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IRREDUCIBLE CONSTITUENTS OF PRINCIPAL SERIES OF $SL_n(k)$

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§1. Introduction. The unitary principal series of the general linear group $GL_n(k)$ or the special linear group $SL_n(k)$ over a nondiscrete locally compact field k of characteristic zero consists of the representations unitarily induced from a continuous unitary character of the upper triangular group. In the case of $GL_n(k)$, Gelfand and Naimark [4] gave a proof that shows that these representations are always irreducible (see also [2]).

Our concern in this paper will be with $SL_n(k)$, where reducibility can occur. We shall describe the irreducible constituents of the unitary principal series of $SL_n(k)$, and we shall relate the reducibility that occurs to the abelian Galois extensions of k. In particular, the irreducible constituents will be parametrized by an abelian Galois group.

Some historical remarks about $SL_n(k)$ will put matters in perspective. For $k = \mathbb{C}$, the proof of irreducibility for $GL_n(\mathbb{C})$ given by Gelfand and Naimark [4] also proves irreducibility for $SL_n(\mathbb{C})$. For $SL_n(\mathbb{R})$, reducibility into two pieces can occur [8], and the irreducible constituents were described in [9]. The method of accounting for the reducibility within $SL_n(\mathbb{R})$ turns out to be a prototype for the classification of irreducible tempered representations of real semisimple groups.

For the case that k is nonarchimedean, Winarsky [18] showed that reducibility into more than two pieces can occur. Howe and Silberger [5] proved that in any event the irreducible constituents all have multiplicity one. Muller [14] and Winarsky [18] independently introduced a finite group, known as the R group, to parallel the case of real groups and obtained, with the aid of a completeness theorem due to Harish-Chandra ([17], Theorem 5.5.3.2), a basis for the commuting algebra. Keys [7] clarified the nature of this basis.

In all this, however, the problem of describing the irreducible constituents remained unsolved. Our intention is to give such a description in this paper for k nonarchimedean.

From [3] this result is known already for the case $n = 2$, but our proof is new even in that case. Briefly we start with a character χ_s of the upper triangular group of $SL_n(k)$, extend it to the upper triangular group of $GL_n(k)$ in a particular way, and use the extension to define a group G_χ intermediate between $SL_n(k)$ and $GL_n(k)$. Using an easy general argument, we show that none of the reducibility is lost in passing from $SL_n(k)$ to G_χ. We can then apply a slight

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variant of the Gelfand-Naimark argument to G_χ to find the irreducible constituents.

The paper is organized as follows. In §2, we take advantage of special properties of $\mathrm{SL}_n(k)$ to introduce a finite abelian group $\bar{L}(\chi)$ as a more convenient version of the R group. This definition is used in §3 to introduce the group G_χ and to show that no reducibility is lost in passing to G_χ. In §4 we give the variant of the Gelfand-Naimark argument that enables us to describe the irreducible constituents. The multiplicity-one result of [5] is a consequence of this argument, and the argument is not simplified by assuming the result of [5].

As already suggested, our approach to the problem contains a bonus in that it points to a close connection between this reducibility problem and abelian class field theory. It turns out to be possible to associate to the given character χ_s a canonical finite abelian Galois extension of k, and the R group and $\bar{L}(\chi)$ are both canonically isomorphic to the dual of the Galois group. Moreover, every finite abelian Galois extension of k arises in this way. These matters are discussed in more detail in §5.

It is known that the irreducible constituents under study are parametrized by the dual of the R group, once a particular constituent is specified as base point, and it follows that the irreducible constituents are parametrized by the Galois group (after a choice of base point). Connections between this fact and the Langlands notion of L-indistinguishability will be discussed in a later paper [1].

§2. The group that indexes reducibility. Let k be a nonarchimedean nondiscrete locally compact field of characteristic 0, i.e., a finite extension of a p-adic number field, and let $k^\times$ be the multiplicative group. We use the notation G for the general linear group $\mathrm{GL}_n(k)$ and G_s for the special linear group

$$G_s = \mathrm{SL}_n(k) = \{ g \in G \mid \det g = 1 \}.$$

Generally a subscript s on a subgroup of G will denote the intersection with G_s, and a subscript s on a function with domain contained in G will denote the restriction to the intersection of the domain with G_s. If A is a locally compact abelian group, $\hat{A}$ will denote the dual group of continuous homomorphisms into the circle.

Let T and T_s be the diagonal subgroups of G and G_s, respectively. We write $(a_1, \ldots, a_n)$ for the member of T with diagonal entries $a_1, \ldots, a_n$. Let χ_s be a member of $\hat{T}_s$. We recall the unitary principal series representation of G_s with parameter χ_s. For this purpose let N be the group of upper-triangular matrices with ones on the diagonal, and let $\mu : T \to \mathbb{R}^+$ be defined by

$$\mu(a_1, \ldots, a_n) = \prod_{i<j} |a_i / a_j|.$$

(Recall that μ, extended trivially on N, describes the modular function of TN.) We define the unitary principal series representation

$$U(\chi_s) = \operatorname*{ind}_{T_s N}^{G_s} (\chi_s \otimes 1)$$

of G_s to operate in the dense subspace

$$H(\chi_s) = \left\{ f : G_s \to \mathbb{C} \,\middle|\, \begin{array}{l} f(tng) = \mu(t)^{1/2}\chi_s(t)f(g) \text{ for } t \in T_s,\, n \in N \\ f \text{ locally constant} \end{array} \right\}$$

of a Hilbert space by

$$(U(\chi_s, g)f)(x) = f(xg).$$

The norm on $H(\chi_s)$ is the L^2 norm of the restriction to the maximal compact subgroup K_s of G_s consisting of integer matrices.

