem 3.2. This is the desired easy proof of unitarity. In more detail, when $\mathfrak{g}_0$ is $\mathfrak{so}(2, 2n-2)$ or $\mathfrak{so}^*(2n)$ or one of the real forms E III of E_6 and E VII of E_7, then λ is in the weakly fair range and $A_\mathfrak{q}(\lambda)$ is infinitesimally unitary as a consequence of the more readily applicable Example 3.5. When $\mathfrak{g}_0$ is $\mathfrak{su}(m, n)$, there are examples where Example 3.5 is not applicable and we have to resort to the full strength of Theorem 3.2.

In the line-bundle cases (i.e., $\dim V_{\lambda'} = 1$), we mentioned that Wallach [**W1**] settled matters by direct computation. We can ask whether Theorem 5.1 handles these cases, too. The answer is "not always." For the line-bundle cases, it turns out that we can define $\mathfrak{q}$ and an associated λ all the time except for two values of z in the real form E III of E_6 and three in the real form E VII of E_7. Again $A_\mathfrak{q}(\lambda)$ is infinitesimally unitary as a consequence of Theorem 3.2, and the more readily applicable Example 3.5 applies except for $\mathfrak{su}(m, n)$.

In the remainder of this section, first we prove enough of Theorem 6.1 below to verify our definitions of $\mathfrak{q}$ and the associated λ when z equals $-B$, the leftmost point of the Wallach set. Second we prove enough of Theorem 6.2 to show in the vector-bundle case how to choose $\mathfrak{q}$ and λ for other values of z. Third we give the proofs of Theorems 6.1e and 6.2e, which say that $A_\mathfrak{q}(\lambda)$ is infinitesimally unitary. We begin with $\mathfrak{g}_0 \neq \mathfrak{su}(m, n)$, and then we consider $\mathfrak{g}_0 = \mathfrak{su}(m, n)$. The argument for $\mathfrak{su}(m, n)$ is the deepest part of the proof of Theorems 6.1 and 6.2 because the hypothesis "weakly fair" is not necessarily valid.

THEOREM 6.1. *Suppose that G/K is Hermitian symmetric, $\Delta^+(\mathfrak{g},\mathfrak{t})$ is chosen compatibly with that hypothesis, $\mathfrak{g}_0$ is noncompact simple, α is the unique simple root, the Dynkin diagram of $\mathfrak{g}$ has only single lines, and G is linear with a simply connected complexification. Let $\lambda' \in \mathfrak{t}^*$ be integral and Δ_K^+ dominant, and let $V_{\lambda'}$ be a K module with highest weight λ'. Define a θ stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ compatible with $\Delta^+(\mathfrak{g},\mathfrak{t})$ as in Theorem 4.1 so that $\mathfrak{l}$ is built from*

$$\{\alpha \text{ and all compact simple } \gamma \text{ with } \gamma \perp \lambda'\}$$

and $\mathfrak{u}$ is built from the positive roots not contributing to $\mathfrak{l}$. If the number $z = 2\langle \lambda' + \delta, \alpha\rangle/|\alpha|^2$ of (4.1) corresponds to the leftmost point of the Wallach set, i.e., is equal to the number $-B = (2\langle 2\delta_{L\cap K}, \alpha\rangle/|\alpha|^2) - 1$, then

(a) *$\lambda = \lambda' + 2\delta(\mathfrak{l} \cap \mathfrak{p}^+)$ is integral and is orthogonal to the roots of $\mathfrak{l}$*
(b) *$\Lambda = \lambda' + 2\delta(\mathfrak{p}^+)$ is integral and is Δ_K^+ dominant*
(c) *the K type Λ occurs with multiplicity one in both $(\mathcal{L}_\mathfrak{k}^\mathfrak{g})_0(V_{\lambda'})$ and $A_\mathfrak{q}(\lambda)$*
(d) *there exists a $(\mathfrak{g}, K)$ map $(\mathcal{L}_\mathfrak{k}^\mathfrak{g})_0(V_{\lambda'}) \to A_\mathfrak{q}(\lambda)$ of K rank one on the K type Λ*
(e) *the $(\mathfrak{g}, K)$ module $A_\mathfrak{q}(\lambda)$ is infinitesimally unitary as a consequence of the continuity argument given as Theorem 3.2.*

PROOF OF ALL BUT (e).

(a) Integrality is trivial. The simple roots of $\mathfrak{l}$ consist of α and those compact simple roots γ orthogonal to λ'. The roots γ are orthogonal also to $2\delta(\mathfrak{l}\cap\mathfrak{p}^+)$ and hence to $\lambda = \lambda' + 2\delta(\mathfrak{l}\cap\mathfrak{p}^+)$. In addition,

$$\begin{aligned}\frac{2\langle\lambda,\alpha\rangle}{|\alpha|^2} &= \frac{2\langle\lambda' + 2\delta(\mathfrak{l}\cap\mathfrak{p}^+),\alpha\rangle}{|\alpha|^2} = \frac{2\langle\lambda' + 2\delta_L,\alpha\rangle}{|\alpha|^2} - \frac{2\langle 2\delta_{L\cap K},\alpha\rangle}{|\alpha|^2} \\ &= \frac{2\langle\lambda' + \delta,\alpha\rangle}{|\alpha|^2} + 1 - \frac{2\langle 2\delta_{L\cap K},\alpha\rangle}{|\alpha|^2} = z + B = 0.\end{aligned}$$

(b) Integrality is again trivial. Both λ' and $2\delta(\mathfrak{p}^+)$ are Δ_K^+ dominant, and hence so is Λ.

(c,d) We shall apply Theorem 5.1. The hypotheses on λ are satisfied by (a) and (b) above since $\Lambda = \lambda + 2\delta(\mathfrak{u}\cap\mathfrak{p}^+)$ as a result of the definitions in (a) and (b). Since $\mathfrak{u}' = \mathfrak{p}^+$, $S' = 0$. Applying (b) and (c) of Theorem 5.1, we obtain (c) and (d) of the present theorem.

