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Geometric interpretations of two branching theorems of D.E. Littlewood

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Abstract

D.E. Littlewood proved two branching theorems for decomposing the restriction of an irreducible finite-dimensional representation of a unitary group to a symmetric subgroup. One is for restriction of a representation of $U(n)$ to the rotation group $SO(n)$ when the given representation τ_λ of $U(n)$ has nonnegative highest weight λ of depth $\leqslant n/2$. It says that the multiplicity in $\tau_\lambda|_{SO(n)}$ of an irreducible representation of $SO(n)$ of highest weight ν is the sum over μ of the multiplicities of τ_λ in the $U(n)$ tensor product $\tau_\mu \otimes \tau_\nu$, the allowable μ's being all *even* nonnegative highest weights for $U(n)$. Littlewood's proof is character-theoretic. The present paper gives a geometric interpretation of this theorem involving the tensor products $\tau_\mu \otimes \tau_\nu$ explicitly. The geometric interpretation has an application to the construction of small infinite-dimensional unitary representations of indefinite orthogonal groups and, for each of these representations, to the determination of its restriction to a maximal compact subgroup. The other Littlewood branching theorem is for restriction from $U(2r)$ to the rank-r quaternion unitary group $Sp(r)$. It concerns nonnegative highest weights for $U(2r)$ of depth $\leqslant r$, and its statement is of the same general kind. The present paper finds an analogous geometric interpretation for this theorem also.

MSC: primary 20G20, 22E46; secondary 14L35, 13A50

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Introduction

In the 1940 edition of his book [Li], D.E. Littlewood obtained a branching theorem describing how certain irreducible representations of a unitary group $U(n)$ reduce upon

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restriction to the subgroup $SO(n)$ of rotation matrices. The statement was combinatorial, and the proof was character-theoretic. There was no hint of any special representation-theoretic significance to the reduction he obtained. In particular, the only subspaces of the given representation space that the reduction pointed to canonically were the isotypic subspaces corresponding to each equivalence class of irreducible representations of $SO(n)$. In this paper we shall see that the reduction can be recast in a concrete and natural setting that exhibits a finer canonical decomposition than the one into isotypic subspaces; this finer decomposition will give a direct explanation for the relationship between the branching that Littlewood was addressing and the tensor products of representations of $U(n)$ that are implicit in the statement of his theorem.

To state Littlewood's theorem, let us consider dominant integral forms λ for $U(n)$. These are linear functionals on the diagonal subalgebra of the complexified Lie algebra $\mathfrak{gl}(n,\mathbb{C})$ of $U(n)$. If e_j denotes evaluation of the jth diagonal entry of a diagonal matrix, the linearity of λ means that λ equals $\sum_{j=1}^n a_j e_j$ for suitable complex numbers a_j. The condition "dominant integral" means that $a_1 \geqslant \cdots \geqslant a_n$ and that the a_j are integers. We shall often write $\lambda = (a_1, \ldots, a_n)$. For λ dominant integral we denote by τ_λ an irreducible representation of $U(n)$ with highest weight λ. We say that λ is *nonnegative* if $a_n \geqslant 0$. If λ is nonnegative, we say that the *depth* of λ is the smallest $k \geqslant 0$ for which $a_l = 0$ whenever $l > k$, and we define $\|\lambda\| = \sum_{k=1}^n a_k$. In classical notation, when λ is nonnegative dominant integral, λ is often viewed as a partition of $\|\lambda\|$.

The nonnegative dominant integral forms ν of depth $\leqslant n/2$ can be regarded as highest weights for $SO(n)$ by dropping 0's in the tuples beyond index $[n/2]$; we write σ_ν for an irreducible representation of $SO(n)$ with highest weight ν. For n odd, all highest weights for $SO(n)$ are obtained by restriction in this way; for n even, there are some other highest weights, but their behavior for current purposes can be deduced from the behavior of the ones we have just described.

The representations in Littlewood's theorem are those τ_λ's whose highest weights are nonnegative and have depth $\leqslant n/2$. Let ν be a nonnegative dominant integral form for $U(n)$ of depth $\leqslant n/2$, and consider the representation σ_ν of $SO(n)$. The theorem is that the multiplicity of σ_ν in $\tau_\lambda|_{SO(n)}$ equals the sum of the Littlewood–Richardson coefficients $c^\lambda_{\mu\nu}$ over all nonnegative dominant integral μ of depth $\leqslant n/2$ such that $\|\mu\| = \|\lambda\| - \|\nu\|$ and μ is *even* in the sense that every entry of μ is even. Here the *Littlewood–Richardson coefficient* $c^\lambda_{\mu\nu}$ is defined to be the multiplicity of τ_λ in $\tau_\mu \otimes \tau_\nu$ and can be computed by a well known combinatorial method that will not concern us. (See [Mac] for the method and a proof of its validity.)

We shall make repeated use of the fact that if λ, μ, and ν are nonnegative dominant integral and if the Littlewood–Richardson coefficient $c^\lambda_{\mu,\nu}$ is greater than 0, then the depths of μ and ν are automatically $\leqslant$ the depth of λ. Also $\|\lambda\| = \|\mu\| + \|\nu\|$ automatically, so that $c^\lambda_{\mu,\nu}$ is well defined in any unitary group $U(p)$ with $p \geqslant \operatorname{depth}\lambda$. Finally the value of $c^\lambda_{\mu,\nu}$ is independent of p in this range.

Now we establish our geometric setting, which will allow an extra parameter m; this m is to be a positive integer satisfying the inequality $m \leqslant n/2$ and two inequalities about depths that we state in a moment. Let $M_{mn} = M_{mn}(\mathbb{C})$ be the vector space of m-by-n matrices over $\mathbb{C}$, and let $S_{mn} = S(M_{mn})$ be the algebra of symmetric tensors on M_{mn}. The groups $U(m)$ and $SO(n)$ act on M_{mn} by matrix multiplication on the appropriate side: $u(x) = ux$

and $r(x) = xr^{-1}$ for $u \in U(m)$, $r \in SO(n)$, and $x \in M_{mn}$. These actions extend naturally to S_{mn}, and S_{mn} becomes the setting for our geometric interpretation. We introduce the expression $[A : B]_G$ for the multiplicity of an irreducible representation B of G in the restriction of A to G.

With λ and ν as above, suppose that m satisfies

$$\operatorname{depth}\lambda \leqslant m \leqslant n/2 \quad \text{and} \quad \operatorname{depth}\nu \leqslant m \leqslant n/2. \tag{0.1}$$

Because of this inequality, λ can be regarded as a highest weight for $U(m)$ as well as $U(n)$. We write τ_λ^m and τ_λ^n for the respective irreducible representations. Results of classical invariant theory, particularly Corollary 4.5.19 and Theorem 5.2.7 of [GoW], justify the last step in the following computation of multiplicities, in which $(\,\cdot\,)^c$ indicates contragredient:

$$\begin{aligned}\left[S_{mn} : \tau_\lambda^m \mathbin{\widehat{\otimes}} \sigma_\nu\right]_{U(m)\times SO(n)} &= \left[S_{mn} : \tau_\lambda^m \mathbin{\widehat{\otimes}} (\sigma_\nu)^c\right]_{U(m)\times SO(n)} \\ &= \left[S_{mn} : \tau_\lambda^m \mathbin{\widehat{\otimes}} \left(\tau_\lambda^n\right)^c\right]_{U(m)\times U(n)}\left[\tau_\lambda^n : \sigma_\nu\right]_{SO(n)} \\ &= \left[\tau_\lambda^n : \sigma_\nu\right]_{SO(n)}.\end{aligned} \tag{0.2}$$

Let $V^{\lambda,\nu}$ be the subspace of S_{mn} where $U(m) \times SO(n)$ acts by $\tau_\lambda^m \mathbin{\widehat{\otimes}} \sigma_\nu$, i.e., where $U(m)$ acts by τ_λ^m and $SO(n)$ acts by σ_ν. Because of the above equality of multiplicities, $V^{\lambda,\nu}$ can be regarded as the tensor product of the space for τ_λ^m with the full ν-isotypic subspace for $\tau_\lambda^n|_{SO(n)}$.

The symmetric-tensor multiplication map $S_{mn} \times S_{mn} \to S_{mn}$ is bilinear and extends to a linear map $\mathcal{M} : S_{mn} \otimes S_{mn} \to S_{mn}$ that respects the group actions. With μ as above, we define $V^{\nu,\nu}$ and $V^{\mu,0}$ in the same way that $V^{\lambda,\nu}$ was defined above. Let $V^{\lambda,\nu,\mu}$ be the intersection of $V^{\lambda,\nu}$ with the image of $V^{\nu,\nu} \otimes V^{\mu,0}$ under $\mathcal{M}$; this is a subspace of $V^{\lambda,\nu}$ stable under $U(m)$ and $SO(n)$; let $(V^{\nu,\nu} \otimes V^{\mu,0})^\lambda$ be the subspace of $V^{\nu,\nu} \otimes V^{\mu,0}$ that transforms according to τ_λ^m on the left.

Theorem 0.1. *Let λ and ν be nonnegative dominant integral forms with* $\operatorname{depth}\nu \leqslant m \leqslant n/2$ *and* $\operatorname{depth}\lambda \leqslant m \leqslant n/2$. *Then the multiplication map*

$$\mathcal{M} : \Bigg(V^{\nu,\nu} \otimes \bigoplus_{\substack{\mu \text{ dominant integral} \\ \mu \text{ even nonnegative} \\ \|\mu\| = \|\lambda\| - \|\nu\|}} V^{\mu,0}\Bigg)^{\lambda} \to V^{\lambda,\nu}$$

within the algebra S_{mn} of symmetric tensors is one–one and onto. Therefore $V^{\lambda,\nu}$ is the direct sum of the subspaces $V^{\lambda,\nu,\mu}$ over all even nonnegative dominant integral μ such that $\operatorname{depth}\mu \leqslant \operatorname{depth}\lambda$.

The proof of Theorem 0.1 will make use of Littlewood's theorem. Conversely, the assertion of Theorem 0.1 easily implies Littlewood's result: since the multiplication map is one–one from $(V^{\nu,\nu} \otimes V^{\mu,0})^\lambda$ onto $V^{\lambda,\nu,\mu}$, the multiplicity of $\tau_\lambda^m \otimes \sigma_\nu$ in $V^{\lambda,\nu,\mu}$ is $c_{\mu\nu}^\lambda$.

From the direct-sum decomposition $V^{\lambda,\nu} = \bigoplus_\mu V^{\lambda,\nu,\mu}$, we see that the sum over μ of the $c^\lambda_{\mu\nu}$ equals the multiplicity of $\tau^m_\lambda \mathbin{\widehat{\otimes}} \sigma_\nu$ in $V^{\lambda,\nu}$. By (0.2) this in turn equals the multiplicity of σ_ν in $\tau^n_\lambda|_{SO(n)}$.