The reducibility of $U(\chi_s)$ is understood on a certain level by knowledge of the commuting algebra $\mathcal{C}(U(\chi_s))$ of $U(\chi_s)$, which is known to have a basis of operators parametrized by a certain finite group $R(\chi_s)$. (See [14], [18], and [10].) For $\mathrm{SL}_n(k)$ we can proceed more directly, using the group $\bar{L}(\chi)$ defined below. The group $\bar{L}(\chi)$ is canonically isomorphic with $R(\chi_s)$, as will be proved in [1], and has the advantage that it leads directly both to an identification of the irreducible constituents of $U(\chi_s)$ (see Theorem 4.1 below) and to connections between reducibility of $U(\chi_s)$ and field extensions of k (see [1]).

Before defining $\bar{L}(\chi)$, we recall the action of the Weyl group W for our situation. The group W can be regarded as the group of all permutations on n letters; it operates on $\hat{T}_s$ (and also on $\hat{T}$) by

$$\sigma\chi_s(a_1, \ldots, a_n) = \chi_s(a_{\sigma(1)}, \ldots, a_{\sigma(n)}).$$

For each pair (i, j) with $1 \leqslant i < j \leqslant n$, let $\psi_{ij} : k^\times \to T_s$ be defined by the recipe that $\psi_{ij}(a)$ is a in the ith entry, a^{-1} in the jth entry, and 1 in the other entries.

To define $\bar{L}(\chi)$, we choose a particular extension χ of χ_s from T_s to T. Namely, define $\chi_i \in (k^\times)\hat{}$ for $1 \leqslant i \leqslant n-1$ by $\chi_i = \chi_s \circ \psi_{in}$, and define $\chi_n = 1$. Our choice is χ in $\hat{T}$ given by

$$\chi(a_1, \ldots, a_n) = \prod_{i=1}^{n} \chi_i(a_i) = \chi_s(a_1, \ldots, a_{n-1}, a_1^{-1} \cdot \ldots \cdot a_{n-1}^{-1}), \qquad (2.1)$$

and it is clear that the restriction of χ to T_s is χ_s. Motivated by Labesse-Langlands [11], we define

$$L(\chi) = \{\sigma \in W \mid \sigma\chi = (\omega \circ \det)\chi \text{ for some } \omega \text{ in } (k^\times)\hat{}\}$$

$$\bar{L}(\chi) = \{\omega \in (k^\times)\hat{} \mid (\omega \circ \det)\chi = \sigma\chi \text{ for some } \sigma \text{ in } W\}.$$

Then $L(\chi)$ and $\bar{L}(\chi)$ are finite groups, $\bar{L}(\chi)$ is abelian, and there is an obvious homomorphism φ of $L(\chi)$ *onto* $\bar{L}(\chi)$ given by associating the unique ω that goes with σ in the definition of $L(\chi)$.

Define N_χ to be the intersection of ker ω for all ω in $\bar{L}(\chi)$. We observe that N_χ contains all nth powers of members of $k^\times$:

$$(k^\times)^n \subseteq N_\chi. \qquad (2.2)$$

In fact, let ω be in $\bar{L}(\chi)$ and choose σ in $L(\chi)$ with $\varphi(\sigma) = \omega$. Then

$$\chi(a, a, \ldots, a) = \sigma\chi(a, \ldots, a) = \omega(a^n)\chi(a, \ldots, a)$$

and so $\omega(a^n) = 1$.

Let us observe the canonical isomorphism

$$\bar{L}(\chi)^\wedge \cong k^\times / N_\chi. \tag{2.3}$$

In fact, we know that $\bar{L}(\chi) \subseteq (k^\times)^\wedge$; hence $\bar{L}(\chi)^\wedge$ is a quotient of $((k^\times)^\wedge)^\wedge \cong k^\times$. By definition the kernel of this quotient map must be N_χ. Hence $\bar{L}(\chi)^\wedge \cong k^\times / N_\chi$ canonically.

§3. **Restriction of induced representations**. The connection between $\bar{L}(\chi)$ and the reducibility problem under study comes in partly through two general lemmas about restricting irreducible unitary representations to open normal subgroups of finite index. Temporarily we set aside our notation G, G_s, etc. Lemma 3.1 is given explicitly as Lemma 1.16 of Miličić [13]. Lemma 3.2 underlies a variant of Lemma 1.2.1 of Jacquet, Piatetski-Shapiro, and Shalika [6]; their result is stated in [6] in the context of admissible representations and is given without proof.

