Equivariant Maps onto Minimal Flows

by

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1. Introduction

Let G be a connected Lie group. (X, G) is a *flow* if X is a compact Hausdorff space with a jointly continuous group action by G. The flow (X, G) is *minimal* if every orbit is dense or, equivalently, if X has no proper closed non-empty invariant set.

We wish to capture two ideas about minimal sets. The first is that if (X, G) is a minimal flow, then not only is each orbit dense, but also each orbit actually winds around X in much the same way that a Kronecker line on the torus winds around the torus. This fact is made precise and explained further in Theorem 2.1 and the discussion immediately following it. As a consequence of this theorem, we obtain as Theorem 2.2 a statement about equivariant maps of flows onto minimal flows, which generalizes the main result of Chu and Geraghty in [2].

The second idea is that the space X of a minimal flow should have some homogeneity property. A known result, due to A. A. Markoff [7], is that if X is finite-dimensional, then X has the same dimension at each point. The conjecture that X has a transitive set of homeomorphisms commuting with G is shown to be false by enlarging the space of Floyd's example [6] and making it into a flow under the reals in the usual way. Instead, our result is of a relative rather than an absolute nature. Namely, if π is an equivariant mapping between minimal flows (X, G) and (Y, G), then under suitable conditions X is the bundle space of a fiber bundle with base space Y and with projection π. Such a result is proved as Theorem 3.1 under the assumption that everything is differentiable.

2. Relation of Orbits to Homotopy

Let (Y, G) be a flow with Y connected and locally arcwise connected, let X be a covering space of Y, and let π be the projection. There is at most one way of lifting the flow on Y to a jointly continuous group action of G on X which commutes with π, and if G is simply connected, there is at least one.

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The uniqueness is trivial, and the existence follows from the homotopy lifting property for covering spaces.

THEOREM 2.1. *Let G be a connected Lie group, let (Y, G) be a minimal flow with Y (connected and) locally arcwise connected, and let X be a covering space of Y with projection π. Suppose the flow on Y lifts to a jointly continuous action of G on X. If X is compact, then (X, G) is a minimal flow. If X is noncompact, then every orbit of G on X is unbounded.*

Proof. The argument for compact X appears on p. 27 of [1], and our proof, given only in the noncompact case, is similar to that argument. Suppose the theorem is false in the noncompact case. Let C be the compact closure of a bounded orbit in X. C is invariant under G and must contain a minimal closed invariant subset M. Then M is nowhere dense. In fact, M intersected with the complement of the interior of M is a closed invariant subset of M and so is empty or is all of M. The first alternative would mean $X = M$ or else X is disconnected. Since neither of these things is so, we conclude that the intersection is M and hence M is nowhere dense. Since π is a local homeomorphism and M is compact nowhere dense, $\pi(M)$ is compact and nowhere dense. The invariance of M then contradicts the fact that (Y, G) is minimal, and the proof is complete.

Theorem 2.1 allows us to make the following description of the relationship between orbits of a minimal flow $(Y, \mathbf{R})$, where $\mathbf{R}$ is the additive group of reals, and the 1-dimensional simplicial homology of Y. Suppose Y admits a finite triangulation. Call a 1-cycle on Y *free* if no multiple of it bounds. Then each orbit of $(Y, \mathbf{R})$ winds arbitrarily often around each free 1-cycle of Y, in a sense that we now make precise.

Choose any free 1-cycle z of Y, and let $[z]$ be its homology class. Let c be any indivisible class in $H_1(Y)$ with $k_0 c = [z]$ for some k_0 in the integers $\mathbf{Z}$, and let $h_c\colon H_1(Y) \to \mathbf{Z}$ be any homomorphism sending c into 1 and sending the other members of a basis for the free part of $H_1(Y)$ into 0. Let $h\colon \pi_1(Y, y_0) \to H_1(Y)$ be factorization by the commutator subgroup, and let $K_c \subseteq \pi_1(Y, y_0)$ be the kernel of the composition $h_c \circ h$. Finally let $p\colon X \to Y$ be a regular covering with a point x_0 such that $p_*(\pi_1(X, x_0)) = K_c$. Since $h_c \circ h$ is onto $\mathbf{Z}$, the covering space X has $\mathbf{Z}$ as its group of deck transformations and X is noncompact.