The subspaces $V^{\lambda,\nu,\mu}$ are canonical, and the direct-sum decomposition $V^{\lambda,\nu} = \bigoplus_\mu V^{\lambda,\nu,\mu}$ produced by the theorem therefore represents a canonical decomposition of the isotypic subspace $V^{\lambda,\nu}$. In a later paper [Kn2] it is shown that this canonical decomposition has an application to the unitarity of certain exotic infinite-dimensional representations of indefinite orthogonal groups and to a description of the splitting of these representations under a maximal compact subgroup. The specific form of the decomposition that is used in the first instance in [Kn2] is an inclusion of $V^{\lambda,\nu}$ into a set of sums of products: $V^{\lambda,\nu} \subseteq \sum_\mu V^{\nu,\nu} V^{\mu,0}$.

Littlewood's branching theorem from $U(n)$ to $SO(n)$ appeared on page 240 of the 1940 edition of [Li]. On page 295 of the 1950 edition, Littlewood stated a companion theorem for branching from $U(2r)$ to the rank-r quaternion unitary group $Sp(r)$. This theorem has a geometric interpretation in the spirit of Theorem 0.1; the statement of the geometric interpretation and an indication of its proof appear in Section 6 below.

There have been other efforts to find representation-theoretic interpretations for Littlewood's two theorems. Let us mention specifically some joint work of Deenen and Quesne [DeQ] and some work of Quesne [Q1,Q2] and Maliakas [Mal1]. These papers are quite different in spirit from the present one, and they do not appear to offer any insight into our main results or help with the proofs. The papers by Quesne are related to Howe's theory of dual reductive pairs [Ho], and the one by Maliakas involves resolutions of modules and application of the Euler–Poincaré principle.

In proving his theorem for branching from $U(n)$ to $SO(n)$, Littlewood was building on ideas in [Mu], but the statement in [Li] is not absolutely clear and the proof is difficult to decipher. Statements of Littlewood's results for $SO(n)$ and $Sp(n)$ with all the hypotheses in place appear in [DeQ] and [Mal2], respectively, and references to modern proofs may be found in [Mal2]. Newell [Ne] showed how Littlewood's theorem for $SO(n)$ could be modified to eliminate the limitation on the depth of the given highest weight; there is no attempt below to give a geometric interpretation of Newell's modification.

The first three sections develop relevant properties of $V^{\nu,\nu}$, $V^{\mu,0}$, and sums of spaces $V^{\mu,0}$. Section 4 contains a proof that $\mathcal{M}$ is one–one. Section 5 completes the proof of Theorem 0.1 by combining Littlewood's Theorem and the result of Section 4 to show that the domain and range of the map $\mathcal{M}$ in Theorem 0.1 have the same dimension, so that $\mathcal{M}$ one–one implies $\mathcal{M}$ is onto. Finally, Section 6 states as Theorem 6.1 a comparable geometric interpretation of Littlewood's other branching theorem, the one concerning restriction from $U(2r)$ to $Sp(r)$.

1. Highest weight vectors for $V^{\nu,\nu}$

We assume throughout Sections 1–5 that $m \leqslant n/2$. The complexified linear Lie algebras of our two groups $U(m)$ and $SO(n)$ are $\mathfrak{gl}(m,\mathbb{C}) = M_{mm}$ and $\mathfrak{so}(n,\mathbb{C}) \subseteq \mathfrak{gl}(n,\mathbb{C}) = M_{nn}$.

We denote general members of $\mathfrak{gl}(m,\mathbb{C})$, $\mathfrak{gl}(n,\mathbb{C})$, and M_{mn} by E, E', and X, respectively, so that the actions are given by

$$E(X) = EX \quad \text{and} \quad E'(X) = -XE',$$

the right side in each case being a matrix product.

Let r be the greatest integer in $n/2$, so that $m \leqslant r$. The row indices for X will be written simply as $\{1,\dots,m\}$, but the column indices will usually be written as

$$\begin{array}{ll} \{1,\dots,r,1',\dots,r'\} & \text{if } n \text{ is even}, \\ \{1,\dots,r,1',\dots,r',\infty\} & \text{if } n \text{ is odd}. \end{array}$$

In this case we shall write matrices out correspondingly in blocks of respective sizes r, r, and 0 if n is even, or r, r, and 1 if n is odd. We reserve the index k for a column index that goes from 1 to n. A symbol E, E', or X with the appropriate kind of subscripts stands for the matrix that is 1 in the indicated entry and 0 elsewhere.

We use the customary Cartan subalgebra, root ordering, and root vectors for $\mathfrak{gl}(m,\mathbb{C})$: The Cartan subalgebra is the diagonal subalgebra $\{\sum_{a=1}^m h_a E_{aa}\}$. We define e_b to be evaluation of the bth diagonal entry, with $e_b(\sum_{a=1}^m h_a E_{aa}) = h_b$. The roots are all $e_a - e_b$ with $a \neq b$, and corresponding root vectors are the E_{ab}. The usual ordering makes $e_a - e_b$ positive if $a < b$.

For the action of $\mathfrak{gl}(m,\mathbb{C})$ on M_{mn}, we have $E_{aa}X_{bk} = \delta_{ab}X_{ak}$. Thus the weights for this action are the various e_a, and the weight space for the weight e_a is

$$\sum_{b=1}^{r} \mathbb{C}X_{ab} \oplus \sum_{b=1}^{r} \mathbb{C}X_{ab'} \oplus \mathbb{C}X_{a\infty}. \tag{1.1}$$

The root vectors affect only the row indices of members of M_{mn}:

$$E_{ab}(X_{ck}) = E_{ab}X_{ck} = \delta_{bc}X_{ak}. \tag{1.2}$$

The corresponding remarks about the action on the right side of M_{mn} are more complicated because (0.2) assumes use for $\mathfrak{gl}(n,\mathbb{C})$ of the diagonal Cartan subalgebra, which meets $\mathfrak{so}(n,\mathbb{C})$ in 0. Let us write members of the diagonal subalgebra of $\mathfrak{gl}(n,\mathbb{C})$ as

$$\operatorname{diag}(H, H', h_\infty) = \begin{pmatrix} H & & \\ & H' & \\ & & h_\infty \end{pmatrix}$$

with H and H' diagonal of size r and with $h_\infty \in \mathbb{C}$. Define an automorphism Φ of $\mathfrak{gl}(n,\mathbb{C})$ by $\Phi(E') = CE'C^{-1}$, where

$$C = \begin{pmatrix} \sqrt{2}/2 & -\mathrm{i}\sqrt{2}/2 & 0 \\ -\mathrm{i}\sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then the set of all

$$\Phi\bigl(\operatorname{diag}(H, H', h_\infty)\bigr) = \begin{pmatrix} \frac{1}{2}(H+H') & \frac{\mathrm{i}}{2}(H-H') & 0 \\ -\frac{\mathrm{i}}{2}(H-H') & \frac{1}{2}(H+H') & 0 \\ 0 & 0 & h_\infty \end{pmatrix} \tag{1.3}$$

is another Cartan subalgebra of $\mathfrak{gl}(n,\mathbb{C})$. Meanwhile we take the set of all

$$\begin{pmatrix} 0 & \mathrm{i}h & 0 \\ -\mathrm{i}h & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{1.4}$$

with h diagonal of size r, as a Cartan subalgebra of $\mathfrak{so}(n,\mathbb{C})$. The matrices (1.3) that are of the form (1.4) are those with $H' = -H$ and $h_\infty = 0$, and equality is achieved in this case by taking $H = h$. Define $\tilde{e}'_k$ on $\operatorname{diag}(H, H', h_\infty)$ by

$$\tilde{e}'_k\bigl(\operatorname{diag}(H, H', h_\infty)\bigr) = \begin{cases} H_{kk} & \text{if } 1 \leqslant k \leqslant r, \\ H'_{k-r,k-r} & \text{if } r+1 \leqslant k \leqslant 2r, \\ h_\infty & \text{if } n \text{ is odd and } k = n, \end{cases}$$

and define e'_a on (1.4) for $1 \leqslant a \leqslant r$ to be h_{aa}. The result is that

$$e'_a\begin{pmatrix} 0 & \mathrm{i}h & 0 \\ -\mathrm{i}h & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \tilde{e}'_a\bigl(\operatorname{diag}(h, -h, 0)\bigr) \quad \text{for } 1 \leqslant a \leqslant r. \tag{1.5}$$

If ν is a nonnegative dominant integral form on the diagonal subalgebra of $\mathfrak{gl}(n,\mathbb{C})$ of depth $\leqslant r$, formula (1.5) allows us to reinterpret ν as a form on the Cartan subalgebra of $\mathfrak{so}(n,\mathbb{C})$. This is the reinterpretation to use in passing from $\mathfrak{gl}(n,\mathbb{C})$ to $\mathfrak{so}(n,\mathbb{C})$ in Littlewood's theorem. Without it one may not find the correct highest weight vectors for $V^{\nu,\nu}$.

The roots for $\mathfrak{so}(n,\mathbb{C})$ are the $\pm e'_a \pm e'_b$ with $a \neq b$ when n is even, and these plus the $\pm e'_a$ when n is odd. With the usual ordering, the positive roots are the $e'_a \pm e'_b$ with $a < b$ when n is even, and these plus the e'_a when n is odd.

For the action of $\mathfrak{so}(n,\mathbb{C})$ on M_{mn}, we have

$$\bigl(\mathrm{i}\bigl(E'_{aa'} - E'_{a'a}\bigr)\bigr)(X_{bk}) = -\mathrm{i}X_{bk}E_{aa'} + \mathrm{i}X_{bk}E_{a'a} = \begin{cases} -\mathrm{i}\delta_{ac}X_{ba'} & \text{if } k = c, \\ +\mathrm{i}\delta_{ac}X_{ba} & \text{if } k = c+r, \\ 0 & \text{if } k = n, \end{cases}$$

from which it follows that

$$\begin{aligned} \bigl(\mathrm{i}\bigl(E'_{aa'} - E'_{a'a}\bigr)\bigr)(X_{bc} - \mathrm{i}X_{bc'}) &= \delta_{ac}(X_{ba} - \mathrm{i}X_{ba'}), \\ \bigl(\mathrm{i}\bigl(E'_{aa'} - E'_{a'a}\bigr)\bigr)(X_{bc} + \mathrm{i}X_{bc'}) &= -\delta_{ac}(X_{ba} - \mathrm{i}X_{ba'}), \\ \bigl(\mathrm{i}\bigl(E'_{aa'} - E'_{a'a}\bigr)\bigr)(X_{b\infty}) &= 0. \end{aligned}$$

Therefore

$$\begin{aligned} X_{bc} - \mathrm{i}X_{bc'} &\text{ has weight } e'_c, \\ X_{bc} + \mathrm{i}X_{bc'} &\text{ has weight } -e'_c, \\ X_{b\infty} &\text{ has weight } 0. \end{aligned} \tag{1.6}$$

Because of our interest in nonnegative highest weights, we define

$$Z_{bc} = X_{bc} - \mathrm{i}X_{bc'} \quad \text{for } 1 \leqslant b \leqslant m \text{ and } 1 \leqslant c \leqslant r.$$