THEOREM 6.2. *Suppose that G/K is Hermitian symmetric, $\Delta^+(\mathfrak{g},\mathfrak{t})$ is chosen compatibly with that hypothesis, $\mathfrak{g}_0$ is noncompact simple, α is the unique simple root, the Dynkin diagram of $\mathfrak{g}$ has only single lines, and G is linear with a simply connected complexification. Let $\lambda' \in \mathfrak{t}^*$ be integral and Δ_K^+ dominant, and let $V_{\lambda'}$ be a K module with highest weight λ'. Suppose that $\dim V_{\lambda'} > 1$ and that $z = 2\langle\lambda' + \delta,\alpha\rangle/|\alpha|^2$ is a negative integer in the Wallach set. Then there exists a subset S of*

$$\widetilde{S} = \{\alpha \text{ and all compact simple } \gamma \text{ with } \gamma \perp \lambda'\}$$

containing α such that if $\mathfrak{q} = \mathfrak{l}\oplus\mathfrak{u}$ is the θ stable parabolic subalgebra compatible with $\Delta^+(\mathfrak{g},\mathfrak{t})$ with $\mathfrak{l}$ built from S and $\mathfrak{u}$ built from the positive roots not contributing to $\mathfrak{l}$ then

(a) *$\lambda = \lambda' + 2\delta(\mathfrak{l} \cap \mathfrak{p}^+)$ is integral and is orthogonal to the roots of $\mathfrak{l}$*
(b) *$\Lambda = \lambda' + 2\delta(\mathfrak{p}^+)$ is integral and is Δ_K^+ dominant*
(c) *the K type Λ occurs with multiplicity one in both $(\mathcal{L}_\mathfrak{k}^\mathfrak{g})_0(V_{\lambda'})$ and $A_\mathfrak{q}(\lambda)$*

(d) *there exists a* $(\mathfrak{g}, K)$ *map* $(\mathcal{L}_{\mathfrak{k}}^{\mathfrak{g}})_0(V_{\lambda'}) \to A_{\mathfrak{q}}(\lambda)$ *of* K *rank one on the* K *type* Λ

(e) *the* $(\mathfrak{g}, K)$ *module* $A_{\mathfrak{q}}(\lambda)$ *is infinitesimally unitary as a consequence of the continuity argument given as Theorem* 3.2.

PROOF OF ALL BUT (e). Reviewing the proof of Theorem 6.1, we see that it is enough to choose S so that the associated λ satisfies $2\langle\lambda, \alpha\rangle/|\alpha|^2 = 0$. Equivalently we want L to have the property that

$$z = \frac{2\langle 2\delta_{L\cap K}, \alpha\rangle}{|\alpha|^2} - 1. \tag{6.2}$$

From the formula for B, there is no loss of generality in assuming that $\widetilde{S}$ is connected.

We turn to the table in the remarks following Theorem 4.1, and we proceed case by case. Suppose $\widetilde{S}$ corresponds to $\mathfrak{su}(m,n)$ with $m \le n$. We are to see that all negative integer values of z from $-(m+n-1)$ to -1 can be achieved in (6.2) by defining S suitably. The point is that (6.2) is just the number $-B$ for the line-bundle case in which the whole Lie algebra is built from S. We obtain $-(m+n-1)$ for $S = \widetilde{S}$. As we delete simple roots from either end of the diagram, one at a time, the table shows that $-B$ increases by 1 each time. We repeat this procedure, always retaining $e_m - e_{m+1}$, and we see that all the required values of z are obtained.

Suppose $\widetilde{S}$ corresponds to $\mathfrak{so}(2, 2n-2)$. We are to see that

$$-2(n-3), -(n-1), -(n-2), \dots, -1$$

can be achieved by defining S suitably. In fact, $-r$ with $1 \le r \le n-1$ is achieved by taking $S = \{e_1 - e_2, \dots, e_r - e_{r+1}\}$, according to the line of the table for $\mathfrak{su}(1,r)$. To achieve $-2(n-3)$, we use $S = \widetilde{S}$.

Suppose $\widetilde{S}$ corresponds to $\mathfrak{so}^*(2n)$. We are to see that

$$-(2n-3), -(2n-5), \dots, -5, -3, -1$$

and also $-(n-1), -(n-2), \dots, -2, -1$ can be achieved by defining S suitably. In fact, $S = \{e_r - e_{r+1}, \dots, e_{n-2} - e_{n-1}, e_{n-1} - e_n, e_{n-1} + e_n\}$ yields $2n - 2r - 1$ if $1 \le r \le n-3$, and this takes care of $-(2n-3), -(2n-5), \dots, -5$ in the first list. Also $S = \{e_r - e_{r+1}, \dots, e_{n-2} - e_{n-1}, e_{n-1} + e_n\}$ yields $-(n-r)$ if $1 \le r \le n-1$, and this yields the whole second list. The second list always contains -3 and -1 since $n \ge 4$, and thus we can achieve all the desired negative integer values.

Suppose $\widetilde{S}$ corresponds to a subset of E III or E VII. Since $\dim V_{\lambda'} > 1$ by assumption, $\widetilde{S}$ cannot correspond to the whole Dynkin diagram. Since α is at the end of the long branch, a proper connected subdiagram containing α must be classical. All the classical cases for $\widetilde{S}$ have been treated above, and thus the proof is complete.

REMARKS.

For $\mathfrak{g}_0$ of type E III, suppose $\dim V_{\lambda'} = 1$. The Wallach set then consists of $-11, -8, -7, -6, \dots, -1$. We can achieve -11 with $S = \widetilde{S}$, -7 with S corresponding to the real form $\mathfrak{so}(2,8)$ of D_5, and $-5, \dots, -1$ with S corresponding to some $\mathfrak{su}(1,r)$ with $1 \le r \le 5$. But we do not achieve -8 and -6.

Similarly for $\mathfrak{g}_0$ of type E VII with $\dim V_{\lambda'} = 1$, the Wallach set consists of $-17, -13, -9, -8, -7, -6, \dots, -1$. We achieve -17 from $S = \widetilde{S}$, -9 with S

corresponding to the real form $\mathfrak{so}(2,10)$ of D_6, and $-6,\ldots,-1$ with S corresponding to some $su(1,r)$. But we do not achieve -13, -8, and -7.