LEMMA 3.1 (Miličić). *Let G be a locally compact unimodular group, let H be an open normal subgroup of finite index, and let π be an irreducible unitary representation of G. Then $\pi|_H$ splits as the finite orthogonal sum of $\leqslant |G/H|$ irreducible pieces. Moreover, there are integers M and m such that the decomposition of $\pi|_H$ into irreducible pieces takes the form*

$$\pi|_H \cong \sum_{j=1}^{M} m\pi_j \tag{3.1}$$

with $\pi_1, \ldots, \pi_M$ mutually inequivalent and all in the same G-orbit of the unitary dual of H.

LEMMA 3.2. (cf. [6]). *Let G be a separable locally compact unimodular group, let H be an open normal subgroup of G with finite index and with G/H abelian, let π be an irreducible unitary representation of G, and define*

$$X_H(\pi) = \{\nu \in (G/H)^\wedge \mid \pi \otimes \nu \cong \pi\}. \tag{3.2}$$

Then

$$\dim \mathcal{C}(\pi|_H) = |X_H(\pi)|.$$

Proof. Let $I(\cdot, \cdot)$ refer to the dimension of the space of intertwining maps between two representations. The unitary equivalence

$$\pi \otimes \operatorname*{ind}_H^G 1 \cong \operatorname*{ind}_H^G (\pi|_H) \tag{3.3}$$

is exhibited by the map carrying an element $v \otimes f$ of the representation space on the left side of (3.3) to the function F on G defined by $F(x) = f(x)\pi(x^{-1})v$. Then

$$\begin{aligned} |X_H(\pi)| &= \sum_{\nu \in (G/H)^\wedge} I(\pi \otimes \nu, \pi) \qquad \text{by irreducibility} \\ &= I\Big(\pi \otimes \sum_{\nu \in (G/H)^\wedge} \nu, \pi\Big) \\ &= I\Big(\pi \otimes \operatorname*{ind}_H^G 1, \pi\Big) \\ &= I\Big(\operatorname*{ind}_H^G (\pi|_H), \pi\Big) \qquad \text{by (3.3)} \\ &= I(\pi|_H, \pi|_H), \end{aligned}$$

the last inequality holding by Frobenius reciprocity (Theorem 4′ of [12]) and an application of Lemma 3.1. The result follows.

Returning to the notation at the beginning of §2, let χ_s be a character of T_s, and let χ be the extension of χ_s to T given by (2.1). We define

$$G_\chi = \{ g \in G \mid \det g \in N_\chi \}$$

$$T_\chi = T \cap G_\chi$$

$$U(\chi) = \operatorname*{ind}_{T_\chi N}^{G_\chi} (\chi \otimes 1), \qquad \text{representation of} \quad G_\chi$$

$H(\chi)$ = dense subspace of locally constant functions on which $U(\chi)$ acts.

LEMMA 3.3. *Restriction of functions in the space $H(\chi)$ from G_χ to G_s maps $H(\chi)$ one-one onto $H(\chi_s)$, and under this identification the restriction $U(\chi)_s$ of $U(\chi)$ to G_s is identified with $U(\chi_s)$.*

Remark. Similarly we can relate $U(\chi)_s$ and $U(\chi)$ to the restrictions to G_s and G_χ, respectively, of the induced representation $\operatorname{ind}_{TN}^G(\chi \otimes 1)$ of G. The proof is substantially unchanged.

Proof. It is clear that the image of this restriction map is in $H(\chi_s)$ and that the map is G_s-equivariant. We prove the map is one-one onto.

If g is in G_χ, then $t = (1, \ldots, 1, \det g)$ is in T_χ, and $g = tg_s$ for suitable g_s in G_s. If f is in $H(\chi)$, then

$$f(g) = \mu^{1/2}(t)\chi(t) f(g_s),$$

and so $f = 0$ on G_s implies $f = 0$ on G_χ. Thus the map is one-one. To see the map is onto, let h on G_s be a given member of $H(\chi_s)$, and define

$$f(g) = \mu^{1/2}(t)\chi(t)h(g_s)$$

whenever $g = tg_s$ as above. If $g = t'g_s'$ also, then $tg_s = t'g_s'$ and hence $g_s'g_s^{-1} = t'^{-1}t$ is in $G_s \cap T_\chi = T_s$. Thus

$$\begin{aligned}\mu^{1/2}(t)\chi(t)h(g_s) &= \mu^{1/2}(t)\chi(t)\,\mu^{1/2}(g_s'g_s^{-1})^{-1}\chi(g_s'g_s^{-1})^{-1}h(g_s'g_s^{-1}g_s)\\ &= \mu^{1/2}(t')\chi(t')h(g_s'),\end{aligned}$$

and f is unambiguously defined. The function f has the correct transformation properties, and the lemma follows.