We shall make use of the effect of root vectors for positive roots on these elements. For root vectors we can take

$$\begin{aligned} E'_{e'_a - e'_b} &= (E'_{ab} - E'_{ba}) + \mathrm{i}(E'_{ab'} - E'_{b'a}) + \mathrm{i}(E'_{ba'} - E'_{a'b}) + (E'_{a'b'} - E'_{b'a'}) \quad \text{if } a < b, \\ E'_{e'_a + e'_b} &= (E'_{ab} - E'_{ba}) - \mathrm{i}(E'_{ab'} - E'_{b'a}) + \mathrm{i}(E'_{ba'} - E'_{a'b}) - (E'_{a'b'} - E'_{b'a'}) \quad \text{if } a < b, \\ E'_{e'_a} &= (E'_{a\infty} - E'_{\infty a}) - \mathrm{i}(E'_{a'\infty} - E'_{\infty a'}). \end{aligned}$$

A little computation then gives

$$\begin{aligned} E'_{e'_a - e'_b}(Z_{cd}) &= 2\delta_{bd} Z_{ca} \quad \text{if } a < b, \\ E'_{e'_a + e'_b}(Z_{cd}) &= 0 \quad \text{if } a < b, \\ E'_{e'_a}(Z_{cd}) &= 0. \end{aligned} \tag{1.7}$$

Lemma 1.1. *Let $f : \mathbb{C}^p \to \mathbb{C}$ be a polynomial function, and let E be a derivation of S_{mn}. If $A_1, \ldots, A_p$ are in S_{mn}, then*

$$E\big(f(A_1, \ldots, A_p)\big) = \sum_{j=1}^{p} \left(\frac{\partial f}{\partial x_j}(A_1, \ldots, A_p) \right) E(A_j).$$

Proof. Direct calculation shows that the two sides match when f is a monomial function $f(x_1, \ldots, x_p) = x_1^{q_1} \cdots x_p^{q_p}$. By linearity the two sides match for a general polynomial function f. $\square$

Lemma 1.2. *Let x_{ij}, for $1 \leqslant i \leqslant p$ and $1 \leqslant j \leqslant p$, be independent complex variables, and let x be the p-by-p matrix $\{x_{ij}\}$. Write $\hat{x}_{ab}$ for the matrix x with the ath row and bth column deleted. Then*

$$\frac{\partial(\det x)}{\partial x_{ab}} = (-1)^{a+b} \det \hat{x}_{ab}.$$

Proof. By linearity of det in the ath row, we have

$$\det\bigl(\{x_{ij}+h\delta_{ai}\delta_{bj}\}\bigr)=\det x+\det\begin{pmatrix} x_{11} & \cdots & x_{1b} & \cdots & x_{1p} \\ \vdots & \ddots & \vdots & & \vdots \\ 0 & \cdots & h & \cdots & 0 \\ \vdots & & \vdots & \ddots & \vdots \\ x_{p1} & \cdots & x_{pb} & \cdots & x_{pp} \end{pmatrix},$$

the exceptional row of the right-hand determinant being the ath. The partial derivative in question is the derivative with respect to h at $h=0$ of the left side. Expanding the right-hand determinant by cofactors about the ath row and differentiating the right side with respect to h, we obtain the result of the lemma. □

Proposition 1.3. *If $1\leqslant p\leqslant m$, then the element*

$$\det\Bigl(\{Z_{ab}\}_{\substack{1\leqslant a\leqslant p\\ 1\leqslant b\leqslant p}}\Bigr)$$

of S_{mn} has weight $(e_1+\cdots+e_p,\ e'_1+\cdots+e'_p)$ under $\mathfrak{gl}(m,\mathbb{C})\oplus\mathfrak{so}(n,\mathbb{C})$ and is a nonzero highest weight vector.

Remarks. The method of proof will be needed again later. Thus we provide details this time and in the future will be able to say that the method is the same as for the proof of Proposition 1.3. This kind of result and proof is not at all new; see Procesi [Pr], especially Section 5.2, and also [DeP]. Similar remarks apply to Proposition 1.4.

Proof. Write Z for the matrix $\{Z_{ij}\}$, and let $\widehat{Z}_{ab}$ be the matrix Z with the ath row and bth column deleted. The expansion of $\det Z$ involves the product $\prod_{a=1}^{p} X_{aa}$, which cancels no other term of the determinant; therefore $\det Z$ is not 0.

For any derivation E of S_{mn}, Lemmas 1.1 and 1.2 give

$$E(\det Z)=\sum_{a,b}(-1)^{a+b}\bigl(\det\widehat{Z}_{ab}\bigr)E(Z_{ab}). \tag{1.8}$$

First take E in (1.8) to be application of a diagonal matrix H in $\mathfrak{gl}(m,\mathbb{C})$. Then (1.1) and the expansion-by-cofactors formula show that the right side is

$$\begin{aligned}&=\sum_a\sum_b(-1)^{a+b}e_a(H)Z_{ab}\bigl(\det\widehat{Z}_{ab}\bigr)=\sum_a e_a(H)\sum_b(-1)^{a+b}Z_{ab}\bigl(\det\widehat{Z}_{ab}\bigr)\\ &=\sum_a e_a(H)\det Z=(e_1+\cdots+e_p)(H)\det Z.\end{aligned}$$

This establishes the weight of $\det Z$ under the action by $\mathfrak{gl}(m,\mathbb{C})$. A similar computation—taking E to be application of a member of the Cartan subalgebra of $\mathfrak{so}(n,\mathbb{C})$, using (1.6), and summing over a and b in the reverse order—establishes the weight of $\det Z$ under the action of $\mathfrak{so}(n,\mathbb{C})$.

To see that $\det Z$ is a highest weight vector for $\mathfrak{gl}(m,\mathbb{C})$ and $\mathfrak{so}(n,\mathbb{C})$, we take E in (1.8) to be application of a root vector for a positive root. In the case of $\mathfrak{gl}(m,\mathbb{C})$, the root vector is E_{ij} for $i<j$. We have $E_{ij}(Z_{ab})=\delta_{ja}Z_{ib}$ by (1.2), and thus (1.8) is

$$=\sum_a \delta_{ja}\sum_b (-1)^{a+b}Z_{ib}\big(\det \widehat{Z}_{ab}\big).$$

This is 0 if j is not one of the indices a between 1 and p. Otherwise it is

$$=\sum_b (-1)^{j+b}Z_{ib}\big(\det \widehat{Z}_{jb}\big),$$

i.e., the determinant of the matrix Z except that the jth row has been replaced by the contents of the ith row. Since $i<j$, the new determinant now has its ith and jth rows equal. It is therefore 0. A similar computation—using (1.7) and summing over a and b in the reverse order—shows that the root vectors for the positive roots of $\mathfrak{so}(n,\mathbb{C})$ act by 0. Therefore $\det Z$ is a highest weight vector. □

Proposition 1.4. *If $\nu=\sum_{a=1}^m \nu_a e_a$ is nonnegative dominant integral for $\mathfrak{gl}(m,\mathbb{C})$, so that* (1.5) *allows it to be reinterpreted as a highest weight $\nu=\sum_{a=1}^m \nu_a e'_a$ for $\mathfrak{so}(n,\mathbb{C})$, then the representation $\tau_\nu \mathbin{\widehat{\otimes}} \sigma_\nu$ occurs with multiplicity one in S_{mn}, and the highest weight vectors for it are the multiples of*

$$Z(\nu)=\prod_{p=1}^m \det\Big(\{Z_{ab}\}_{\substack{1\leqslant a\leqslant p\\ 1\leqslant b\leqslant p}}\Big)^{\nu_p-\nu_{p+1}}$$

if ν_{m+1} is interpreted as 0.

Proof. We apply Proposition 1.3 to each determinant factor. The product of highest weight vectors in S_{mn} is a highest weight vector, and the weights add. Therefore $Z(\nu)$ is a highest weight vector, and the representation that it generates is of type $\tau_\nu \mathbin{\widehat{\otimes}} \sigma_\nu$.

For a proof that the multiplicity is one, an easy argument is to quote Littlewood's theorem: the only allowable μ is $\mu=0$, and the relevant sum of Littlewood–Richardson coefficients reduces to $c^\nu_{0\nu}=1$. For a more elementary argument, one can use formulas (1.3)–(1.5) to see that the highest weight of $\tau_\nu \mathbin{\widehat{\otimes}} \sigma_\nu$ arises from $\tau^m_\nu \mathbin{\widehat{\otimes}} (\tau^n_\nu)^c$ by restriction in only one way. □

2. Some linear independence

Let S^d_{mn} be the subspace of S_{mn} of elements homogeneous of degree d. This finite-dimensional subspace is stable under the actions by $U(m)$ and $SO(n)$. It is therefore the direct sum of its weight spaces. In view of (1.1), the weight spaces that contribute to S^d are those with any weight

$$\sum_{i=1}^{m} a_i e_i + \sum_{j=1}^{r} b_j e'_j \quad \text{for which } \sum_{i=1}^{m} a_i = d. \tag{2.1}$$

Let X_a be the formal "row vector" with the n entries X_{ak}, $1 \leqslant k \leqslant n$, and define the *dot product* of two of these to be the element in S^2_{mn} given by

$$X_a \cdot X_b = \sum_k X_{ak} X_{bk} = \sum_{c=1}^{r} X_{ac} X_{bc} + \sum_{c=1}^{r} X_{ac'} X_{bc'} + X_{a\infty} X_{b\infty}.$$

Such a dot product is in the subalgebra $(S_{mn})^{SO}$ of $SO(n)$ invariants in S_{mn} and hence is in $(S^2_{mn})^{SO} = S^2_{mn} \cap (S_{mn})^{SO}$. Dot product is of course symmetric. Formula (1.2) and the definitions show that the action of $\mathfrak{gl}(m, \mathbb{C})$ on the left is given by

$$E_{pq}(X_a \cdot X_b) = \delta_{qa}(X_p \cdot X_b) + \delta_{qb}(X_a \cdot X_p). \tag{2.2}$$

In particular,

$$X_a \cdot X_b \text{ is a weight vector under } \mathfrak{gl}(m, \mathbb{C}) \text{ of weight } e_a + e_b. \tag{2.3}$$

We are going to prove that the monomials in the $m(m+1)/2$ dot products $X_a \cdot X_b$, $a \leqslant b$, are linearly independent. We shall obtain this result as a corollary of a stronger result that will be needed also. We begin with a precise notation for monomials. Let $D = \{D_{ab}\}_{a \leqslant b}$ be a tuple of $m(m+1)/2$ nonnegative integers and put $\|D\| = \sum_{a \leqslant b} D_{ab}$. Define the monomial $P(D)$ of dot products by

$$P(D) = \prod_{a \leqslant b} (X_a \cdot X_b)^{D_{ab}}. \tag{2.4}$$

By (2.1) and (2.3),

$$P(D) \text{ lies in } S^d \text{ with } d = 2\|D\|. \tag{2.5}$$

The proof of the linear independence will take us outside the realm of polynomials in dot products. Define a linear mapping $\varphi : M_{mn} \to M_{mn}$ by its values on a basis:

$$\varphi(X_{ab}) = \begin{cases} X_{ab} & \text{if } a \leqslant b, \\ 0 & \text{if } a > b, \end{cases} \qquad \varphi(X_{ab'}) = \begin{cases} X_{ab'} & \text{if } a = b, \\ 0 & \text{if } a \neq b, \end{cases} \qquad \varphi(X_{a\infty}) = 0. \tag{2.6}$$

Then extend φ, without changing its name, to an algebra endomorphism of S_{mn} sending 1 to 1. Since φ sends each monomial into itself or into 0, φ carries S^d_{mn} into itself for each d.