PROOF OF THEOREMS 6.1e AND 6.2e WHEN $\mathfrak{g}_0 \neq \mathfrak{su}(m,n)$. By Corollary 3.4 and Example 3.5 it is enough to show that

$$\langle \lambda + \delta(\mathfrak{u}), \beta \rangle \geq 0 \quad \text{for all } \beta \in \Delta(\mathfrak{u}). \tag{6.3}$$

For $\beta \in \Delta^+(\mathfrak{l},\mathfrak{t})$, we have $\langle \lambda + \delta(\mathfrak{u}), \beta \rangle = 0$. Thus we may as well prove (6.3) for all $\beta \in \Delta^+(\mathfrak{g},\mathfrak{t})$. It is then enough to prove (6.3) for all simple roots β in $\Delta^+(\mathfrak{g},\mathfrak{t})$. Suppose that the Dynkin diagram of $\mathfrak{l}$ has a compact component, so that we can write $\mathfrak{l}_0 = \mathfrak{l}_0^{\text{cpt}} + \mathfrak{l}_0^{\#}$, where $\mathfrak{l}_0^{\text{cpt}}$ is compact semisimple plus abelian, $\mathfrak{l}_0^{\#}$ is reductive with α in its Dynkin diagram, and $\mathfrak{l}_0^{\text{cpt}} \cap \mathfrak{l}_0^{\#}$ equals $\mathfrak{t}_0$. Let $\mathfrak{q}^{\#} = \mathfrak{l}^{\#} \oplus \mathfrak{u}^{\#}$ be the θ stable parabolic subalgebra built from $\mathfrak{l}_0^{\#}$ such that $\Delta(\mathfrak{u}^{\#}) \subseteq \Delta^+(\mathfrak{g},\mathfrak{t})$. First we observe that the definition of λ as $\lambda' + 2\delta(\mathfrak{l} \cap \mathfrak{p}^+)$ does not change in passing from $\mathfrak{q}$ to $\mathfrak{q}^{\#}$ since $\mathfrak{l} \cap \mathfrak{p}^+ = \mathfrak{l}^{\#} \cap \mathfrak{p}^+$. Now suppose we can prove that

$$\langle \lambda + \delta(\mathfrak{u}^{\#}), \beta \rangle \geq 0 \qquad \text{for all simple } \beta. \tag{6.4}$$

We shall prove that

$$\langle \lambda + \delta(\mathfrak{u}), \beta \rangle \geq 0 \qquad \text{for all simple } \beta. \tag{6.5}$$

If β is in $\Delta(\mathfrak{l}^{\text{cpt}},\mathfrak{t})$, then β is in $\Delta(\mathfrak{l},\mathfrak{t})$ and (6.5) follows from the orthogonality of β to λ and $\delta(\mathfrak{u})$. Thus suppose β is simple but is not in $\Delta(\mathfrak{l}^{\text{cpt}},\mathfrak{t})$. Then $\Delta(\mathfrak{u}^{\#})$ consists of the members of $\Delta(\mathfrak{u})$ and the members of $\Delta^+(\mathfrak{l}^{\text{cpt}},\mathfrak{t})$, and the latter roots do not involve β in their expansions in terms of simple roots. Thus

$$\delta(\mathfrak{u}^{\#}) = \delta(\mathfrak{u}) + (\text{a sum of positive roots with no } \beta \text{ involved}),$$

$$\langle \delta(\mathfrak{u}^{\#}), \beta \rangle = \langle \delta(\mathfrak{u}), \beta \rangle + (\leq 0).$$

Hence

$$\langle \lambda + \delta(\mathfrak{u}), \beta \rangle = \langle \lambda + \delta(\mathfrak{u}^{\#}), \beta \rangle + (\geq 0) \geq 0,$$

the final inequality following from (6.4). This proves (6.5).

Consequently there is no loss of generality in assuming that the Dynkin diagram of $\mathfrak{l}$ is a *connected* subset of simple roots containing α. Now we write

$$\lambda + \delta(\mathfrak{u}) = \lambda' + 2\delta(\mathfrak{l} \cap \mathfrak{p}^+) + \delta(\mathfrak{u}) = \lambda' + \delta - \delta_{L\cap K} + \delta(\mathfrak{l} \cap \mathfrak{p}^+) = \lambda' + \delta + \delta_L - 2\delta_{L\cap K}.$$

We are to prove (6.5), and we may assume that β is not in $\Delta(\mathfrak{l},\mathfrak{t})$. Then β is a compact root, say γ, and we have

$$\begin{aligned} \frac{2\langle \lambda + \delta(\mathfrak{u}), \gamma \rangle}{|\gamma|^2} &= \frac{2\langle \lambda', \gamma \rangle}{|\gamma|^2} + 1 + \frac{2\langle \delta_L, \gamma \rangle}{|\gamma|^2} - \frac{2\langle 2\delta_{L\cap K}, \gamma \rangle}{|\gamma|^2} \\ &\geq 1 + \frac{2\langle \delta_L, \gamma \rangle}{|\gamma|^2} - \frac{2\langle 2\delta_{L\cap K}, \gamma \rangle}{|\gamma|^2} \end{aligned} \tag{6.6}$$

since λ' is Δ_K^+ dominant by assumption.

It is enough to prove that the right side of (6.6) is ≥ 0 for all possible $\mathfrak{l}$ having α as a root and having connected Dynkin diagram, and we do this case by case. Since γ is not in $\Delta^+(\mathfrak{l},\mathfrak{t})$, we may always assume the Dynkin diagram of $\mathfrak{l}$ is a proper subset of the Dynkin diagram of $\mathfrak{g}$.

First consider $\mathfrak{g}_0 = \mathfrak{so}(2, 2n-2)$ with $\alpha = e_1 - e_2$. Up to the outer automorphism $e_n \to -e_n$, the possibilities for the simple roots of $\mathfrak{l}$, as was observed in the proof of Theorem 6.2 above, are

$$\{e_1 - e_2, \ldots, e_r - e_{r+1}\}, \qquad 1 \leq r \leq n-1,$$

and the only γ's of interest are $e_{r+1}-e_{r+2}$ if $r<n-1$, and $e_{n-1}+e_n$ if $r\geq n-2$. Then we have

$$\delta_L=\left(\tfrac{r}{2},\tfrac{r-2}{2},\ldots,-\left(\tfrac{r}{2}\right),0,\ldots,0\right),$$
$$\delta_{L\cap K}=\left(0,\tfrac{r-1}{2},\ldots,-\left(\tfrac{r-1}{2}\right),0,\ldots,0\right).$$

The right side of (6.6) is

$$\begin{cases} 1-\frac{r}{2}+2\left(\frac{r-1}{2}\right)=\frac{r}{2} & \text{if } \gamma=e_{r+1}-e_{r+2}\\ 1-\frac{n-2}{2}+2\left(\frac{n-3}{2}\right)=\frac{n}{2}-1 & \text{if } r=n-2 \text{ and } \gamma=e_{n-1}+e_n\\ 1-\frac{n-3}{2}-\frac{n-1}{2}+2\left(\frac{n-4}{2}\right)+2\left(\frac{n-2}{2}\right)=n-3 & \text{if } r=n-1 \text{ and } \gamma=e_{n-1}+e_n,\end{cases}$$

and this is ≥ 0 in each case. This handles $\mathfrak{so}(2,2n-2)$.