LEMMA 3.4. *With $U(\chi)_s$ denoting the restriction of $U(\chi)$ to G_s, the commuting algebra $\mathcal{C}(U(\chi))$ for G_χ equals the commuting algebra $\mathcal{C}(U(\chi)_s)$ for G_s. Moreover,*

$$\dim \mathcal{C}(U(\chi)_s) = |\bar{L}(\chi)| = |k^\times / N_\chi|. \tag{3.4}$$

Proof. Since $G_\chi \supseteq G_s$, we have $\mathcal{C}(U(\chi)) \subseteq \mathcal{C}(U(\chi)_s)$. To get equality, it is thus enough to show that these commuting algebras have equal finite dimension. Let $U = \operatorname{ind}_{TN}^G(\chi \otimes 1)$; U is known to be an irreducible representation of $G = GL_n(k)$ from Gelfand-Naimark [4]. Lemma 3.3, together with the remark after its statement, gives us isomorphisms

$$\mathcal{C}(U(\chi)) \cong \mathcal{C}(U|_{G_\chi})$$

and

$$\mathcal{C}(U(\chi)_s) \cong \mathcal{C}(U(\chi_s)) \cong \mathcal{C}(U|_{G_s}).$$

Thus the desired equality of commuting algebras will follow if we show that

$$\dim \mathcal{C}(U|_{G_s}) = \dim \mathcal{C}(U|_{G_\chi}). \tag{3.5}$$

To prove (3.5), let Z be the set of scalar matrices and apply Lemma 3.2 with $H = G_sZ$ and $\pi = U$. Then the left side of (3.5) is identified as $|X_H(U)|$, since Z acts as scalars. Define

$$H' = \{g \in G \mid \nu(g) = 1 \quad \text{for all } \nu \text{ in } X_H(\pi)\}.$$

Then $H' \supseteq H$, and we can apply Lemma 3.2 to H'. The definition of H' makes $X_{H'}(U) = X_H(U)$, and thus the lemma gives

$$|X_H(U)| = \dim \mathcal{C}(U|_{H'}).$$

Then (3.5) will follow if it is shown that $H' = G_\chi$.

Now $X_H(U)$ consists of all one-dimensional characters $\nu(=\omega \circ \det)$ of G such that

$$\left(\operatorname*{ind}_{TN}^{G} \chi\right) \otimes \nu \cong \operatorname*{ind}_{TN}^{G} \chi.$$

Since $(\operatorname{ind} \chi) \otimes \nu \cong \operatorname{ind}(\nu\chi)$, $X_H(U)$ consists of all ν such that $\operatorname{ind}(\nu\chi) \cong \operatorname{ind}\chi$. By global character theory (or by [14, 18] and Theorem 2.5.8 of [17]) this condition occurs if and only if $\nu\chi = \sigma\chi$ for some σ in W, hence if and only if $\nu = \omega \circ \det$ has ω in $\bar{L}(\chi)$. Briefly $X_H(U) = \bar{L}(\chi) \circ \det$. Passing to kernels, we obtain $H' = G_\chi$, and (3.5) follows.

We have seen in the course of the argument that $\dim \mathcal{C}(U(\chi)_s)$ equals $|X_H(U)|$, which in turn equals $|\bar{L}(\chi)|$. In combination with (2.3), this observation proves (3.4).

§4. **Irreducible constituents.** We shall describe the irreducible constituents of $U(\chi_s)$ in terms of supports of Fourier transforms of functions. When $n = 2$ and reducibility occurs, the constituents are known from [3] to be characterized as follows. One restricts the functions in $U(\chi_s)$ to the lower triangular group $V = \{(\begin{smallmatrix} 1 & 0 \\ x & 1 \end{smallmatrix})\}$, identifies V with the field k, and takes Fourier transforms. The two irreducible subspaces are then realized as functions with support on one or the other coset of $k^\times / N_\chi$.

Our theorem will generalize this characterization, using an iterated construction with $n-1$ iterations for $\mathrm{SL}_n(k)$. However, as we remarked in §1, our proof is new even for $n = 2$. The new twist is that we deal with $U(\chi)$ and G_χ in place of $U(\chi_s)$ and G_s, taking advantage of Lemma 3.4; a consequence for $n = 2$ is that we need not examine $U(\chi_s, (\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}))$ at all.

Our construction is canonical once we fix a definition of Fourier transform. Thus let ψ be an additive character of k and define

$$\hat{f}(y) = \int_k \psi(yx) f(x)\, dx \qquad \text{for } f \in L^1(k, dx) \qquad \text{and} \quad y \in k.$$

We shall state the main theorem in a qualitative form, but the proof gives more precise information.

THEOREM 4.1. *By means of a sequence of $n-1$ Fourier transforms, the unitary principal series representation $U(\chi_s)$ has a canonical realization in which the commuting algebra $\mathcal{C}(U(\chi_s))$ is canonically isomorphic with the algebra of complex-valued functions on k that are constant on multiplicative cosets of the norm group N_χ.*

Remarks. It is an important part of the conclusion that the isomorphisms are canonical. (Otherwise the theorem would add nothing new beyond formula (3.4) and the results of [5].) To be more specific, we note that the irreducible projections in the commuting algebra correspond to the cosets of N_χ. Consequently one can trace through the canonical isomorphisms in the proof of

Theorem 4.1 below to obtain an explicit (albeit complicated) description of the irreducible subspaces themselves.