For example,

$$\varphi(Z_{ab}) = \varphi(X_{ab}) - \mathrm{i}\varphi(X_{ab'}) = \begin{cases} Z_{ab} & \text{if } a = b, \\ X_{ab} & \text{if } a < b, \\ 0 & \text{if } a > b. \end{cases} \tag{2.7}$$

If $p \leqslant m$, then the determinant of Proposition 1.3 satisfies

$$\varphi\Big(\det\Big(\{Z_{ab}\}_{\substack{1\leqslant a\leqslant p\\ 1\leqslant b\leqslant p}}\Big)\Big) = \sum_{\pi\in\mathfrak{S}_p} (\operatorname{sgn}\pi) \prod_{a=1}^{p} \varphi(Z_{a\pi(a)}) = \prod_{a=1}^{p} Z_{aa},$$

where $\mathfrak{S}_p$ is the symmetric group on $\{1, \dots, p\}$, since each nontrivial permutation π has $a > \pi(a)$ for some a. Therefore the element $Z(\nu)$ of Proposition 1.4 satisfies

$$\varphi\big(Z(\nu)\big) = \prod_{p=1}^{m} \prod_{a=1}^{p} Z_{aa}^{\nu_p-\nu_{p+1}} = \prod_{p=1}^{m} (Z_{pp})^{\nu_p}. \tag{2.8}$$

We define

$$Q(D) = \varphi\big(P(D)\big). \tag{2.9}$$

Theorem 2.1. *The members $Q(D)$ of S_{mn} are linearly independent.*

Proof. It is enough to restrict attention to those D's with $2\|D\|$ equal to a particular degree d, since such D's have $Q(D)$ in S_{mn}^d and the spaces S_{mn}^d are linearly independent as d varies. We shall prove by induction on d that the $Q(D)$'s with $2\|D\| = d$ are linearly independent. The base case is that $d = 0$. Then the only $Q(D)$ is 1, corresponding to having $D_{ab} = 0$ for all $a \leqslant b$. The set $\{1\}$ is linearly independent, and this disposes of the case $d = 0$.

Assume the linear independence for $S_{mn}^0, \dots, S_{mn}^{d-1}$ with d now > 0. We shall do an induction on an index p with $1 \leqslant p \leqslant m$. The assertion to be proved by induction on p is that whenever

$$\sum_{2\|D\|=d} a_D Q(D) = 0, \tag{2.10}$$

then $a_D = 0$ for all $D = \{D_{ab}\}_{a\leqslant b}$ with $D_{ab} \neq 0$ for some pair (a, b) with $a \leqslant b < p$. This assertion is empty when $p = 1$. We shall prove that an identity (2.10) implies that $a_D = 0$ whenever D has $D_{ab} \neq 0$ for some pair (a, b) with $a \leqslant b < p + 1$, i.e., $a \leqslant b \leqslant p$. At the end of this inner induction, we shall have proved concerning any identity (2.10) that $a_D = 0$ whenever D has $D_{ab} \neq 0$ for some pair (a, b) with $a \leqslant b \leqslant m$; since $d > 0$, each D with $2\|D\| = d$ has such a pair (a, b), and therefore $a_D = 0$ for all D.

With the assertion about p assumed by induction, the new thing that we have to prove about an identity (2.10) is that $a_D = 0$ if $D_{ab} \neq 0$ for some pair (a, b) with $a \leqslant b = p$, i.e., with (a, b) equal to $(1, p), \dots, (p - 1, p)$, or (p, p).

First we consider the a's with $1 \leqslant a < p$. Fix such an a. If D has $D_{ap} \neq 0$, then $\varphi(X_a \cdot X_p)$ is a factor of $Q(D)$. Since

$$\begin{aligned}\varphi(X_a \cdot X_p) &= \varphi(X_{a1}X_{p1} + \cdots + X_{ar}X_{pr} + X_{a1'}X_{p1'} + \cdots + X_{ar'}X_{pr'} + X_{a\infty}X_{p\infty}) \\ &= X_{ap}X_{pp} + \cdots + X_{ar}X_{pr},\end{aligned}$$

X_{ap} has to occur in the expansion of $Q(D)$. Conversely, consider all D's such that X_{ap} occurs in the expansion of $Q(D)$, and suppose that $a_D \neq 0$. For X_{ap} to occur in the Dth term, it must occur in a term $\varphi(X_{ap}X_{cp})$ of $\varphi(X_a \cdot X_c)$ for some c, and we must have $c \leqslant p$ by (2.6). If $a \leqslant c$, this means $D_{ac} > 0$; if $a > c$, it means $D_{ca} > 0$. In either case, if $c < p$ then a and c are $< p$, and our inductive hypothesis says that $a_D = 0$, contradiction; so $c \geqslant p$, and we conclude $c = p$. Thus the D's in (2.10) for which $a_D \neq 0$ and the expansion of $Q(D)$ contains X_{ap} are exactly those with $D_{ap} > 0$. Reviewing this argument, we see that the D's in (2.10) for which $a_D \neq 0$ and the expansion of $Q(D)$ contains the lth power of X_{ap} but no higher power are exactly those with $D_{ap} = l$ and that $Q(D)/(\varphi(X_a \cdot X_p))^l$ does not have X_{ap} in its expansion. Consequently, we can rewrite (2.10) as

$$\sum_{l=0}^{L} \bigl(\varphi(X_a \cdot X_p)\bigr)^l \sum_{\substack{D \text{ with} \\ D_{ap}=l}} a_D \frac{Q(D)}{(\varphi(X_a \cdot X_p))^l} = 0, \tag{2.11}$$

and no X_{ap} occurs in the expansion of any $Q(D)/(\varphi(X_a \cdot X_p))^l$ having $a_D \neq 0$. Equating the coefficients of $(X_{ap})^L$ on the two sides of (2.11), we obtain

$$\sum_{\substack{D \text{ with} \\ D_{ap}=L}} a_D \frac{Q(D)}{(\varphi(X_a \cdot X_p))^L} = 0. \tag{2.12}$$

If $L > 0$, then (2.12) is a relation like (2.10) but with d replaced by $d - 2L$. Thus the outer induction shows that all the coefficients a_D in (2.12) are 0, and the term of (2.11) with $D_{ap} = L$ can be dropped. Arguing similarly for l equal to $L-1, L-2, \ldots, 1$, we see that all the coefficients a_D are 0 except possibly some of the ones in the $l = 0$ term of (2.11). We are thus reduced to an identity $\sum_{D \text{ with } D_{ap}=0} a_D Q(D) = 0$. That is, we have succeeded in showing that $D_{ap} = 0$ for all coefficients in (2.10) for which $a_D \neq 0$. Since a is arbitrary with $1 \leqslant a < p$, we obtain $D_{1p} = D_{2p} = \cdots = D_{p-1,p} = 0$ in all terms of (2.10) with $a_D \neq 0$.

Now we consider the case $a = p$. If D has $D_{pp} \neq 0$, then $\varphi(X_p \cdot X_p)$ is a factor of $Q(D)$. Since

$$\varphi(X_p \cdot X_p) = X_{pp}X_{pp} + \cdots + X_{pr}X_{pr} + X_{pp'}X_{pp'},$$

X_{pp} has to occur in the expansion of $Q(D)$. Conversely consider all D's such that X_{pp} occurs in the expansion of $Q(D)$, and suppose that $a_D \neq 0$. For X_{pp} to occur in the Dth term, it must occur in a term $\varphi(X_{pp}X_{cp})$ of $\varphi(X_p \cdot X_c)$ for some c, and we must have $c \leqslant p$

by (2.6). If $c < p$, this means $D_{cp} > 0$; by what we have just shown, a_D must be 0. Thus if $a_D \neq 0$, then X_{pp} occurs in the expansion of $Q(D)$ if and only if $D_{pp} > 0$. Reviewing this argument, we see that the D's in (2.10) for which $a_D \neq 0$ and the expansion of $Q(D)$ contains the lth power of X_{pp} but no higher power have the properties that l is even, $D_{pp} = l/2$, and $Q(D)/(\varphi(X_p \cdot X_p))^{l/2}$ does not have X_{pp} in its expansion. Consequently, we can rewrite (2.10) as

$$\sum_{l=0}^{L} \bigl(\varphi(X_p \cdot X_p)\bigr)^l \sum_{\substack{D \text{ with} \\ D_{pp}=l}} a_D \frac{Q(D)}{(\varphi(X_p \cdot X_p))^l} = 0, \tag{2.13}$$

and no X_{pp} occurs in the expansion of any $Q(D)/(\varphi(X_p \cdot X_p))^l$ having $a_D \neq 0$. Equating the coefficients of $(X_{pp})^{2L}$ on the two sides of (2.13), we obtain

$$\sum_{\substack{D \text{ with} \\ D_{pp}=L}} a_D \frac{Q(D)}{(\varphi(X_p \cdot X_p))^L} = 0. \tag{2.14}$$

If $L > 0$, then (2.14) is a relation like (2.10) but with d replaced by $d - 2L$. Thus the outer induction shows that all the coefficients a_D in (2.14) are 0, and the term of (2.13) with $D_{pp} = L$ can be dropped. Arguing similarly for l equal to $L-1, L-2, \ldots, 1$, we see that all the coefficients a_D are 0 except possibly some of the ones in the $l = 0$ term of (2.13). We are thus reduced to an identity $\sum_{D \text{ with } D_{pp}=0} a_D Q(D) = 0$. That is, we have succeeded in showing that $D_{pp} = 0$ for all coefficients in (2.10) for which $a_D \neq 0$. This completes the induction on p, the induction on d, and the proof of the theorem. □

Corollary 2.2. *The members $P(D)$ of $(S_{mn})^{SO}$ are linearly independent.*

Remark. This result was known already. See [DeP, Theorem 5.1]. For our application, however, we need the stronger result listed above as Theorem 2.1.

Proof. It is enough to consider the finite set of D's with $2\|D\|$ equal to a fixed d. If $\sum_D a_D P(D) = 0$, then application of φ gives $\sum_D a_D Q(D) = 0$, and Theorem 2.1 shows that all the a_D are 0. □

3. Structure of $V^{\mu,0}$

In this section we shall identify the highest weight vectors in $V^{\mu,0}$ and examine how all the members of $V^{\mu,0}$ are propagated from the highest weight vectors.

Proposition 3.1. *If $1 \leqslant p \leqslant m$, then the element*

$$\det\Bigl(\{X_a \cdot X_b\}_{\substack{1 \leqslant a \leqslant p \\ 1 \leqslant b \leqslant p}}\Bigr)$$

of S_{mn} has weight $(2e_1+\cdots+2e_p,0)$ *under* $\mathfrak{gl}(m,\mathbb{C})\oplus\mathfrak{so}(n,\mathbb{C})$ *and is a nonzero highest weight vector.*

Proof. The determinant in question is a polynomial in dot products and is therefore invariant under $\mathfrak{so}(n,\mathbb{C})$. One of the terms from the determinant is $\prod_{a=1}^{p} X_a\cdot X_a$, and Corollary 2.2 shows that this term does not fully cancel when the determinant is expanded out; therefore the determinant is not 0.