Next consider $\mathfrak{g}_0=\mathfrak{so}(2n)^*$ with $\alpha=e_{n-1}+e_n$. The diagram for $\mathfrak{l}$ can be of type D_{n-r}, with simple roots

(6.7a) $\quad \{e_r-e_{r+1},\ldots,e_{n-2}-e_{n-1},e_{n-1}-e_n,e_{n-1}+e_n\},\quad 2\leq r\leq n-2,$

or of type A_{n-r-1}, with simple roots

(6.7b) $\quad \{e_r-e_{r+1},\ldots,e_{n-2}-e_{n-1},e_{n-1}+e_n\},\quad 1\leq r\leq n-2,$

or of type A_1, with simple root $e_{n-1}+e_n$. For (6.7a) we may restrict attention to $\gamma=e_{r-1}-e_r$ and

$$\delta_L=(0,\ldots,n-r,n-r-1,\ldots,1,0)$$
$$\delta_{L\cap K}=\left(0,\ldots,\tfrac{n-r}{2},\tfrac{n-r-2}{2},\ldots,-\left(\tfrac{n-r-2}{2}\right),-\left(\tfrac{n-r}{2}\right)\right).$$

The right side of (6.6) is

$$1-(n-r)+(n-r)=1\geq 0.$$

For (6.7b) we may restrict attention to $\gamma=e_{r-1}-e_r$ and $\gamma=e_{n-1}-e_n$, and

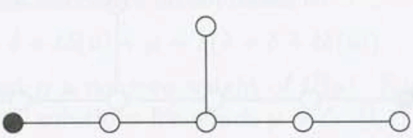
$$\delta_L=\left(0,\ldots,\tfrac{n-r}{2},\tfrac{n-r-2}{2},\ldots,-\left(\tfrac{n-r-2}{2}\right),+\left(\tfrac{n-r}{2}\right)\right),$$
$$\delta_{L\cap K}=\left(0,\ldots,\tfrac{n-r-1}{2},\tfrac{n-r-3}{2},\ldots,-\left(\tfrac{n-r-1}{2}\right),0\right).$$

The right side of (6.6) is

$$\begin{cases} 1-\frac{n-r}{2}+(n-r-1)=\frac{1}{2}(n-r) & \text{if } \gamma=e_{r-1}-e_r\\ 1-\frac{n-r-2}{2}-\frac{n-r}{2}+(n-r-1)=1 & \text{if } \gamma=e_{n-1}-e_n,\end{cases}$$

and this is ≥ 0. Finally if the diagram of $\mathfrak{l}$ is just A_1, then $\gamma=e_{n-2}-e_{n-1}$, $\delta_L=(0,\ldots,0,\frac{1}{2},\frac{1}{2})$, and $\delta_{L\cap K}=0$. The right side of (6.6) is $1-\frac{1}{2}+0=\frac{1}{2}$, and this is ≥ 0. This handles $\mathfrak{so}^*(2n)$.

Next consider $\mathfrak{g}_0=\mathrm{E\ III}$ of type E_6, in which the Vogan diagram is



We let the simple roots on the horizontal A_5 be $\alpha_3,\alpha_4,\alpha_5,\alpha_7,\alpha_8$, and we call the simple root on the short branch α_6. If $\mathfrak{l}$ corresponds to D_5, then we may take

$\gamma = \alpha_8$. We calculate the right side of (6.6) with $\gamma = \alpha_8$ by counting how many times α_7 occurs in δ_L and $\delta_{L\cap K}$:

$$\#(\text{times } \alpha_7 \text{ occurs in } 2\delta_L) = \#\Delta^+(\mathrm{D}_5) - \#\Delta^+(\mathrm{A}_4) = 20 - 10 = 10,$$
$$\#(\text{times } \alpha_7 \text{ occurs in } 2\delta_{L\cap K}) = \#\Delta^+(\mathrm{D}_4) - \#\Delta^+(\mathrm{A}_3) = 12 - 6 = 6,$$
$$\frac{2\langle \delta_L, \alpha_8\rangle}{|\alpha_8|^2} = -5, \qquad \frac{2\langle 2\delta_{L\cap K}, \alpha_8\rangle}{|\alpha_8|^2} = -6,$$
$$(\text{right side of (6.6)}) = 1 - 5 + 6 = 2 \geq 0.$$

Next let $\mathfrak{l}$ correspond to a horizontal A_n, and consider $\gamma = \alpha_6$. For this situation let us use 5-tuples to denote the respective values of quantities for $n = 1, 2, 3, 4, 5$. Then

$$\#(\text{times } \alpha_5 \text{ occurs in } 2\delta_L) = (0, 0, 3, 6, 9),$$
$$\#(\text{times } \alpha_5 \text{ occurs in } 2\delta_{L\cap K}) = (0, 0, 2, 4, 6),$$
$$\frac{2\langle \delta_L, \alpha_6\rangle}{|\alpha_6|^2} = -(0, 0, \tfrac{3}{2}, 3, \tfrac{9}{2}), \qquad \frac{2\langle \delta_{L\cap K}, \alpha_6\rangle}{|\alpha_6|^2} = -(0, 0, 2, 4, 6),$$
$$(\text{right side of (6.6)}) = 1 - (0, 0, \tfrac{3}{2}, 3, \tfrac{9}{2}) + (0, 0, 2, 4, 6) = (1, 1, \tfrac{3}{2}, 2, \tfrac{5}{2}).$$

Thus the right side of (6.6) is ≥ 0 for each choice of n. The other possibility to check with the same A_n is that γ is the root on the horizontal just beyond the A_n. Then we have

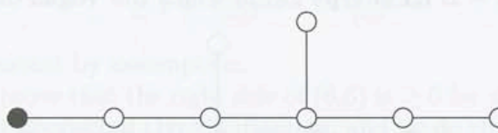
$$\#(\text{times last root of } \mathrm{A}_n \text{ occurs in } 2\delta_L) = n,$$
$$\#(\text{times last root of } \mathrm{A}_n \text{ occurs in } 2\delta_{L\cap K}) = n - 1,$$
$$\frac{2\langle \delta_L, \gamma\rangle}{|\gamma|^2} = -\tfrac{n}{2}, \qquad \frac{2\langle 2\delta_{L\cap K}, \gamma\rangle}{|\gamma|^2} = -(n-1),$$
$$(\text{right side of (6.6)}) = 1 - \tfrac{n}{2} + (n-1) = \tfrac{n}{2} \geq 0.$$

Finally let $\mathfrak{l}$ correspond to $\{\alpha_3, \alpha_4, \alpha_5, \alpha_6\}$ of type A_4, and let $\gamma = \alpha_7$. Then we have

$$\#(\text{times } \alpha_5 \text{ occurs in } 2\delta_L) = 6$$
$$\#(\text{times } \alpha_5 \text{ occurs in } 2\delta_{L\cap K}) = 4,$$
$$\frac{2\langle \delta_L, \gamma\rangle}{|\gamma|^2} = -3, \qquad \frac{2\langle 2\delta_{L\cap K}, \gamma\rangle}{|\gamma|^2} = -4,$$
$$(\text{right side of (6.6)}) = 1 - 3 + 4 = 2 \geq 0.$$

This handles E III.