Preparation. We shall work with the representation $U(\chi)$ of G_χ. Define

$$M_j = G_\chi \cap \left[\begin{array}{c|ccc} * & & 0 & \\ \hline & * & & 0 \\ 0 & & \ddots & \\ & 0 & & * \end{array}\right]\Big\}j, \qquad 1 \leqslant j \leqslant n$$

$$N_j = G_\chi \cap \left[\begin{array}{c|c|c} I & * & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & 0 & I \end{array}\right]\begin{array}{l}\}j \\ \}1,\\ \end{array} \qquad 1 \leqslant j \leqslant n-1$$

$$V_j = G_\chi \cap \left[\begin{array}{c|c|c} I & 0 & 0 \\ \hline * & 1 & 0 \\ \hline 0 & 0 & I \end{array}\right]\begin{array}{l}\}j \\ \}1,\\ \end{array} \qquad 1 \leqslant j \leqslant n-1$$

$$M_j' = G_\chi \cap \left[\begin{array}{c|ccc} * & & 0 & \\ \hline & m_0 & & 0 \\ 0 & & \ddots & \\ & 0 & & m_0 \end{array}\right]\Big\}j, \qquad 1 \leqslant j \leqslant n-1$$

Observe that

$$M_1 = T_\chi, \qquad M_n = G_\chi, \qquad M_{n-1}' = M_{n-1},$$

$$M_j' \subseteq M_j \quad \text{for} \quad 1 \leqslant j \leqslant n-1.$$

Define inductively

$$\xi_1 = \chi|_{T_\chi} \quad \text{on} \quad M_1$$

$$\xi_j = \operatorname*{ind}_{M_{j-1}N_{j-1}}^{M_j} (\xi_{j-1} \otimes 1), \qquad 2 \leqslant j \leqslant n.$$

The representation ξ_j operates by right translation in the space H^{ξ_j} given by

$$\left\{ f: M_j \to H^{\xi_{j-1}} \,\middle|\, \begin{array}{l} f(mnx) = \mu_j(m)^{1/2}\xi_{j-1}(m)f(x) \\ \text{for} \quad m \in M_{j-1}, n \in N_{j-1}, x \in M_j \end{array} \right\}. \tag{4.1}$$

Here $\mu_j(m)$ is given in terms of the action of M_{j-1} on Haar measure of N_{j-1} by

$$dx' = \mu_j(m)\,dx \qquad \text{if} \quad x' = m^{-1}xm.$$

By the double induction formula, $U(\chi) \cong \xi_n$ canonically.

LEMMA 4.2. *There is a natural algebra inclusion of the commuting algebra* $\mathcal{C}(\xi_j|_{M'_{j-1}V_{j-1}})$ *into* $\mathcal{C}(\xi_{j-1}|_{M'_{j-2}V_{j-2}})$ *for* $3 \leqslant j \leqslant n$, *given as follows: If B is in* $\mathcal{C}(\xi_j|_{M'_{j-1}V_{j-1}})$, *then B in the noncompact Fourier transform picture is given by*

$$(Bf)\hat{}(y) = b(y)\hat{f}(y),$$

where y is in the dual V'_{j-1}, $b(y)$ *is a bounded operator on* $H^{\xi_{j-1}}$, *and* $b(y)$ *varies continuously in y for* $y \neq 0$. *The inclusion is given by* $B \to b(y_0)$, *where* y_0 *picks off the* $(j, j-1)$*th entry of members of* V_{j-1}.

Proof. Restriction of the functions in (4.1) from M_j to V_{j-1} gives the noncompact picture of ξ_j. (The Gelfand-Naimark decomposition $M_j \doteq N_{j-1}M_{j-1}V_{j-1}$ allows one to recover the functions (4.1) from their restrictions.) In this realization

$$\xi_j(v_0)f(v) = f(vv_0) \qquad \text{for} \quad v \in V_{j-1}, v_0 \in V_{j-1} \tag{4.2}$$

$$\xi_j(m)f(v) = \mu_j(m)^{1/2}\xi_{j-1}(m)f(m^{-1}vm) \quad \text{for} \quad v \in V_{j-1}, m \in M_{j-1}. \tag{4.3}$$

The noncompact Fourier transform picture $\hat{\xi}_j$ of ξ_j is defined by

$$\hat{\xi}_j(g)\hat{f}(y) = \int_{V_{j-1}} \psi\langle y, x\rangle \xi_j(g)f(x)\,dx \qquad \text{for} \quad y \in V'_{j-1}, g \in M_j. \tag{4.4}$$

For $g = v_0$ in V_{j-1}, (4.4) becomes

$$\hat{\xi}_j(v_0)\hat{f}(y) = \int_{V_{j-1}} \psi\langle y, x\rangle f(xv_0)\,dx = \overline{\psi\langle y, v_0\rangle}\,\hat{f}(y). \tag{4.5}$$