Lift the flow $(Y, \mathbf{R})$ to an action of $\mathbf{R}$ on X, and let T be the orbit through x_0. Find a compact set $C \subseteq X$ containing x_0 with $p(C) = Y$ and define, for $k \geqslant 0$,

$$B_k = \bigcup_{n=-k}^{k} n \cdot C,$$

where n acts as a deck transformation. B_k is compact and $\cup_k B_k = X$. Theorem 2.1 implies that the orbit T is contained in no B_k. That is, T reaches arbitrarily remote sheets of the covering X. In this sense the corresponding orbit in $(Y, \mathbf{R})$ wraps arbitrarily often around the cycle z.

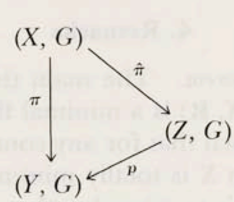
The next theorem generalizes the main result of [2]. The details of this implication are given at the beginning of §4.

THEOREM 2.2. *Let G be a connected Lie group, let (X, G) and (Y, G) be flows with (Y, G) minimal, and let π be a continuous equivariant map of X onto Y with $\pi(x_0) = y_0$. Suppose that X is compact, connected and locally arcwise connected and that Y is locally arcwise connected and semi-locally 1-connected. Then $\pi_*(\pi_1(X, x_0))$ has finite index in $\pi_1(Y, y_0)$.*

Proof. It suffices to prove the result for G simply connected, since the hypotheses and conclusion are unchanged when G is replaced by its universal covering group and X and Y are considered as flows under the covering group.

With G simply connected, let $p\colon Z \to Y$ be a covering with a point z_0 in Z such that $p_*(\pi_1(Z, z_0)) = \pi_*(\pi_1(X, x_0))$. The flow (Y, G) lifts uniquely to a jointly continuous action of G on Z. The condition on p_* is such that $\pi\colon X \to Y$ lifts to a unique continuous function $\hat{\pi}\colon X \to Z$ that satisfies $\hat{\pi}(x_0) = z_0$, and it is readily checked that $\hat{\pi}$ is equivariant. Z is compact if and only if $\pi_*(\pi_1(X, x_0))$ has finite index in $\pi_1(Y, y_0)$ and we may therefore assume that Z is noncompact. Then $\hat{\pi}(X)$ is a compact subset of Z containing the orbit of z_0 under G, and we have arrived at a conclusion contradicting Theorem 2.1.

COROLLARY 2.3. *Let G be a simply connected Lie group, let (X, G) and (Y, G) be flows with (Y, G) minimal, and let π be a continuous equivariant map of X onto Y with $\pi(x_0) = y_0$. Suppose that X is compact, connected and locally arcwise connected and that Y is locally arcwise connected and semi-locally 1-connected. Then there exist a minimal flow (Z, G), continuous equivariant maps $\hat{\pi}\colon X \to Z$ and $p\colon Z \to Y$, and a point z_0 in Z such that the diagram*



commutes, such that $\hat{\pi}_*\colon \pi_1(X, x_0) \to \pi_1(Z, z_0)$ is onto, and such that p is a covering map (and hence $p_*\colon \pi_1(Z, z_0) \to \pi_1(Y, y_0)$ is one-to-one).

Proof. Let (Z, G), $\hat{\pi}$, and z_0 be as in the proof of Theorem 2.2. Since $\pi_*(\pi_1(X, x_0))$ has finite index in $\pi_1(Y, y_0)$, Z is compact. By Theorem 2.1, (Z, G) is minimal. The remaining statements follow easily from standard facts about covering spaces.

COROLLARY 2.4. *Let G be a connected Lie group, let (X, G) and (Y, G) be flows with (Y, G) minimal, and let π be a continuous equivariant map of X onto Y. Suppose that X is compact, connected and locally arcwise connected and that Y is locally arcwise connected and semi-locally 1-connected. Then as a map on singular cohomology, $\pi^*\colon H^1(Y) \to H^1(X)$ is one-to-one.*

Proof: $H^1(W) = \mathrm{Hom}\,(\pi_1(W), \mathbf{Z})$ if $W = X$ or Y. Apply Theorem 2.2.