Last consider $\mathfrak{g}_0 = $ E VII of type E_7. The Vogan diagram is



We let the simple roots on the horizontal A_6 be $\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_7, \alpha_8$, and we call the root on the short branch α_6. If $\mathfrak{l}$ corresponds to D_6, then we may take $\gamma = \alpha_8$.

We have

$$\#(\text{times } \alpha_7 \text{ occurs in } 2\delta_L) = \#\Delta^+(\mathrm{D}_6) - \#\Delta^+(\mathrm{A}_5) = 30 - 15 = 15,$$
$$\#(\text{times } \alpha_7 \text{ occurs in } 2\delta_{L\cap K}) = \#\Delta^+(\mathrm{D}_5) - \#\Delta^+(\mathrm{A}_4) = 20 - 10 = 10,$$
$$\frac{2\langle \delta_L, \gamma\rangle}{|\gamma|^2} = -\tfrac{15}{2}, \qquad \frac{2\langle 2\delta_{L\cap K}, \gamma\rangle}{|\gamma|^2} = -10,$$
$$(\text{right side of } (6.6)) = 1 - \tfrac{15}{2} + 10 = \tfrac{7}{2} \geq 0.$$

Next let $\mathfrak{l}$ correspond to a horizontal A_n, and consider $\gamma = \alpha_6$. For this situation let us use 6-tuples to denote the respective values of quantities for $n = 1, 2, 3, 4, 5, 6$. Then

$$\#(\text{times } \alpha_5 \text{ occurs in } 2\delta_L) = (0, 0, 0, 4, 8, 12),$$
$$\#(\text{times } \alpha_5 \text{ occurs in } 2\delta_{L\cap K}) = (0, 0, 0, 3, 6, 9),$$
$$\frac{2\langle \delta_L, \alpha_6\rangle}{|\alpha_6|^2} = -(0, 0, 0, 2, 4, 6), \qquad \frac{2\langle \delta_{L\cap K}, \alpha_6\rangle}{|\alpha_6|^2} = -(0, 0, 0, 3, 6, 9),$$
$$(\text{right side of } (6.6)) = 1 - (0, 0, 0, 2, 4, 6) + (0, 0, 0, 3, 6, 9) = (1, 1, 1, 2, 3, 4).$$

Thus the right side of (6.6) is ≥ 0 for each choice of n. The other possibility to check with the same A_n is that γ is the root on the horizontal just beyond the A_n. Then we have

$$\#(\text{times last root of } \mathrm{A}_n \text{ occurs in } 2\delta_L) = n,$$
$$\#(\text{times last root of } \mathrm{A}_n \text{ occurs in } 2\delta_{L\cap K}) = n - 1,$$
$$\frac{2\langle \delta_L, \gamma\rangle}{|\gamma|^2} = -\tfrac{n}{2}, \qquad \frac{2\langle 2\delta_{L\cap K}, \gamma\rangle}{|\gamma|^2} = -(n-1),$$
$$(\text{right side of } (6.6)) = 1 - \tfrac{n}{2} + (n-1) = \tfrac{n}{2} \geq 0.$$

Finally let $\mathfrak{l}$ correspond to $\{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$ of type A_5, and let $\gamma = \alpha_7$. Then we have

$$\#(\text{times } \alpha_5 \text{ occurs in } 2\delta_L) = 8$$
$$\#(\text{times } \alpha_5 \text{ occurs in } 2\delta_{L\cap K}) = 6,$$
$$\frac{2\langle \delta_L, \gamma\rangle}{|\gamma|^2} = -4, \qquad \frac{2\langle 2\delta_{L\cap K}, \gamma\rangle}{|\gamma|^2} = -6,$$
$$(\text{right side of } (6.6)) = 1 - 4 + 6 = 3 \geq 0.$$

This handles E VII and completes the proof of Theorem 6.1e for $\mathfrak{g}_0 \neq \mathfrak{su}(m, n)$.

Proof of Theorems 6.1e and 6.2e when $\mathfrak{g}_0 = \mathfrak{su}(m, n)$. We are considering $A_\mathfrak{q}(\lambda) = \mathcal{L}_S(\mathbb{C}_\lambda)$, and the infinitesimal character of $\mathbb{C}_\lambda$ is $\lambda_0 = \lambda + \delta_L$. By Corollary 3.6 it is enough to prove that there are no solutions to

$$\lambda + \delta + t\delta(\mathfrak{u}) + \mu = s(\lambda + \delta + t\delta(\mathfrak{u})) \tag{6.8}$$

with $t \geq 0$, $s \in W(\mathfrak{g}, \mathfrak{t})$, and μ a nonzero weight of $U(\mathfrak{u})$. Since $\mathbb{C}_\lambda$ is one dimensional, we need consider only solutions for which μ is $\Delta^+(\mathfrak{l}, \mathfrak{t})$ dominant.

To avoid complications that are only technical, we shall assume until near the end that the Dynkin diagram of $\mathfrak{l}$ is connected. We let the simple roots of $\mathfrak{g}_0 = \mathfrak{su}(m, n)$ be

$$e_1 - e_2,\ e_2 - e_3, \ldots,\ e_{m+n-1} - e_{m+n}$$

as usual, with $e_m - e_{m+1}$ noncompact and the others compact. We may suppose that the simple roots of $\mathfrak{l}$ are

$$e_r - e_{r+1}, \ldots, e_m - e_{m+1}, \ldots, e_{s-1} - e_s \qquad \text{with } 1 \leq r \leq m \leq s-1 \leq m+n-1.$$

We begin by obtaining some inequalities for the inner product of $\lambda + \delta$ with various positive roots. To simplify the notation, let us normalize all root lengths squared to be 2. Referring to (6.6), we see that

$$\langle \lambda + \delta, \gamma \rangle \geq 1 + \langle 2\delta_L, \gamma \rangle - \langle 2\delta_{L\cap K}, \gamma \rangle. \tag{6.9}$$