For m in M_{j-1}, define $m^{-1}ym$ for y in V'_{j-1} by

$$\langle m^{-1}ym, x\rangle = \langle y, mxm^{-1}\rangle.$$

Then (4.4) with $g = m$ becomes

$$\hat{\xi}_j(m)\hat{f}(y) = \mu_j(m)^{-1/2}\xi_{j-1}(m)\left[\hat{f}(m^{-1}ym)\right] \tag{4.6}$$

because the left side equals

$$\int_{V_{j-1}} \psi\langle y, x\rangle \mu_j(m)^{1/2}\xi_{j-1}(m)\left[f(m^{-1}xm)\right]dx$$

$$= \int_{V_{j-1}} \psi\langle y, mx'm^{-1}\rangle \mu_j(m)^{-1/2}\xi_{j-1}(m)\left[f(x')\right]dx'$$

$$\text{under} \quad x' = m^{-1}xm$$

$$= \int_{V_{j-1}} \psi\langle m^{-1}ym, x'\rangle \mu_j(m)^{-1/2}\xi_{j-1}(m)\left[f(x')\right]dx,$$

which is the right side of (4.6).

Let B be given in $\mathcal{C}(\xi_j|_{M'_{j-1}V_{j-1}})$. Here B is acting on $L^2(V_{j-1}, H^{\xi_{j-1}})$, and (4.2) implies B commutes with translations. Since B is bounded, we have

$$(Bf)\hat{}(y) = b(y)\hat{f}(y),$$

with $b(y)$ a bounded operator on $H^{\xi_{j-1}}$ varying measurably in y and with $|b(y)|$ in L^∞. The commutativity of B with $\xi_j|_{M'_{j-1}}$ implies, in view of (4.6),

$$\begin{aligned}\mu_j(m)^{-1/2}\xi_{j-1}(m)b(m^{-1}ym)\hat{f}(m^{-1}ym) &= \hat{\xi}_j(m)b(y)\hat{f}(y)\\ &= b(y)\hat{\xi}_j(m)\hat{f}(y)\\ &= \mu_j(m)^{-1/2}b(y)\xi_{j-1}(m)\hat{f}(m^{-1}ym)\end{aligned}$$

for m in M'_{j-1}. Since $\hat{f}$ is essentially arbitrary, for each m in M'_{j-1} we have

$$\xi_{j-1}(m)b(m^{-1}ym) = b(y)\xi_{j-1}(m) \quad \text{for a.e.} \quad y \in V'_{k-1}. \tag{4.7}$$

By Fubini's Theorem almost every y has the property that (4.7) holds for almost every m. Finding one such y and redefining $b(m^{-1}ym)$ for the exceptional values of m by (4.7), we are led to conclude that (4.7) holds for all m and for every y in any open orbit of V'_{j-1}:

$$\xi_{j-1}(m)b(m^{-1}ym) = b(y)\xi_{j-1}(m) \qquad \text{for} \quad m \in M'_{j-1}, y \in \text{any open orbit}. \tag{4.8}$$

Now we compute $m^{-1}ym$ for m in M_{j-1}. For x in V_{j-1}, denote by $\bar{x}$ the row vector consisting of the first $j-1$ entries of the jth row of x. For m in M_{j-1}, let $\bar{m}$ be the upper left $(j-1)$-by-$(j-1)$ block, and let m_0 be the jth diagonal entry. If y is in V'_{j-1}, let $\bar{y}$ be the $(j-1)$-dimensional column vector such that $\langle y, x\rangle = \bar{x}\bar{y}$. We readily compute that

$$\overline{mxm^{-1}} = m_0\bar{x}\bar{m}^{-1}$$

Then

$$\begin{aligned}\bar{x}\,\overline{m^{-1}ym} = \langle m^{-1}ym, x\rangle = \langle y, mxm^{-1}\rangle &= \overline{mxm^{-1}}\,\bar{y}\\ &= m_0\bar{x}\bar{m}^{-1}\bar{y} = \bar{x}\big(m_0\bar{m}^{-1}\bar{y}\big),\end{aligned}$$

and we conclude that

$$\overline{m^{-1}ym} = m_0\bar{m}^{-1}\bar{y} \qquad \text{for} \quad m \in M_{j-1}. \tag{4.9}$$

We shall use (4.9) to show that M_{j-1} acts in V'_{j-1} with just one orbit, apart from a set of measure 0. Let e_r, $1 \leqslant r \leqslant j-1$, be the $(j-1)$-dimensional column vector that is 1 in the rth entry and 0 elsewhere. The linear functional y_0 in the statement of the lemma has $\bar{y}_0 = e_{j-1}$. Let us specialize to elements m in M'_{j-1} that have $m_0 = 1$ and $\bar{m}$ of determinant 1. Then (4.9) simplifies to

$$\overline{m^{-1}ym} = \bar{m}^{-1}\bar{y}. \tag{4.10}$$

Let a column vector $\mathbf{a} = a_1e_1 + \cdots + a_{j-1}e_{j-1}$ be given with $a_r \neq 0$. In (4.10) choose $\bar{y} = \bar{y}_0 = e_{j-1}$, and let $\bar{m}^{-1}$ be the matrix given by

$$\bar{m}^{-1} = \begin{cases} -e_{j-1} & \text{in } r\text{th column} \\ e_r & \text{in } (j-1)\text{st column} \\ e_i & \text{in } i\text{th column for other } i. \end{cases}$$