3. Equivariant Maps of Differentiable Flows

We say that a flow (X, G) under the connected Lie group G is *differentiable* if X is a compact connected C^∞ manifold (without boundary) and if, for each g in G, the map $x \to gx$ is a diffeomorphism of X.

THEOREM 3.1. *Let (X, G) and (Y, G) be differentiable flows with (Y, G) minimal, and let π be a differentiable equivariant map of X onto Y. Then X is the bundle space of a differentiable fiber bundle with base space Y, projection π, and a differentiable manifold (not necessarily connected) as fiber. Consequently* $\dim Y \leqslant \dim X$, *and, if equality holds, then $\pi: X \to Y$ is a covering map.*

Proof. Let C be the set of x in X for which the differential $d\pi_x$ is not onto. C is closed. Since π is equivariant,

$$d\pi_{gx} = dg_{\pi(x)} \circ d\pi_x \circ (dg_x)^{-1},$$

from which it follows that C is G-invariant. Since (Y, G) is minimal, $\pi(C)$ is empty or $\pi(C) = Y$. Sard's Theorem precludes the second possibility, and thus $\pi: X \to Y$ is an onto map with $d\pi_x$ onto for all x in X. The rest follows from the proposition on p. 31 of [4].

In the differentiable case we can strengthen Corollary 2.3 as follows.

COROLLARY 3.2. *Let (X, G) and (Y, G) be differentiable flows with (Y, G) minimal, let G be simply connected, and let π be a differentiable equivariant map of X onto Y. Then there exist a differentiable minimal flow (Z, G) and differentiable equivariant maps $\hat{\pi}: X \to Z$ and $p: Z \to Y$ such that $p \circ \hat{\pi} = \pi$, p is a covering map, and $\hat{\pi}$ is the projection in a differentiable fiber bundle with bundle space X, base space Z, and a connected differentiable manifold as fiber.*

Proof. Apply Corollary 2.3 and then Theorem 3.1.

4. Remarks

1. *The Chu-Geraghty Theorem.* The main theorem proved by Chu and Geraghty in [2] is that if $(X, \mathbf{R})$ is a minimal flow with X (connected and) locally arcwise connected such that for any continuous $f: X \to S^1$ the image of $\pi_1(X)$ under f_* is 0, then X is totally minimal. (Totally minimal means that if the action by $\mathbf{R}$ on X is restricted to that of any subgroup $c\mathbf{Z}$, then it is still true that every orbit is dense. It is easy to see that $(X, \mathbf{R})$ is totally minimal if and only if $(X, \mathbf{R})$ does not have any $(S^1, \mathbf{R})$ as a quotient flow with $\mathbf{R}$ acting on S^1 by rotations.)

If X is not totally minimal, there is a continuous equivariant map π of $(X, \mathbf{R})$ onto some $(S^1, \mathbf{R})$, and Theorem 2.2 shows that $\pi_*(\pi_1(X))$ has finite index in an infinite cyclic group. The image under π_* of $\pi_1(X)$ is therefore not 0, and we have arrived at a contradiction. Thus the Chu-Geraghty Theorem follows from Theorem 2.2 with $G = \mathbf{R}$ and $Y = S^1$.

2. *Ellis's Generalization of the Chu-Geraghty Theorem.* Ellis in Theorem 2 of [5] and its corollaries generalized the Chu-Geraghty theorem to other pairs of groups than $\mathbf{R}$ and $\mathbf{Z}$. Under suitable conditions, if (X, G) is a minimal flow, Ellis concludes that $\pi_1(X) \neq 0$. When his group G is a connected

Lie group, his corollaries are special cases of our Theorem 2.2 with (Y, G) taken as certain compact homogeneous spaces of G.

3. *Images of* π_1 *and* H_1. Simple examples in Theorem 2.2 show that π_* need not map $\pi_1(X)$ onto $\pi_1(Y)$ or $H_1(X)$ onto $H_1(Y)$. For instance, if $Y = S^1$ and X is a double covering of Y by S^1 and $\mathbf{R}$ acts by rotations on each in such a way that the projection π is equivariant, then π_* is not onto.