From this it follows that

$$\langle \lambda + \delta, \gamma \rangle \geq 1 \qquad \text{for } \gamma \text{ simple in } \Delta(\mathfrak{u}) \text{ if } \gamma \neq e_{r-1} - e_r \text{ and } \gamma \neq e_s - e_{s+1}.$$

Direct calculation using (6.9) yields also

$$\langle \lambda + \delta,\ e_{r-1} - e_r \rangle \geq s - m + 1,$$
$$\langle \lambda + \delta,\ e_s - e_{s+1} \rangle \geq r - m.$$

In addition,

$$\langle \lambda + \delta, \gamma \rangle = 1 \qquad \text{for } \gamma \text{ simple in } \Delta^+(\mathfrak{l}, \mathfrak{t}),$$

by Theorem 6.1a or 6.2a. Writing

$$e_r - e_{s+1} = (e_r - e_{r+1}) + \cdots + (e_{s-1} - e_s) + (e_s - e_{s+1}),$$

we obtain

$$\langle \lambda + \delta,\ e_r - e_{s+1} \rangle \geq 1 + \cdots + 1 + (r - m) = (s - r) + (r - m) = s - m \geq 1.$$

This is the crucial step of the proof.

We combine these inequalities with the fact that $t\delta(\mathfrak{u})$ is dominant for $\Delta^+(\mathfrak{g}, \mathfrak{t})$, and we obtain

$$\begin{aligned} \langle \lambda + \delta + t\delta(\mathfrak{u}), \gamma \rangle &\geq 1 \qquad \text{for } \gamma \text{ simple if } \gamma \neq e_s - e_{s+1} \\ \langle \lambda + \delta + t\delta(\mathfrak{u}),\ e_r - e_{s+1} \rangle &\geq 1. \end{aligned} \tag{6.10}$$

Let us write members of $\mathfrak{t}^*$ as $(m+n)$-tuples; if σ is in $\mathfrak{t}^*$, the j^{th} entry of the tuple for σ is just $\langle \sigma, e_j \rangle$. Arguing by contradiction, suppose that (6.8) holds. We shall obtain the contradiction $\mu = 0$. Write

$$\lambda + \delta + t\delta(\mathfrak{u}) = (a_1, a_2, \ldots, a_{m+n}).$$

From (6.10) we have

$$\begin{gathered} a_1 > a_2 > \cdots > a_{r-1} > a_r > \cdots > a_s, \\ a_{s+1} > \cdots > a_{m+n}, \\ a_r > a_{s+1}. \end{gathered} \tag{6.11}$$

It follows that the largest r of the a_j's, in strict decreasing order, are $a_1, \ldots, a_r$. Therefore

$$\langle \lambda + \delta + t\delta(\mathfrak{u}),\ e_1 + \cdots + e_p \rangle$$

gives the sum of the largest p entries a_j if $p \leq r$. Since μ is a sum of positive roots and $e_1 + \cdots + e_p$ is dominant,

$$\begin{aligned} \langle \lambda + \delta + t\delta(\mathfrak{u}),\ e_1 + \cdots + e_p \rangle &\leq \langle \lambda + \delta + t\delta(\mathfrak{u}) + \mu,\ e_1 + \cdots + e_p \rangle \\ &= \langle s(\lambda + \delta + t\delta(\mathfrak{u})),\ e_1 + \cdots + e_p \rangle \\ &= \langle \lambda + \delta + t\delta(\mathfrak{u}),\ s^{-1}(e_1 + \cdots + e_p) \rangle. \end{aligned} \tag{6.12a}$$

The right side of (6.12a) is

$$\leq \langle \lambda+\delta+t\delta(\mathfrak{u}),\ e_1+\cdots+e_p\rangle \tag{6.12b}$$

because the right side of (6.12a) gives the sum of p specific entries a_j but not necessarily the sum of the largest p entries. Therefore equality must hold throughout in (6.12), and we conclude that

$$\langle \mu,\ e_1+\cdots+e_p\rangle = 0 \qquad \text{for } 1\leq p\leq r.$$

From this condition we see that

$$\langle \mu, e_1\rangle = \cdots = \langle \mu, e_r\rangle = 0. \tag{6.13}$$

Inductively let us see that

$$\langle \mu, e_r\rangle = \cdots = \langle \mu, e_s\rangle = 0.$$

Suppose $r<p\leq s$ and $\langle \mu, e_r\rangle = \cdots = \langle \mu, e_{p-1}\rangle = 0$. Since μ is a sum of positive roots and $e_1+\cdots+e_p$ is dominant, $\langle \mu,\ e_1+\cdots+e_p\rangle\geq 0$. Thus $\langle \mu, e_p\rangle \geq 0$ by inductive hypothesis. Since μ is dominant for $\Delta^+(\mathfrak{l},\mathfrak{t})$, $\langle \mu,\ e_r-e_p\rangle\geq 0$. But $\langle \mu, e_r\rangle = 0$ by (6.13), and thus $\langle \mu, e_p\rangle\leq 0$. We conclude that $\langle \mu, e_p\rangle = 0$, and the induction is complete. Therefore $\langle \mu, e_p\rangle = 0$ for $1\leq p\leq s$.

Inductively let us see next that the remaining entries of μ are 0. Thus suppose $p>s$ and $\langle \mu, e_1\rangle = \cdots = \langle \mu, e_{p-1}\rangle = 0$. The entries a_j of $\lambda+\delta+t\delta(\mathfrak{u})$ with $a_j\geq a_p$ all have $j\leq p$, by (6.11). Suppose these j's are of the form

$$j_1<j_2<\cdots<j_q<p.$$

Then we have

$$\begin{aligned}
&\langle \lambda+\delta+t\delta(\mathfrak{u}),\ e_{j_1}+\cdots+e_{j_q}+e_p\rangle\\
&\quad\leq \langle \lambda+\delta+t\delta(\mathfrak{u}),\ e_{j_1}+\cdots+e_{j_q}+e_p\rangle + \langle \mu, e_1+\cdots+e_p\rangle \quad \text{since } e_1+\cdots+e_p \text{ is dominant}\\
&\quad= \langle \lambda+\delta+t\delta(\mathfrak{u})+\mu,\ e_{j_1}+\cdots+e_{j_q}+e_p\rangle \quad \text{by inductive hypothesis}\\
&\quad= \langle s(\lambda+\delta+t\delta(\mathfrak{u})),\ e_{j_1}+\cdots+e_{j_q}+e_p\rangle\\
&\quad= \langle \lambda+\delta+t\delta(\mathfrak{u}),\ s^{-1}(e_{j_1}+\cdots+e_{j_q}+e_p)\rangle,
\end{aligned}$$

and this is

$$\leq \langle \lambda+\delta+t\delta(\mathfrak{u}),\ e_{j_1}+\cdots+e_{j_q}+e_p\rangle$$

because of the choice of $j_1,\ldots j_q$. Equality must hold throughout and in particular at the first step. We conclude that $\langle \mu, e_p\rangle = 0$. Thus $\langle \mu, e_j\rangle = 0$ for all j, and $\mu=0$. This completes the proof when the Dynkin diagram of $\mathfrak{l}$ is connected.