It is consequently enough to prove the statements about the action by $\mathfrak{gl}(m,\mathbb{C})$. For that purpose, we argue as in the proof of Proposition 1.3, using (2.2) and (2.3) to handle the terms $E(A_j)$ in Lemma 1.1. □

Proposition 3.2. *If* $\mu=\sum_{a=1}^{m}\mu_a e_a$ *is nonnegative dominant integral for* $\mathfrak{gl}(m,\mathbb{C})$ *and is even, then the representation* $\tau_\mu\,\widehat{\otimes}\,1$ *occurs with multiplicity one in* S_{mn}*, and the highest weight vectors for it are the multiples of*

$$\prod_{p=1}^{m}\left(\det\Bigl(\{X_a\cdot X_b\}_{\substack{1\leqslant a\leqslant p\\1\leqslant b\leqslant p}}\Bigr)^{(\mu_p-\mu_{p+1})/2}\right)$$

if μ_{m+1} *is interpreted as* 0.

Proof. We apply Proposition 3.1 to each determinant factor. The product of highest weight vectors in S_{mn} is a highest weight vector, and the weights add. Therefore the indicated product is a highest weight vector, and the representation that it generates is of type $\tau_\mu\,\widehat{\otimes}\,1$. For the limitation on multiplicities, it is not hard to give a direct proof by means of a theorem of Cartan and Helgason in [He, Section III.3] (or apply [Kn1, Theorem 9.70] to the compact symmetric space $U(n)/SO(n)$), but it is easier to quote Littlewood's theorem. Here (λ,ν) is $(\mu,0)$, and the relevant sum of Littlewood–Richardson coefficients reduces to $c^{\mu}_{\mu 0}=1$. □

For $1\leqslant p\leqslant m$, let us use the notation

$$[a_1,\dots,a_p;b_1,\dots,b_p] \tag{3.1}$$

to denote the determinant of the p-by-p matrix whose (i,j)th entry is $X_{a_i}\cdot X_{b_j}$. These determinants are members of S_{mn} invariant under the action of $SO(n)$ on the right, and we need to know the effect of the $U(m)$ action on the left. Such a determinant is of course 0 if the list $a_1,\dots,a_p$ contains any repetitions or if the list $b_1,\dots,b_p$ contains any repetitions. The determinant is sent into its negative if two of the a's are interchanged or if two of the b's are interchanged. It is unchanged if the a's are exchanged with the b's.

Proposition 3.3. *If* $a_1,\dots,a_p$ *are distinct and* $b_1,\dots,b_p$ *are distinct, then the member* E_{uv} *of* $\mathfrak{gl}(m,\mathbb{C})$ *acts by*

$$\begin{aligned}E_{uv}\bigl([a_1,\dots,a_p;b_1,\dots,b_p]\bigr)&=[a_1,\dots,a_{i-1},u,a_{i+1},\dots,a_p;b_1,\dots,b_p]\\&\quad+[a_1,\dots,a_p;b_1,\dots,b_{j-1},u,b_{j+1},\dots,b_p]\end{aligned}$$

if $a_i = v$ *and* $b_j = v$. *If no* b_j *equals* v, *the second term on the right is absent. If no* a_i *equals* v, *the first term on the right is absent. If no* $a_i = v$ *and no* b_j *equals* v, *the right side is* 0.

Proof. Suppose that $a_1, \ldots, a_p$ are distinct and that $b_1, \ldots, b_p$ are distinct. Suppose also that $a_i = v$ and $b_j = v$. Let A be the matrix whose determinant is $[a_1, \ldots, a_p; b_1, \ldots, b_p]$, and let $\widehat{A}_{st}$ be the matrix A with the sth row and tth column deleted. Then Lemmas 1.1 and 1.2, in combination with (2.2), give

$$\begin{aligned}
E_{uv}(\det A) &= \sum_{s,t} (-1)^{s+t} \big(\det \widehat{A}_{st}\big) E_{uv}(X_{a_s} \cdot X_{b_t}) \\
&= \sum_{s,t} (-1)^{s+t} \big(\det \widehat{A}_{st}\big) E_{uv}(X_{a_s}) \cdot X_{b_t} + \sum_{s,t} (-1)^{s+t} \big(\det \widehat{A}_{st}\big) X_{a_s} \cdot E_{uv}(X_{b_t}) \\
&= \delta_{a_i v} \sum_{t} (-1)^{i+t} \big(\det \widehat{A}_{it}\big) X_u \cdot X_{b_t} + \delta_{v b_j} \sum_{s} (-1)^{s+j} \big(\det \widehat{A}_{sj}\big) X_{a_s} \cdot X_u \\
&= \delta_{a_i v}[a_1, \ldots, a_{i-1}, u, a_{i+1}, \ldots, a_p; b_1, \ldots, b_p] \\
&\quad + \delta_{v b_j}[a_1, \ldots, a_p; b_1, \ldots, b_{j-1}, u, b_{j+1}, \ldots, b_p].
\end{aligned}$$

The other cases are handled similarly. □

Proposition 3.4. *If* d *is odd, the space* $(S^d_{mn})^{SO}$ *of* $SO(n)$ *invariants homogeneous of degree* d *is* 0. *If* d *is even, the following three spaces coincide*:

(a) *the space* $(S^d_{mn})^{SO}$,
(b) *the linear span of the* $P(D)$ *with* $\|D\| = d/2$,
(c) *the direct sum of the spaces* $V^{\mu,0}$ *for even* μ *nonnegative dominant integral with* $\|\mu\| = d$, *these spaces being of multiplicity one.*

Remark. Cf. [GoW, Section 4.2].

Proof. The highest weight vectors of each space $V^{\mu,0}$ in (c) are in the space (b) by Proposition 3.2, and it follows from Proposition 3.3 that each whole space $V^{\mu,0}$ is in the space (b). It is clear that the space (b) is contained in the space (a).

For any d, odd or even, $(S^d_{mn})^{SO}$ is the direct sum of irreducible subspaces under $U(m) \times SO(n)$, and the highest weights have to be of the form $(\mu, 0)$ for some nonnegative dominant integral form μ for $U(m)$. The theorem of Cartan and Helgason mentioned in the proof of Proposition 3.2 shows that μ has to be even and the multiplicity is at most 1. This completes the proof. □

Corollary 3.5. *The* $P(D)$ *form a basis of* $(S_{mn})^{SO}$.

Proof. The linear independence is given by Corollary 2.2. The spanning is immediate from the equivalence of (a) and (c) in Proposition 3.4 since S_{mn} is the direct sum of the S^d_{mn}. □

For $1 \leqslant p \leqslant m$, let $V_p = V^{\mu,0}$ for μ equal to $2e_1 + \cdots + 2e_p$, and let U_p be the linear span of all p-by-p minors of $\det(\{X_a \cdot X_b\}_{1\leqslant a\leqslant m,\,1\leqslant b\leqslant m})$. Proposition 3.1 shows that the highest weight vector of V_p lies in U_p, and Proposition 3.3 shows that U_p is an invariant subspace. Consequently $V_p \subseteq U_p$.

The reverse inclusion is more subtle. A look at Proposition 3.3 shows that application of members of $\mathfrak{gl}(m,\mathbb{C})$ to the highest weight vector of V_p in Proposition 3.1 quickly results in sums of minors rather than individual minors, and there is no apparent way of obtaining the individual minors. One may then suspect that the inclusion of V_p in U_p is sometimes proper. The following example indicates complications: let us say that two nonzero p-by-p minors as in (3.1) are "really different" if the one cannot be converted into the other by permuting the a indices, permuting the b indices, and possibly exchanging the a indices with the b indices. The number of really different 2-by-2 minors can be seen to be $\binom{m}{2}\bigl(\binom{m}{2}+1\bigr)/2$, which is 21 in the case that $m = 4$. On the other hand, the dimension of V_2 for $m = 4$ is 20, by the Weyl Dimension Formula. If, under the influence of Corollary 2.2, one expects that the really different minors are linearly independent, one is led to expect that the inclusion of V_2 in U_2 for $m = 4$ is indeed proper. However, the really different minors are not linearly independent, as the identity

$$\det\begin{pmatrix} X_1\cdot X_3 & X_1\cdot X_4 \\ X_2\cdot X_3 & X_2\cdot X_4 \end{pmatrix} + \det\begin{pmatrix} X_1\cdot X_2 & X_1\cdot X_3 \\ X_4\cdot X_2 & X_4\cdot X_3 \end{pmatrix} = \det\begin{pmatrix} X_1\cdot X_2 & X_1\cdot X_4 \\ X_3\cdot X_2 & X_3\cdot X_4 \end{pmatrix}$$

shows.

Corollary 3.6. *For $1 \leqslant p \leqslant m$, the space V_p equals the linear span of all p-by-p minors of* $\det(\{X_a \cdot X_b\}_{1\leqslant a\leqslant m,\,1\leqslant b\leqslant m})$.

Proof. Let U_p be the linear span of the p-by-p minors. Arguing by contradiction, suppose that V_p is a proper subspace of U_p. Then an invariant complement of V_p in U_p is a nonzero invariant subspace of the space (b) of Proposition 3.4 with $d = 2p$. By (c) in the proposition, U_p contains some $V^{\mu,0}$ for an even nonnegative dominant integral $\mu = \sum_{a=1}^{m} \mu_a e_a$ with $\|\mu\| = 2p$ but with $\mu \neq 2e_1 + \cdots + 2e_p$. This μ must have $\mu_p = 0$ and therefore $\mu_1 \geqslant 4$. The space U_p must contain a nonzero highest weight vector for $V^{\mu,0}$, and this has to be of the form given in Proposition 3.2. Expanding out this expression as a linear combination of $P(D)$'s, we obtain coefficient 1 for

$$P(D) = (X_1\cdot X_1)^{\mu_1/2}(X_2\cdot X_2)^{\mu_2/2}\cdots(X_p\cdot X_p)^{\mu_p/2}.$$

On the other hand, no p-by-p minor, when expanded as a linear combination of $P(D')$'s, can contain $X_1 \cdot X_1$ to a power greater than 1. By Corollary 2.2 the highest weight vector in question cannot be in U_p, and we have a contradiction. □

4. Proof that $\mathcal{M}$ is one–one

Now we are ready to start the proof of Theorem 0.1. In this section we shall prove that $\mathcal{M}$ is one–one, and in the next section we shall prove a dimensional equality that implies the map is onto $V^{\lambda,\nu}$.