Then $\bar{m}^{-1}e_{j-1} = e_r$. Next define $\bar{m}^{-1}$ by

$$\bar{m}^{-1} = \begin{cases} \mathbf{a} & \text{in } r\text{th column} \\ a_r^{-1}e_l & \text{in } l\text{th column for some } l \neq r \\ e_i & \text{in } i\text{th column for other } i. \end{cases}$$

(This definition makes sense as soon as $j - 1 \geqslant 2$, i.e., $j \geqslant 3$ as in the hypothesis.) This $\bar{m}^{-1}$ satisfies $\bar{m}^{-1}e_r = \mathbf{a}$. Thus $j \geqslant 3$ implies that the M'_{j-1} orbit of y_0 is $V'_{j-1} - \{0\}$. By (4.8), $b(y_0)$ determines $b(y)$ on $V'_{j-1} - \{0\}$, hence almost everywhere. Consequently the algebra homomorphism $B \to b(y_0)$ is injective into bounded operators on $H^{\xi_{j-1}}$.

To complete the proof, we show that $b(y_0)$ is in $\mathcal{C}(\xi_{j-1}|_{M'_{j-2}V_{j-2}})$. This conclusion will follow from (4.8) if we show that the isotropy subgroup in M'_{j-1} at y_0 is exactly $M'_{j-2}V_{j-2}$. Let m in M'_{j-1} be built from the $(j-1)$-by-$(j-1)$ matrix $\bar{m}$ and the scalar m_0, as above. The isotropy condition for y_0 in (4.8), in view of (4.9), is that $\bar{y}_0 = m_0\bar{m}^{-1}\bar{y}_0$, hence that $\bar{m}\bar{y}_0 = m_0\bar{y}_0$, hence that e_{j-1} be an eigenvector of $\bar{m}$ with eigenvalue m_0. This condition means that the last column of $\bar{m}$ is m_0e_{j-1}, and the result follows. The proof of the lemma is complete.

LEMMA 4.3. *There is a natural algebra inclusion of the commuting algebra $\mathcal{C}(\xi_2|_{M'_1V_1})$ into the algebra of complex-valued functions on $V'_1 \cong k$ that are constant on multiplicative cosets of the norm group N_χ, given as follows: If B is in $\mathcal{C}(\xi_2|_{M'_1V_1})$, then B in the noncompact Fourier transform picture is given by*

$$(Bf)\hat{}(y) = b(y)\hat{f}(y),$$

where y is in V'_1, b is complex-valued, and b is constant on cosets of N_χ. The inclusion is given by $B \to b$.

Proof. The proof of Lemma 4.2 remains valid with $j = 2$ through the proof of (4.10). The mapping $B \to b$ is certainly an algebra inclusion, and we have to show b is constant on cosets as indicated. Since (4.8) is a scalar equation, we have

$$b(m^{-1}ym) = b(y) \qquad \text{for} \quad m \in M'_1, y \in \text{any open orbit}.$$

In view of (4.9), we have to know the orbit structure under $\bar{y} \to m_0 \bar{m}^{-1} \bar{y}$, i.e., under multiplication in k by the scalar $m_0 m^{-1}$ subject to the condition that the diagonal matrix $m = (\bar{m}, m_0, \ldots, m_0)$ is in M_1'. Rewriting $\bar{m}$ as $\bar{m}' m_0$, we see that $\bar{m}'$ is to be in N_χ, since m_0^n is in N_χ by formula (2.2). Then the multiplicative scalar $m_0 \bar{m}^{-1}$ is $\bar{m}'^{-1}$, and the multiplication is by an arbitrary member of N_χ. This completes the proof of the lemma.

Proof of Theorem 4.1. We have established canonical isomorphisms and inclusions

$$\begin{aligned}
\mathcal{C}(U(\chi_s)) &= \mathcal{C}(U(\chi)_s) && \text{by Lemma 3.3}\\
&= \mathcal{C}(\xi_n|_{G_\chi}) && \text{by Lemma 3.4}\\
&\subseteq \mathcal{C}(\xi_n|_{M_{n-1}'V_{n-1}}) && \text{trivially}\\
&\subseteq \mathcal{C}(\xi_{n-1}|_{M_{n-2}'V_{n-2}}) && \text{by Lemma 4.2}\\
&\subseteq \cdots \subseteq \mathcal{C}(\xi_2|_{M_1'V_1}) && \text{by Lemma 4.2}\\
&\subseteq \left\{ \begin{array}{c} \text{functions on } k \text{ constant} \\ \text{on cosets of } N_\chi \end{array} \right\} && \text{by Lemma 4.3.}
\end{aligned}$$

The dimensions of the first and last spaces are equal, by Lemma 3.4, and hence all the inclusions are equalities.

§5. Interpretation in terms of abelian field extensions. Theorem 4.1 shows that the group $k^\times / N_\chi$ acts canonically as a simply transitive permutation group on the set of irreducible constituents of $U(\chi_s)$. We can reinterpret this fact in terms of abelian field extensions of k. By the fundamental theorem of local class field theory (see [16], Chapter XIV), there is a one-one correspondence between closed subgroups H of finite index in $k^\times$ (such as $H = N_\chi$) and finite abelian Galois extensions K of k, the correspondence being that H is the norm group of K over k; moreover the degree of the extension is $|k^\times / H|$, and the Galois group is canonically isomorphic to $k^\times / H$.