For general $\mathfrak{l}$, we observe that $\lambda = \lambda' + 2\delta(\mathfrak{l}\cap\mathfrak{p}^+)$ is unaffected when we discard compact components from the roots of $\mathfrak{l}$. The inequalties (6.10) hold when all compact components of $\mathfrak{l}$ have been discarded, and we readily check that they remain valid for $\mathfrak{l}$ itself. The remainder of the above proof depended only on (6.10), and thus the proof goes through for general $\mathfrak{l}$.

7. Application to a nonholomorphic case

In this section we apply Theorem 5.1 to continuations of nonholomorphic discrete series, again expecting that the theory will yield small representations that may be unitary. We treat here only the situation in which the Vogan diagram of

$\mathfrak{g}_0$ is connected and has just one noncompact simple root α, with α having coefficient 2 in the largest root. This situation was considered originally by Enright-Parthasarathy-Wallach-Wolf [**EPWW**]. We take $\Delta(\mathfrak{l}')$ to be generated by all the simple roots that are compact, we let λ' be integral and $\Delta^+(\mathfrak{l}')$ dominant, and we study $\mathcal{L}_{S'}(V_{\lambda'})$. Since the largest root contains α with coefficient 2, it is compact but is not a root of $\mathfrak{l}'$. Thus $S' = \dim(\mathfrak{u}' \cap \mathfrak{k}) > 0$.

The "line-bundle cases" are those with λ' orthogonal to $\Delta(\mathfrak{l}')$. Much is known about these, beginning with partial results in [**EPWW**]. The authors of [**EPWW**] were interested in all cases where the bottom layer of $\mathcal{L}_{S'}(V_{\lambda'})$ is not empty, i.e., $\Lambda = \lambda' + 2\delta(\mathfrak{u}' \cap \mathfrak{p})$ is Δ_K^+ dominant. It is known [**K3**, Corollary 8] that Λ must be a minimal K type in the sense of Vogan. The results in [**EPWW**] concerning unitarity are in effect those that can be obtained by direct continuity arguments. They depend on explicit computations that require that the associated bundle over G/L' be a line bundle. Some of the cases that [**EPWW**] did not settle have been treated by Brylinski-Kostant, Gross-Wallach, Binegar-Zierau, Kazhdan-Savin, and others.

Since our objective is to reduce the number of small representations whose unitarity is to be studied manually, we shall largely leave aside the line-bundle cases and concentrate on "vector-bundle cases," those with λ' nonorthogonal to $\Delta(\mathfrak{l}')$. In this paper we shall limit our discussion to one group in order to highlight some phenomena that occur. We choose an example with a double line in its Dynkin diagram. Experience shows that a double line introduces complications not seen in single-line cases, and thus we should expect to see a broader range of phenomena than might be expected for a single-line case.

The example we shall work with is $\mathfrak{g}_0 = \mathfrak{so}(4,5)$ with Vogan diagram

and with respective simple roots $e_1 - e_2,\ e_2 - e_3,\ e_3 - e_4,\ e_4$. The simple roots of $\mathfrak{l}'$ are $e_1 - e_2$, $e_3 - e_4$, and e_4. We see directly that $\Delta(\mathfrak{u}' \cap \mathfrak{k}) = \{e_1 + e_2\}$, so that $S' = 1$.

We are going to be using Theorem 5.1, and the group L will have to correspond to a Hermitian case. To have anything interesting happen, we want L to be able to be relatively large. In addition, Theorem 5.1 produces a one-dimensional representation of L, and thus λ' should be orthogonal to the compact roots of any prospective L. The paper [**EPWW**] studied λ' of the form $\lambda' = (a, a, 0, 0)$ with $a \in \frac{1}{2}\mathbb{Z}$. This yields the line-bundle case. We shall study

$$\lambda' = (a + b, a, 0, 0)$$

with $a \in \frac{1}{2}\mathbb{Z}$, $b \in \mathbb{Z}$, and $b \geq 0$. The vector-bundle cases are those with $b > 0$. Actually, in any event, we restrict attention to $a \in \mathbb{Z}$ in order to be able to use a small representation on L that is one-dimensional.

We form $(\mathcal{L}_{\mathfrak{l}'}^{\mathfrak{g}})_1(V_{\lambda'})$. This has infinitesimal character

$$\lambda' + \delta = \left(a + b + \tfrac{7}{2},\, a + \tfrac{5}{2},\, \tfrac{3}{2},\, \tfrac{1}{2}\right).$$

The weakly good cases are those with $a \geq -1$. Then $(\mathcal{L}_{\mathfrak{l}'}^{\mathfrak{g}})_1(V_{\lambda'})$ is irreducible unitary, is in fact a discrete series or limit of discrete series representation. We discard these cases as uninteresting for our purposes.

The K parameter is

$$\Lambda = \lambda' + 2\delta(\mathfrak{u}' \cap \mathfrak{k}) = (a+b+5,\, a+5,\, 0,\, 0).$$

When Λ is Δ_K^+ dominant, the K type Λ occurs in $(\mathcal{L}_{\mathfrak{l}'}^{\mathfrak{g}})_1(V_{\lambda'})$ with multiplicity one. A sufficient condition for Δ_K^+ dominance is that $a \geq -5$. In the line-bundle case, this condition is also necessary, and we have noted that Λ is then a minimal K type of $(\mathcal{L}_{\mathfrak{l}'}^{\mathfrak{g}})_1(V_{\lambda'})$. In the vector-bundle case, values of $a \leq -6$ can lead to Δ_K^+ dominance, as with $a=-6$ and $b=2$. But Λ then does not appear to be a minimal K type in $(\mathcal{L}_{\mathfrak{l}'}^{\mathfrak{g}})_1(V_{\lambda'})$, and unitarity of the irreducible subquotient containing the K type Λ seems unlikely. We shall therefore concentrate on $a \geq -5$, and then it is easy to check by direct calculation that Λ is a minimal K type of $(\mathcal{L}_{\mathfrak{l}'}^{\mathfrak{g}})_1(V_{\lambda'})$.