Theorem 4.1. *Any nonzero highest weight vector of type (λ,ν) in the tensor product $V^{\nu,\nu}\otimes\bigoplus_\mu V^{\mu,0}$ is of the form*

$$\phi_{\lambda,\nu}=Z(\nu)\otimes\sum_{\substack{D\text{ with}\\ \|D\|=\frac{1}{2}(\|\lambda\|-\|\nu\|)}}a_D P(D)+\sum_\gamma\Bigg(Z_\gamma(\nu)\otimes\sum_{\substack{D\text{ with}\\ \|D\|=\frac{1}{2}(\|\lambda\|-\|\nu\|)}}b_{\gamma,D}P(D)\Bigg),$$

where $Z(\nu)$ is the highest weight vector of $V^{\nu,\nu}$ given in Proposition 1.4, $\sum a_D P(D)$ is not 0, γ ranges over nonzero sums of positive roots of $\mathfrak{gl}(m,\mathbb{C})$ (the sums possibly being repeated), and each $Z_\gamma(\nu)$ is a member of $V^{\nu,\nu}$ of weight $\nu-\gamma$. If $\phi_{\lambda,\nu}$ lies in $V^{\nu,\nu}\otimes\bigoplus_{\mu\in F}V^{\mu,0}$ for a subset F of μ's, then $\sum_D a_D P(D)$ lies in $\bigoplus_{\mu\in F}V^{\mu,0}$. Moreover, if φ is the homomorphism defined in (2.6) and if $Q(D)$ is defined as $\varphi(P(D))$, then

$$\varphi\big(\mathcal{M}(\phi_{\lambda,\nu})\big)=\Bigg(\prod_{p=1}^m (Z_{pp})^{\nu_p}\Bigg)\Bigg(\sum_D a_D Q(D)\Bigg),$$

where $\mathcal{M}$ denotes multiplication.

Proof. By abstract character theory the multiplicity of $\tau_\lambda\,\widehat{\otimes}\,\sigma_\nu$ in the tensor product $V^{\nu,\nu}\otimes\bigoplus_\mu V^{\mu,0}$ is the sum over μ of the multiplicity in each $V^{\nu,\nu}\otimes V^{\mu,0}$. Thus we may compute the highest weight vectors of type (λ,ν) by computing them within each $V^{\nu,\nu}\otimes V^{\mu,0}$ and then taking sums.

Fix μ. It is known for any compact connected Lie group that any nonzero highest weight vector of type ω in a tensor product of the form $\tau_\xi\otimes\tau_\eta$ is of the form

$$v_\xi\otimes v_{\omega-\xi}+\sum_\gamma v_{\xi-\gamma}\otimes v_{\omega-\xi+\gamma},$$

where each vector has the indicated weight in the space for τ_ξ or τ_η as appropriate, $v_\xi\otimes v_{\omega-\xi}$ is not 0, and the γ's are nonzero sums of positive roots, the sums possibly repeated. (See [Kn1, second edition, Proposition 9.72 and its proof].[1]) If we fix μ, we can apply this fact to the group $U(m)\times SO(n)$ with $\omega=(\lambda,\nu)$, $\xi=(\nu,\nu)$, and $\eta=(\mu,0)$. Since Proposition 1.4 shows $Z(\nu)$ to be a nonzero highest weight vector of $V^{\nu,\nu}$ and since Proposition 3.4 shows every element of $V^{\mu,0}$ to be a linear combination of the $P(D)$, the formula follows for $\phi_{\lambda,\nu}$ in the case of a single μ, provided we interpret γ as a

[1] In the first edition, see instead Problem 16, p. 285, and its solution on p. 554.

nonzero sum of positive roots for $\mathfrak{gl}(m,\mathbb{C}) \oplus \mathfrak{so}(n,\mathbb{C})$. We need to see that the positive roots of $\mathfrak{so}(n,\mathbb{C})$ are not involved. For fixed γ, the vector $\sum_D b_{\gamma,D} P(D)$ is a weight vector of $\tau_{\mu,0} \widehat{\otimes} 1$, and its weight is therefore of the form $(\mu', 0)$. The weight of the term $Z_\gamma(\nu) \otimes \sum_D b_{\gamma,D} P(D)$ is then $(\nu,\nu) - \gamma + (\mu',0)$, and this must match (λ,ν). Therefore γ has $\mathfrak{so}(n,\mathbb{C})$ component 0 and is a sum of positive roots from $\mathfrak{gl}(m,\mathbb{C})$ alone. We shall write $\gamma = \sum_{a=1}^m \gamma_a e_a$; at least one component γ_a of γ is nonzero, and the first nonzero component is positive.

Taking sums in the case that $\phi_{\lambda,\nu}$ lies in $V^{\nu,\nu} \otimes \bigoplus_{\mu\in F} V^{\mu,0}$ for a subset F of μ's, we see that $\sum_D a_D P(D)$ lies in $\bigoplus_{\mu\in F} V^{\mu,0}$. We still have to see that $\sum_D a_D P(D)$ is not 0. If the first term in the expansion for each single μ is $Z(\nu) \otimes \sum_D a_{D,\mu} P(D)$, then the first term in the expansion for all μ can be written $Z(\nu) \otimes \sum_D a_D P(D)$ with $a_D = \sum_\mu a_{D,\mu}$. Here the various terms $\sum_D a_{D,\mu} P(D)$ are nonzero members of their respective spaces $V^{\mu,0}$, and their sum cannot be 0 since the spaces $V^{\mu,0}$ are independent.

To complete the proof, we have to verify the formula for $\varphi(\mathcal{M}(\phi_{\lambda,\nu}))$. Application of $\mathcal{M}$ to $\phi_{\lambda,\nu}$ is accomplished by replacing the tensor-product signs by multiplication signs. We then apply the homomorphism φ of (2.6) to the result and use (2.8) to obtain

$$\varphi\bigl(\mathcal{M}(\phi_{\lambda,\nu})\bigr) = \prod_{p=1}^m (Z_{pp})^{\nu_p} \sum_D a_D Q(D) + \sum_\gamma \biggl(\varphi\bigl(Z_\gamma(\nu)\bigr) \sum_D b_{\gamma,D} Q(D)\biggr).$$

We shall show that $\varphi(Z_\gamma(\nu)) = 0$ for every γ, and then our expression reduces to

$$\varphi\bigl(\mathcal{M}(\phi_{\lambda,\nu})\bigr) = \prod_{p=1}^m (Z_{pp})^{\nu_p} \sum_D a_D Q(D),$$

as required.

To begin the proof that $\varphi(Z_\gamma(\nu)) = 0$ for every γ, we examine properties of monomials in the Z_{ab}'s. By (1.1) and (1.6) the weight of $\prod_{a,b} Z_{ab}^{p_{a,b}}$ is $\sum_{a,b} p_{ab}(e_a + e'_b)$. Meanwhile from (1.2) it follows that every root vector for $\mathfrak{gl}(m,\mathbb{C})$ carries monomials in the Z_{ab}'s to multiples of monomials in the Z_{ab}'s.

The expression $Z(\nu)$ is a linear combination of monomials $\prod_{a,b} Z_{ab}^{p_{a,b}}$ of weight $\sum_a \nu_a(e_a + e'_a)$. Since $Z_\gamma(\nu)$ can be obtained $Z(\nu)$ by applying a linear combination of products of root vectors of negative roots for $\mathfrak{gl}(m,\mathbb{C})$, we may assume that $Z_\gamma(\nu)$ is a linear combination of monomials in the Z_{ab}'s of weight $\sum_a (\nu_a - \gamma_a) e_a + \sum_a \nu_a e'_a$. To prove that $\varphi(Z_\gamma(\nu)) = 0$, it is enough to prove that each monomial $\prod_{a,b} Z_{ab}^{p_{a,b}}$ of this weight is annihilated by φ.

Assume the contrary. By (2.7) each $p_{a,b}$ must be 0 when $a > b$. Thus we have a monomial of the form $\prod_{a\leqslant b} Z_{ab}^{p_{a,b}}$ with

$$\sum_{a\leqslant b} p_{ab}\bigl(e_a + e'_b\bigr) = \sum_a (\nu_a - \gamma_a) e_a + \sum_a \nu_a e'_a. \tag{4.1}$$

Here some γ_a is nonzero, and the first nonzero one is positive. Rewriting (4.1), we obtain

$$\sum_a \Biggl(-\nu_a + \gamma_a + \sum_{b\geqslant a} p_{ab}\Biggr) e_a + \sum_a \Biggl(-\nu_a + \sum_{b\leqslant a} p_{ba}\Biggr) e'_a = 0.$$

Each coefficient of e_a or e'_a must be 0, and thus

$$-\nu_a + \gamma_a + \sum_{b\geqslant a} p_{ab} = 0, \qquad -\nu_a + \sum_{b\leqslant a} p_{ba} = 0$$

for all a. Subtracting these equations yields

$$\gamma_a + \sum_{b\geqslant a} p_{ab} = \sum_{b\leqslant a} p_{ba} \quad \text{for all } a.$$

The diagonal terms in the sums on the two sides cancel, and the result is

$$\gamma_a + \sum_{b>a} p_{ab} = \sum_{b<a} p_{ba} \quad \text{for all } a. \tag{4.2}$$

Let us prove by induction on $0 \leqslant i \leqslant m$ that

$$\gamma_1 = \cdots = \gamma_i = 0 \quad \text{if } i > 0, \quad \text{and} \quad p_{ab} = 0 \quad \text{whenever } 1 \leqslant a \leqslant i \text{ and } a < b. \tag{4.3}$$

The base case of the induction is $i = 0$, and there is nothing to prove. Assume (4.3) for $i - 1$ with $i > 0$. We use (4.2) with $a = i$. Since $\gamma_1 = \cdots = \gamma_{i-1} = 0$, we must have $\gamma_i \geqslant 0$. Thus (4.2) shows that

$$\sum_{b>i} p_{ib} \leqslant \sum_{b<i} p_{bi} \tag{4.4}$$

with equality only if $\gamma_i = 0$. Each term on the right side of (4.4) is 0 by inductive assumption (4.3) (in which i is to be replaced by $i - 1$). Since each p_{ib} is $\geqslant 0$, we conclude from (4.4) that each p_{ib} is 0 for $b > i$ and that equality holds in (4.4). Since equality holds, $\gamma_i = 0$; this proves the first half of (4.3) at the inductive step. The new assertion at the inductive step in the second half of (4.3) is that $p_{ab} = 0$ for $a = i$ when $a < b$, and we have just proved that as well. This completes the induction.

Since (4.3) is now known to hold for $i = m$, we see that $\gamma = 0$. But this contradicts the assumption throughout that γ is a nonzero sum of positive roots. The conclusion is that $\varphi(Z_\gamma(\nu)) = 0$, and therefore the proof of the theorem is complete. □

Corollary 4.2. *The mapping* $\mathcal{M}$ *of Theorem* 0.1 *is one–one.*

Proof. Since $\mathcal{M}$ is equivariant for the action of $U(m) \times SO(n)$, it is enough to check that $\mathcal{M}$ is one–one on the highest weight vectors of type (λ, ν) in $V^{\nu,\nu} \otimes \bigoplus_\mu V^{\mu,0}$. A nonzero such vector is of the form $\phi_{\lambda,\nu}$ in Theorem 4.1. Arguing by contradiction, suppose that $\mathcal{M}(\phi_{\lambda,\nu}) = 0$. Following $\mathcal{M}$ with the homomorphism φ of (2.6) yields

$$0 = \varphi\big(\mathcal{M}(\phi_{\lambda,\nu})\big) = \left(\prod_{p=1}^{m} (Z_{pp})^{\nu_p}\right)\left(\sum_D a_D\, Q(D)\right),$$

by Theorem 4.1. Since $\prod_{p=1}^{m} (Z_{pp})^{\nu_p}$ is manifestly not zero and since $\sum_D a_D P(D)$ nonzero implies $\sum_D a_D Q(D)$ nonzero by Theorem 2.1, we obtain 0 as a product of nonzero elements of S_{mn}, and we have obtained a contradiction. □

5. Dimensional equality

The final step in the proof of Theorem 0.1 is to show that the domain and range have the same dimension. Since Corollary 4.2 has shown $\mathcal{M}$ to be one–one, it then follows that $\mathcal{M}$ is onto, and the proof is complete. Actually the dimensional equality, which is given as Proposition 5.1, makes use of Corollary 4.2 and therefore does not stand on its own.