Applying this correspondence to $H = N_\chi$ and using the isomorphism $\bar{L}(\chi)^\wedge \cong k^\times / N_\chi$ of (2.3), we obtain a finite abelian Galois extension K_χ associated to χ_s for which the group $\bar{L}(\chi)$ is canonically isomorphic to the dual of the Galois group of K_χ over k. The map $\chi_s \to K_\chi$ is onto the set of such abelian field extensions, as is implied by the following theorem.

THEOREM 5.1.
(a) *The map* $\chi_s \to K_\chi$ *carries* $\hat{T}_s$ onto *the set of finite abelian Galois extensions.*
(b) *Two members of* $\hat{T}_s$ *in the same orbit under* W *lead to the same field extension.*
(c) *If the map is restricted to* W*-orbits of characters whose* $\bar{L}$ *group is of order* n*, then it carries the set of such orbits in a one-one fashion onto the set of abelian Galois extensions of degree* n*.*

Proof. We prove (b) first. Let χ be the extension of χ_s to T defined by equation (2.1), and let ϵ be in W. An easy calculation shows that the extension (2.1) of $\epsilon\chi_s$ to T is

$$\chi' = (\omega_0 \circ \det)\epsilon\chi,$$

where ω_0 is the composition of $(\epsilon\chi_s)^{-1}$ and the function $\psi_{\epsilon(n),n}$. (Here $\psi_{\epsilon(n),n} = 1$ if $\epsilon(n) = n$.) If σ in W exhibits ω as in $\bar{L}(\chi)$ by the equation

$$\sigma\chi = (\omega \circ \det)\chi,$$

then

$$\epsilon\sigma\epsilon^{-1}\chi' = (\omega_0 \circ \det)\epsilon\sigma\chi = (\omega_0\omega \circ \det)\epsilon\chi = (\omega \circ \det)\chi'$$

and $\epsilon\sigma\epsilon^{-1}$ exhibits ω as in $\bar{L}(\chi')$. And conversely. Thus $\bar{L}(\chi) = \bar{L}(\chi')$, and (b) follows.

Now fix n. To prove (a) in the sharp form given in (c), it is enough to prove that if H is any subgroup of index n in $k^\times$, then there is a character χ_s for $\mathrm{SL}_n(k)$ such that $\bar{L}(\chi) = (k^\times/H)\hat{}$. Thus, given H, let $\omega_1, \ldots, \omega_n$ be an enumeration of the characters of $k^\times$ that are trivial on H, and let us arrange that $\omega_n = 1$. Define $\chi_s(a_1, \ldots, a_n) = \prod_{i=1}^n \omega_i(a_i)$. Then χ is given by the same formula since $\omega_n = 1$. If ω is in $\bar{L}(\chi)$, choose σ in W with $\sigma\chi = (\omega \circ \det)\chi$. Applying this formula to $(1, 1, \ldots, 1, a)$, we obtain

$$\begin{aligned}\omega(a) = \omega(a)\omega_n(a) = \omega(a)\chi(1, 1, \ldots, 1, a) &= \sigma\chi(1, \ldots, 1, a)\\ = \omega_{\sigma^{-1}(n)}(a). &\qquad (5.1)\end{aligned}$$

Hence $\bar{L}(\chi) \subseteq \{\omega_1, \ldots, \omega_n\}$. For the reverse inclusion, we use the fact that the ω_j's form a group. Given ω_i, define σ by $\sigma^{-1}(r) = s$ if $\omega_i\omega_r = \omega_s$. Then

$$\begin{aligned}\sigma\chi(a_1, \ldots, a_n) = \prod_{r=1}^n \omega_{\sigma^{-1}(r)}(a_r) &= \prod_{r=1}^n \omega_i(a_r)\omega_r(a_r)\\ &= \omega_i(a_1 \cdot \ldots \cdot a_n)\chi(a_1, \ldots, a_n).\end{aligned}$$

Hence ω_i is in $\bar{L}(\chi)$, and $\bar{L}(\chi) = \{\omega_1, \ldots, \omega_n\}$. This proves (a).

Finally we prove the remaining part of (c), that the map is one-one. Suppose χ_s extends to T as χ, suppose χ'_s extends as χ', and suppose $\bar{L}(\chi) = \bar{L}(\chi')$ with $|\bar{L}(\chi)| = n$. Recall that $\chi_i = \chi \circ \psi_{in}$. The same calculation as in (5.1) shows that $\bar{L}(\chi) \subseteq \{\chi_1, \ldots, \chi_n\}$, and a similar result holds for χ'. Since $\bar{L}(\chi) = \bar{L}(\chi')$ and since both sets have n elements, we conclude that the χ_i are distinct, the χ'_i are distinct, and the χ'_i are a permutation of the χ_i, say $\chi'_i = \chi_{\sigma(i)}$. Then $\chi' = \sigma\chi$, as required.

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