4. *Another Version of Corollary 2.4.* Many nontrivial examples of pairs of flows (X, G) and $(Y, G) = \pi(X, G)$ with (Y, G) minimal are such that Y satisfies the connectedness conditions of Corollary 2.4 but X does not. Among the simplest of these is one for which X is a solenoid, Y is S^1, π is the usual projection of X onto Y, and G is $\mathbf{R}$ acting by translation. Corollary 2.4 may be modified to include these examples. (In the case of the solenoid, there is a simple direct proof by duality for abelian groups.)

Specifically *we may remove all connectedness restrictions on X and still conclude that $\pi_*\colon H^1(Y) \to H^1(X)$ is one-to-one, provided that we replace singular cohomology with Čech cohomology.*

For the proof we use the isomorphism of $H^1(Z)$ with $\pi(Z, S^1)$, valid for all paracompact spaces Z ([3], Theorem 8.1). Here $\pi(Z, S^1)$ is the group of homotopy classes of maps of Z into S^1. We are thus required to show that if $f\colon Y \to S^1$ is essential, then $f \circ \pi\colon X \to S^1$ is essential. Assuming that $f \circ \pi$ is inessential, we lift it to a map h from X to the universal covering space R of S^1. Fix x_0 in X and let $y_0 = \pi(x_0)$ and $r_0 = h(x_0)$. Construct the regular covering $\hat{Y}$ of Y corresponding to the kernel of $f_*\colon \pi_1(Y, y_0) \to \pi_1(S^1, f(y_0))$, choose $\hat{y}_0$ in $\hat{Y}$ covering y_0, and lift f to the unique continuous map $\hat{f}\colon \hat{Y} \to R$ satisfying $f(\hat{y}_0) = r_0$. This lifting exists because $r_0 = h(x_0)$ covers $f \circ \pi(x_0) = f(y_0)$.

Now let $T \subseteq X$ be the orbit through x_0, and let $\hat{T}$ be the orbit through $\hat{y}_0$ covering $\pi(T)$. It is easy to show from the covering homotopy property that $h(T) = \hat{f}(\hat{T})$. But $\hat{Y}$ has image $f_* \cong \mathbf{Z}$ for its group of deck transformations and so is noncompact. By Theorem 2.1, $\hat{T}$ is unbounded, that is, it reaches arbitrarily remote sheets of $\hat{Y}$. Since f is essential, $\hat{f}(\hat{T})$ reaches arbitrarily remote sheets of R (covering S^1). Thus $\hat{f}(\hat{T}) = h(T)$ is unbounded, and the larger set $h(X)$ must be unbounded, in contradiction to the compactness of X.

5. *An Example for Theorem 3.1.* Suppose that S^3 admits a differentiable minimal action by $\mathbf{R}$. Let us examine what the possible differentiable quotient flows $(Y, \mathbf{R})$ of $(S^3, \mathbf{R})$ are if the projection π of S^3 onto Y is required to be differentiable. The claim is that either Y is one point or else Y is 3-dimensional and π is a covering map. In fact, $(Y, \mathbf{R})$ must be minimal and have Euler characteristic 0, and Theorem 3.1 thus shows that the only other possibilities for Y are the circle, the 2-torus, and the Klein bottle; since H^1 of each of these spaces is infinite, Corollary 2.4 shows they are not possible quotients.

Hence if Y is nontrivial, $\pi\colon S^3 \to Y$ is a covering map. This fact suggests that in order to construct a minimal flow on S^3, it may not be possible to build the flow from simpler ones, that it may be necessary to work with S^3 directly.

6. *A Note about Dimensions in Theorem 3.1.* The conclusion $\dim Y \leq \dim X$ in Theorem 3.1 follows without the differentiability of the action of G as long as π is known to be differentiable since the image of a differentiable map cannot have dimension greater than the dimension of the domain. It would be interesting to know to what extent Theorem 3.1 is valid if all the assumptions of differentiability are removed.

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