Proposition 5.1. *Let λ and ν be nonnegative dominant integral forms with* depth $\nu \leqslant m \leqslant n/2$ *and* depth $\lambda \leqslant m \leqslant n/2$. *Then*

$$\dim\left(V^{\nu,\nu} \otimes \bigoplus_{\substack{\mu \text{ dominant integral} \\ \mu \text{ even nonnegative} \\ \operatorname{depth}\mu \leqslant \operatorname{depth}\lambda \\ \|\mu\| = \|\lambda\| - \|\nu\|}} V^{\mu,0}\right)^{\lambda} = \dim V^{\lambda,\nu}.$$

Proof. Fix ν and let $L = \|\lambda\|$ for such a λ. Since $\mathcal{M}$ is known from Corollary 4.2 to be one–one, we have

$$\dim\left(V^{\nu,\nu} \otimes \bigoplus_{\mu} V^{\mu,0}\right)^{\lambda} \leqslant \dim V^{\lambda,\nu} \tag{5.1}$$

with μ restricted as in the statement of the proposition.

We make the following computation, in which ν is fixed and (λ, μ) is understood to range over all ordered pairs of nonnegative dominant integral forms of depth $\leqslant m$ such that $\|\lambda\| = L$, $\|\mu\| + \|\nu\| = L$, and μ is even. We shall justify the steps after the computation is complete:

$$\sum_{\lambda} \dim\left(V^{\nu,\nu} \otimes \bigoplus_{\mu} V^{\mu,0}\right)^{\lambda} = \sum_{\mu} (\dim V^{\nu,\nu})\big(\dim V^{\mu,0}\big) \tag{5.2a}$$

$$= \sum_{\mu} \big(\dim \tau_{\nu}^{m} \,\widehat{\otimes}\, \sigma_{\nu}\big)\big(\dim \tau_{\mu}^{m}\big) \tag{5.2b}$$

$$= \sum_{\mu} \bigl(\dim \tau_\mu^m \otimes \tau_\nu^m\bigr)(\dim \sigma_\nu) \tag{5.2c}$$

$$= \sum_{\mu,\lambda} \bigl[\tau_\mu^m \otimes \tau_\nu^m : \tau_\lambda^m\bigr]_{U(m)} \bigl(\dim \tau_\lambda^m\bigr)(\dim \sigma_\nu) \tag{5.2d}$$

$$= \sum_{\lambda} \bigl(\dim \tau_\lambda^m \mathbin{\widehat{\otimes}} \sigma_\nu\bigr) \sum_{\mu} c_{\mu\nu}^{\lambda} \tag{5.2e}$$

$$= \sum_{\lambda} \bigl(\dim \tau_\lambda^m \mathbin{\widehat{\otimes}} \sigma_\nu\bigr) \bigl[\tau_\lambda^n|_{SO(n)} : \sigma_\nu\bigr]_{SO(n)} \tag{5.2f}$$

$$= \sum_{\lambda} \bigl(\dim \tau_\lambda^m \mathbin{\widehat{\otimes}} \sigma_\nu\bigr) \bigl[S_{mn} : \tau_\lambda^m \mathbin{\widehat{\otimes}} \sigma_\nu\bigr]_{U(m)\times SO(n)} \tag{5.2g}$$

$$= \sum_{\lambda} \dim V^{\lambda,\nu}. \tag{5.2h}$$

Step (5.2a) represents Fourier analysis in the λ variable; the contribution of the λ term to the left side is automatically 0 unless λ is nonnegative with $\|\lambda\| = L$. Step (b) substitutes the irreducible representations that are involved, taking into account the multiplicity-one results given in Propositions 1.4 and 3.2. Step (c) is just a regrouping. Step (d) represents Fourier analysis in the λ variable again, and again there is no contribution from the λ term unless λ is nonnegative with $\|\lambda\| = L$. Step (e) substitutes the definition of the Littlewood–Richardson coefficient $c_{\mu\nu}^{\lambda}$, and step (f) follows by application of Littlewood's theorem. Step (g) is an application of (0.2), and step (h) uses the fact that the dimension of an isotypic subspace is the product of the multiplicity and the dimension of the relevant irreducible representation.

Comparing the sum of (5.1) over λ with the result of (5.2), we see that equality must hold in (5.1). This completes the proof of Proposition 5.1. □

6. Littlewood's other branching theorem

The other Littlewood branching theorem concerns restriction from the unitary group $U(2r)$ to the group $Sp(r)$ of r-by-r unitary matrices over the quaternions. The latter group is to be realized as the intersection of $U(2r)$ with

$$Sp(r,\mathbb{C}) = \bigl\{ g \in SL(2r,\mathbb{C}) \bigm| g^{\mathrm{tr}} J g = J \bigr\},$$

J being the $2r$-by-$2r$ matrix given in block form as $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The statement is formally rather similar to Littlewood's theorem concerning branching from $U(n)$ to $SO(n)$ except that the evenness condition on the μ's is changed. A dominant integral form μ will be said to have *paired entries* if $\mu_1 = \mu_2$, $\mu_3 = \mu_4$, etc. For the statement in the case of $U(2r)$ and $Sp(r)$, let λ and ν be nonnegative dominant integral forms of depth $\leqslant r$, let σ_ν be an irreducible representation of $Sp(r)$ with highest weight ν, and again let τ_λ be an irreducible representation of $U(2r)$ with highest weight λ. The theorem is that the multiplicity of σ_ν in

$\tau_\lambda|_{Sp(r)}$ equals the sum of the Littlewood–Richardson coefficients $c^\lambda_{\mu\nu}$ over all nonnegative dominant integral μ of depth $\leqslant r$ such that $\|\mu\| = \|\lambda\| - \|\nu\|$ and μ has paired entries.

Let $n = 2r$, and let m be a positive integer with $m \leqslant r$. The geometric setting again concerns $M_{mn} = M_{mn}(\mathbb{C})$ and its symmetric algebra S_{mn}. The groups $U(m)$ and $Sp(r)$ act on M_{mn} with $U(m)$ acting by left multiplication and $Sp(r)$ acting by right multiplication by inverse elements. These actions extend to S_{mn}. Suppose that both the above linear forms λ and ν have depth $\leqslant m$, which is $\leqslant r$. By the same kind of argument as in (0.2), we have

$$\left[S_{mn} : \tau^m_\lambda \mathbin{\widehat{\otimes}} \sigma_\nu\right]_{U(m)\times Sp(r)} = \left[\tau^n_\lambda : \sigma_\nu\right]_{Sp(r)}. \tag{6.1}$$

Let $\mathcal{M}$ be the multiplication mapping from $S_{mn} \otimes S_{mn}$ to S_{mn}. With λ and ν as above, suppose that μ is nonnegative dominant integral, has depth $\leqslant r$, and has paired entries. We define $V^{\lambda,\nu}$, $V^{\nu,\nu}$, and $V^{\mu,0}$ as invariant subspaces of S_{mn} in analogous fashion to the rotation case: the first superscript in each case refers to the transformation law under $U(m)$, and the second superscript refers to the transformation law under $Sp(r)$.

Theorem 6.1. *Let λ and ν be nonnegative dominant integral forms with* depth $\nu \leqslant m \leqslant n/2$ *and* depth $\lambda \leqslant m \leqslant n/2$. *Then the multiplication map*

$$\mathcal{M} : \Bigg(V^{\nu,\nu} \otimes \bigoplus_{\substack{\mu \text{ dominant integral} \\ \mu \text{ nonnegative} \\ \mu \text{ with paired entries} \\ \|\mu\| = \|\lambda\| - \|\nu\|}} V^{\mu,0}\Bigg)^{\lambda} \to V^{\lambda,\nu}$$

within the algebra S_{mn} of symmetric tensors is one–one and onto.

The rest of this section is devoted to a sketch of how Sections 1–5 need to be adjusted to prove Theorem 6.1. There are only a few serious adjustments.

Throughout this section we assume that $m \leqslant r = n/2$. The complexified Lie algebra of $Sp(r)$ is $\mathfrak{sp}(r, \mathbb{C}) = \{x \in \mathfrak{sl}(n, \mathbb{C}) \mid x^{\mathrm{tr}} J + Jx = 0\}$. We index the rows of $X \in M_{mn}$ by $\{1, \dots, m\}$ and the columns by $\{1, \dots, r, 1', \dots, r'\}$.

The action of $\mathfrak{gl}(m, \mathbb{C})$ on M_{mn} is unchanged from Section 1. Thus the weight spaces are still given by (1.1), and the root vectors act on M_{mn} as in (1.2).

We take as Cartan subalgebra for $\mathfrak{sp}(r, \mathbb{C})$ the set of all diagonal matrices given in block form by $\left(\begin{smallmatrix} H & -0 \\ 0 & -H \end{smallmatrix}\right)$. Evaluation of the pth diagonal entry of H is denoted e'_p, $1 \leqslant p \leqslant r$. The roots for $\mathfrak{sp}(r, \mathbb{C})$ are all $\pm e'_a \pm e'_b$ with $a \neq b$ and all $\pm 2e_a$; here a and b range from 1 to r. We take all $e'_a \pm e'_b$ with $a < b$ and all $2e'_a$ as the positive roots. Taking into account the minus sign built into the action on the right of M_{mn}, we can compute the effect of $E'_{aa} - E'_{a'a'}$ on each X_{bc} and $X_{bc'}$ when a and c extend from 1 to r and b extends from 1 to m. We find that

$$X_{bc} \text{ has weight } -e'_c, \qquad X_{bc'} \text{ has weight } +e'_c. \tag{6.2}$$

Formulas for root vectors may be found in [Kn1, Example 3 of Section II.1]. A little computation then gives

$$\begin{aligned} &E'_{e'_a-e'_b}(X_{cd'}) = \delta_{bd} X_{ca'} && \text{if } a \neq b,\\ &E'_{e'_a+e'_b}(X_{cd'}) = 0 && \text{if } a \neq b,\\ &E'_{2e'_a}(X_{cd'}) = 0 && \text{for all } a. \end{aligned} \tag{6.3}$$

The same arguments as for Propositions 1.3 and 1.4 establish two propositions giving highest weight vectors for $V^{\nu,\nu}$:

Proposition 6.2. *If $1 \leqslant p \leqslant m$, then the element*

$$\det\Big(\{X_{ab'}\}_{\substack{1\leqslant a\leqslant p\\ 1\leqslant b\leqslant p}}\Big)$$

of S_{mn} has weight $(e_1 + \cdots + e_p, e'_1 + \cdots + e'_p)$ under $\mathfrak{gl}(m,\mathbb{C}) \oplus \mathfrak{sp}(r,\mathbb{C})$ and is a nonzero highest weight vector.