The integer values of a of interest are therefore $a = -2, -3, -4, -5$. In the line-bundle cases ($b=0$), the "weakly fair" condition of (3.4) is satisfied for $a=-2$ and $a=-3$, and Corollary 3.4 shows that $(\mathcal{L}_{\mathfrak{l}'}^{\mathfrak{g}})_1(V_{\lambda'})$ is infinitesimally unitary for those values of a. For $a=-4$, even the more powerful Corollary 3.6 does not settle matters.

Corollary 3.4 is not applicable in the vector-bundle cases, and Corollary 3.6 does not seem to be helpful. One other tool in the literature is Theorem 9 of [**K3**], which combines double induction and use of the weakly fair condition. It handles $a=-2$ with arbitrary b, but it does not handle any cases with $a \leq -3$.

Let us now use Theorem 5.1. We shall see that Theorem 5.1 provides an interesting map $(\mathcal{L}_{\mathfrak{l}'}^{\mathfrak{g}})_1(V_{\lambda'}) \to A_{\mathfrak{q}}(\lambda)$ when $a=-2$, $a=-3$, and $a=-5$. Unitarity can then be proved by applying Corollary 3.4 to $A_{\mathfrak{q}}(\lambda)$ when $a=-2$ and $a=-3$. It applies also when $a=-5$ if $b>1$. Our technique will be to examine all possible L's to see what they give. Using an L whose Dynkin diagram is disconnected does not help, and thus there are only three possibilities.

Possibility 1. Take $\Delta(\mathfrak{l}) \leftrightarrow \{e_2 - e_3\}$.
Then $2\delta(\mathfrak{l} \cap \mathfrak{p}^+) = (0,1,-1,0)$, and Theorem 5.1 tells us to define

$$\lambda = \lambda' + 2\delta(\mathfrak{l} \cap \mathfrak{p}^+) = (a+b,\, a+1,\, -1,\, 0).$$

In order to be able to apply Theorem 5.1, λ has to be orthogonal to $\Delta(\mathfrak{l})$, hence to $e_2 - e_3$. We are led to consider $a=-2$. Then Theorem 5.1 is applicable, and we obtain a map $(\mathcal{L}_{\mathfrak{l}'}^{\mathfrak{g}})_1(V_{\lambda'}) \to A_{\mathfrak{q}}(\lambda)$. Since $\delta(\mathfrak{u}) = \left(\frac{7}{2}, 2, 2, \frac{1}{2}\right)$, we have

$$\lambda + \delta(\mathfrak{u}) = \left(\tfrac{3}{2}+b,\, 1,\, 1,\, \tfrac{1}{2}\right).$$

This is dominant, and $A_{\mathfrak{q}}(\lambda)$ is infinitesimally unitary by Corollary 3.4.

Possibility 2. Take $\Delta(\mathfrak{l}) \leftrightarrow \{e_2 - e_3,\, e_3 - e_4\}$.
Then $2\delta(\mathfrak{l} \cap \mathfrak{p}^+) = (0,2,-1,-1)$, and Theorem 5.1 tells us to define

$$\lambda = \lambda' + 2\delta(\mathfrak{l} \cap \mathfrak{p}^+) = (a+b,\, a+2,\, -1,\, -1).$$

Having λ orthogonal to $\Delta(\mathfrak{l})$ forces $a=-3$. Then Theorem 5.1 is applicable, and we obtain a map $(\mathcal{L}_{\mathfrak{l}'}^{\mathfrak{g}})_1(V_{\lambda'}) \to A_{\mathfrak{q}}(\lambda)$. We readily check that

$$\lambda + \delta(\mathfrak{u}) = \left(\tfrac{1}{2}+b,\, \tfrac{1}{2},\, \tfrac{1}{2},\, \tfrac{1}{2}\right),$$

and hence $A_{\mathfrak{q}}(\lambda)$ is infinitesimally unitary.

Possibility 3. Take $\Delta(\mathfrak{l}) \leftrightarrow \{e_2 - e_3,\, e_3 - e_4,\, e_4\}$.
Then $2\delta(\mathfrak{l} \cap \mathfrak{p}^+) = (0,5,0,0)$, and Theorem 5.1 tells us to define

$$\lambda = \lambda' + 2\delta(\mathfrak{l} \cap \mathfrak{p}^+) = (a+b,\, a+5,\, 0,\, 0).$$

Having λ orthogonal to $\Delta(\mathfrak{l})$ forces $a = -5$. Then Theorem 5.1 is applicable, and we obtain a map $(\mathcal{L}^{\mathfrak{g}}_{\mathfrak{l}'})_1(V_{\lambda'}) \to A_{\mathfrak{q}}(\lambda)$. We readily check that

$$\lambda + \delta(\mathfrak{u}) = \left(-\tfrac{3}{2} + b,\, 0,\, 0,\, 0\right),$$

and hence $A_{\mathfrak{q}}(\lambda)$ is infinitesimally unitary if $b \geq 2$. We do not know what happens when $b = 1$, but we expect unitarity.

A natural question is what happens when $a = -4$. Although Theorem 5.1 does not provide us a mapping, we can give an answer about unitarity. Recall that $\Lambda = (b+1, 1, 0, 0)$ is actually a minimal K type of $(\mathcal{L}^{\mathfrak{g}}_{\mathfrak{l}'})_1(V_{\lambda'})$. Using Proposition 4.1 of **[V1]** and knowledge of the infinitesimal character $\lambda' + \delta$, we can determine a set of Langlands parameters (MAN, σ, ν) of the irreducible subquotient of interest. Here MAN is a cuspidal parabolic subgroup of our G, σ is a discrete series or nondegenerate limit of discrete series of M, and ν is a suitable positive parameter in the complexified dual of the Lie algebra of A. What the computation shows is that A can be taken to be one-dimensional and σ is a holomorphic limit of discrete series representation of M, $\mathfrak{m}_0$ being $\mathfrak{sp}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$. The part of the unitary dual for which $\dim A = 1$ is completely known, largely from **[BSK1]**. Let us write $\nu = \frac{1}{2} c\widetilde{\alpha}$, where $\widetilde{\alpha}$ is the real root and c is a positive number; in each case the representation of interest is the Langlands quotient when $c = 3$. Exceptions (v) and (vi) of the main theorem of **[BSK1]** are what to use to decide unitarity. When $b \geq 2$, exception (v) applies, and the unitary points are those with $0 < c \leq 2$. Thus the representation of interest is not infinitesimally unitary when $b \geq 2$. When $b = 1$, exception (vi) applies, and the unitary points are those with $0 < c \leq 2$ and $c = 3$. Thus the representation of interest *is* infinitesimally unitary when $b = 1$.

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