Proposition 6.3. *If $\nu = \sum_{a=1}^m \nu_a e_a$ is nonnegative dominant integral for $\mathfrak{gl}(m,\mathbb{C})$ and is reinterpreted as a highest weight $\nu = \sum_{a=1}^m \nu_a e'_a$ for $\mathfrak{sp}(r,\mathbb{C})$, then the representation $\tau_\nu^m \mathbin{\widehat{\otimes}} \sigma_\nu$ occurs with multiplicity one in S_{mn}, and the highest weight vectors for it are the multiples of*

$$X(\nu) = \prod_{p=1}^m \det\Big(\{X_{ab'}\}_{\substack{1\leqslant a\leqslant p\\ 1\leqslant b\leqslant p}}\Big)^{\nu_p-\nu_{p+1}}$$

if ν_{m+1} is interpreted as 0.

Dealing with $V^{\mu,0}$ involves some changes. With rows X_a of X defined as in Section 2, define the *alternating product* of two rows to be the element in S^2_{mn} given by

$$\langle X_a, X_b\rangle = X_a J X_b^{\mathrm{tr}} = \sum_{c=1}^r X_{ac} X_{bc'} - \sum_{c=1}^r X_{ac'} X_{bc}.$$

The alternating product is skew symmetric. Elements of $\mathfrak{gl}(m,\mathbb{C})$ act on the left by

$$E_{pq}\big(\langle X_a, X_b\rangle\big) = \delta_{qa}\langle X_p, X_b\rangle + \delta_{qb}\langle X_a, X_p\rangle. \tag{6.4}$$

In particular,

$$\langle X_a, X_b\rangle \text{ is a weight vector under } \mathfrak{gl}(m,\mathbb{C}) \text{ of weight } e_a + e_b. \tag{6.5}$$

Since $\langle X_a, X_a\rangle = 0$, the number of basic alternating products is $m(m-1)/2$ rather than $m(m+1)/2$. Let $D = \{D_{ab}\}_{a<b}$ be a tuple of $m(m-1)/2$ nonnegative integers and put $\|D\| = \sum_{a<b} D_{ab}$. Define a monomial of alternating products to be

$$R(D) = \prod_{a<b} \langle X_a, X_b\rangle^{D_{ab}}. \tag{6.6}$$

This lies in S^d with $d = 2\|D\|$. In place of φ, we define a linear mapping $\psi : M_{mn} \to M_{mn}$ by

$$\psi(X_{ab'}) = \begin{cases} X_{ab'} & \text{if } a \leqslant b, \\ 0 & \text{if } a > b, \end{cases} \qquad \psi(X_{ab}) = \begin{cases} X_{ab} & \text{if } a = b, \\ 0 & \text{if } a \neq b. \end{cases} \tag{6.7}$$

This is to be extended to an endomorphism of S_{mn} sending 1 to 1. Just as with (2.8), we work out that

$$\psi\big(X(\nu)\big) = \prod_{p=1}^{m} (X_{pp'})^{\nu_p}. \tag{6.8}$$

As an endomorphism of S_{mn}, ψ takes a surprisingly simple form on alternating products, namely

$$\psi\big(\langle X_a, X_b\rangle\big) = -X_{ab'}X_{bb} \quad \text{if } a < b.$$

We obtain

$$\psi\big(R(D)\big) = (-1)^{\|D\|} \prod_{a<b} X_{ab'}^{D_{ab}} X_{bb}^{D_{ab}},$$

and the analog of Theorem 2.1 now follows by inspection:

Theorem 6.4. *The members $\psi(R(D))$ of S_{mn} are linearly independent.*

Remark. As a consequence the members $R(D)$ of S_{mn} are linearly independent. This consequence was already known; see [DeP, Section 6].

The tools are all in place to prove an analog of Proposition 3.1.

Proposition 6.5. *If $1 \leqslant 2p \leqslant m$, then the element*

$$\det\Big(\big\{\langle X_a, X_b\rangle\big\}_{\substack{1\leqslant a\leqslant 2p \\ 1\leqslant b\leqslant 2p}}\Big)$$

of S_{mn} has weight $(2e_1 + \cdots + 2e_{2p}, 0)$ under $\mathfrak{gl}(m, \mathbb{C}) \oplus \mathfrak{sp}(r, \mathbb{C})$ and is a nonzero highest weight vector.

This is not a good enough result for handling the μ's with paired entries because Littlewood's theorem says that $(e_1 + \cdots + e_{2p}, 0)$ should be a highest weight. Taking a cue from linear algebra, we define a kind of size-$2p$ Pfaffian $\operatorname{Pff}(X_{a_1}, \ldots, X_{a_{2p}})$ to be

$$= \sum_{\pi \in \mathfrak{S}_{2p}} (\operatorname{sgn} \pi)\big\langle X_{a_{\pi(1)}}, X_{a_{\pi(2)}}\big\rangle\big\langle X_{a_{\pi(3)}}, X_{a_{\pi(4)}}\big\rangle \cdots \big\langle X_{a_{\pi(2p-1)}}, X_{a_{\pi(2p)}}\big\rangle,$$

where $\mathfrak{S}_{2p}$ is the symmetric group on $\{1, \ldots, 2p\}$. This is not 0 since the $R(D)$'s are linearly independent, and it has the desired weight $(e_1 + \cdots + e_{2p}, 0)$. A simple change of variables in the sum over $\mathfrak{S}_{2p}$ shows that Pff has the key property

$$\operatorname{Pff}\bigl(X_{a_{\omega(1)}}, \ldots, X_{a_{\omega(2p)}}\bigr) = (\operatorname{sgn}\omega)\operatorname{Pff}\bigl(X_{a_1}, \ldots, X_{a_{2p}}\bigr)$$

for any member ω of $\mathfrak{S}_{2p}$, and it follows that

$$\operatorname{Pff} \text{ is } 0 \quad \text{if two of its arguments are equal.} \tag{6.9}$$

In the reverse direction the second conclusion of Theorem 6.4 shows that

$$\operatorname{Pff} \text{ is nonzero} \quad \text{if its arguments are distinct.} \tag{6.10}$$

The analog of Proposition 3.3 is

Proposition 6.6. *The root vector E_{ca_j} acts on Pfaffians by*

$$E_{ca_j}\bigl(\operatorname{Pff}\bigl(X_{a_1}, \ldots, X_{a_{2p}}\bigr)\bigr) = \operatorname{Pff}\bigl(X_{a_1}, \ldots, X_{a_{j-1}}, X_c, X_{a_{j+1}}, \ldots, X_{a_{2p}}\bigr).$$

This result is an improvement over Proposition 3.3 because a Pfaffian leads to another Pfaffian, while in Proposition 3.3 a determinant led possibly to the sum of two determinants. Use of (6.9), (6.10), and Proposition 6.6 allows us to improve upon Proposition 6.5:

Proposition 6.7. *If $1 \leqslant 2p \leqslant m$, then the element*

$$\operatorname{Pff}(X_1, \ldots, X_{2p})$$

of S_{mn} has weight $(e_1 + \cdots + e_{2p}, 0)$ under $\mathfrak{gl}(m, \mathbb{C}) \oplus \mathfrak{sp}(r, \mathbb{C})$ and is a nonzero highest weight vector.

Corollary 6.8. *If $\mu = \sum_{a=1}^{2[m/2]} \mu_a e_a$ is nonnegative dominant integral for $\mathfrak{gl}(m, \mathbb{C})$ and has paired entries, then the representation $\tau_\mu \mathbin{\widehat{\otimes}} 1$ occurs with multiplicity one in S_{mn}, and the highest weight vectors for it are the multiples of*

$$\prod_{p=1}^{[m/2]} \bigl(\operatorname{Pff}(X_1, \ldots, X_{2p})^{\mu_{2p} - \mu_{2p+1}}\bigr)$$

if $\mu_{2[m/2]+1}$ is interpreted as 0.

Comparison of Corollary 6.8 and Proposition 6.5 yields

Corollary 6.9. *If* $1 \leqslant 2p \leqslant m$, *then the square of* $\mathrm{Pff}(X_1, \ldots, X_{2p})$ *equals*

$$\det\Bigl(\bigl\{\langle X_a, X_b\rangle\bigr\}_{\substack{1\leqslant a\leqslant 2p\\ 1\leqslant b\leqslant 2p}}\Bigr),$$

apart from a nonzero multiplicative constant.

For $1 \leqslant 2p \leqslant m$, let us redefine V_{2p} to be $V^{\mu,0}$ for $\mu = e_1 + \cdots + e_{2p}$. Propositions 6.6 and 6.7 combine immediately to give

Corollary 6.10. *For* $1 \leqslant 2p \leqslant m$, *the space* V_{2p} *equals the linear span of all Pfaffians* $\mathrm{Pff}(X_{a_1}, \ldots, X_{a_{2p}})$. *Here the indices* $a_1, \ldots, a_{2p}$ *are between* 1 *and* m.

There is also an analog of Proposition 3.4:

Proposition 6.11. *If* d *is odd, the space* $(S^d_{mn})^{Sp}$ *of* $Sp(r)$ *invariants homogeneous of degree* d *is* 0. *If* d *is even, the following three spaces coincide*:

(a) *the space* $(S^d_{mn})^{Sp}$,
(b) *the linear span of the* $R(D)$ *with* $\|D\| = d/2$,
(c) *the direct sum of the spaces* $V^{\mu,0}$ *for* μ *nonnegative dominant integral with* $\|\mu\| = d$ *and with paired entries, these spaces being of multiplicity one.*

Remark. Cf. [GoW, Section 4.2].

With these tools in place, it is a simple matter to prove analogs of Theorem 4.1, Corollary 4.2, and Proposition 5.1; then Theorem 6.1 follows. For the analog of Theorem 4.1, any nonzero highest weight vector in $V^{\nu,\nu} \otimes \bigoplus_\mu V^{\mu,0}$ is of the form

$$\phi_{\lambda,\nu} = X(\nu) \otimes \sum_{\substack{D \text{ with}\\ \|D\|=\frac{1}{2}(\|\lambda\|-\|\nu\|)}} a_D R(D) + \sum_\gamma \Biggl(X_\gamma(\nu) \otimes \sum_{\substack{D \text{ with}\\ \|D\|=\frac{1}{2}(\|\lambda\|-\|\nu\|)}} b_{\gamma,D} R(D)\Biggr),$$

and the effect of ψ is given by

$$\psi\bigl(\mathcal{M}(\phi_{\lambda,\nu})\bigr) = \Biggl(\prod_{p=1}^{m} (X_{pp'})^{\nu_p}\Biggr)\Biggl(\sum_D a_D \psi\bigl(R(D)\bigr)\Biggr).$$

To convert the proof of Proposition 5.1 into a proof in the current setting, only minor notational changes are needed.

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