CLASSIFICATION THEOREMS FOR REPRESENTATIONS OF SEMISIMPLE LIE GROUPS

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Let G be a connected linear real semisimple Lie group with maximal compact subgroup K . We shall discuss progress on three classification problems for irreducible representations of G :

a) Irreducible quasisimple representations. A representation of the Lie algebra $\mathfrak{g}$ of G is quasisimple if it is finitely-generated over the universal enveloping algebra, if the action of K is well-defined and every vector is K-finite, and if the representation has an infinitesimal character. Such representations have global characters, defined as distributions on $C^{\infty}_{com}(G)$. The irreducible quasisimple representations have been classified by Langlands [14] , modulo a classification of the irreducible tempered representations.

b) Irreducible tempered representations. A tempered representation is one whose global character extends to Harish-Chandra's Schwartz space [3] on G. The authors gave a classification of the irreducible tempered representations in [11] . The present paper includes a more intrinsic classification, based on a criterion for equivalence of two irreducible tempered representations. (See Theorem 4.)

c) Irreducible unitary representations. Progress in classifying the irreducible unitary representations is limited. We shall give in §4 a theorem that at least tells what the problem is. (See Theorem 7.) Then we show how the theorem relates to the known examples. Finally in §5 we give a technique for fitting known unitary representations into a classification.

It turns out, for irreducible representations, that tempered implies unitary and unitary implies quasisimple. We now consider the three classifica-

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tions in turn.

1. The Langlands classification

Following [14] , we first construct a list of the irreducible quasisimple representations. There are three parameters :

(a) a parabolic subgroup $P = MAN$ containing a fixed minimal parabolic subgroup P_0 ,

(b) an equivalence class of irreducible tempered representations of M , with π as a representative, and

(c) a complex-valued linear functional ν_A on the Lie algebra $\mathfrak{a}$ of A such that $\mathrm{Re}\, \nu_A$ is strictly in the positive Weyl chamber.

We construct the Langlands representation $J_P(\pi:\nu_A)$ as a particular nonzero irreducible quotient of the quasisimple representation.

$$U_P(\pi:\nu_A) = \mathrm{ind}_{MAN}^{G}(\pi \otimes e^{\nu_A}) ,$$

where the induced representation is defined in such a way that G acts on the left and that the representation is unitary if ν_A is imaginary. If $A(\bar{P}:P:\pi:\nu_A)$ is the intertwining operator from $U_P(\pi:\nu_A)$ to $U_{\bar{P}}(\pi:\nu_A)$ given by a convergent integral on K-finite functions as

$$A(\bar{P}:P:\pi:\nu_A)f(x) = \int_{\bar{N}} f(x\bar{n})\,d\bar{n} ,$$

then we define

$$J_P(\pi:\nu_A) = U_P(\pi:\nu_A)/\mathrm{kernel}\, A(\bar{P}:P:\pi:\nu_A)$$
$$\cong \mathrm{image}\, A(\bar{P}:P:\pi:\nu_A) .$$

<u>Théorème 1</u>. (Langlands [14]) .

<u>The representations $J_P(\pi:\nu_A)$ are irreducible quasisimple, are infinitesimally inequivalent, and exhaust the irreducible quasisimple representations of G .</u>

Thus, unless $P = G$, $J_P(\pi:\nu_A)$ is nontempered.

If we do not insist that P contain P_0 , then $J_P(\pi:\nu_A)$ and $J_{P'}(\pi':\nu'_A)$ are infinitesimally equivalent if and only if there is an element g of G carrying P to P' , π to π' (up to equivalence), and ν_A to ν'_A . In any event, the theorem explicitly reduces the classification of irreducible quasisimple representations of semisimple groups to the classification of irreducible tempered representations of a certain class of reductive groups.

2. Irreducible tempered representations

The group M need not be connected or semisimple, but it falls into a class of groups to which the theories of [14] and [11] apply. Motivated by Theorem 1 , we now regard M as the total group in question and write G for it. We examine tempered representations of G .

Examples. Suppose MAN is a parabolic subgroup in G such that M has a compact Cartan subgroup T . (Such a parabolic subgroup is called cuspidal.) In this case, and only in this case, M possesses discrete series representations. By results of Harish-Chandra, such a representation is determined by a nonsingular linear form on $i\mathfrak{t}$, where $\mathfrak{t}$ is the Lie algebra of T , and a character η on the center Z_M of M . (The conditions on λ and η are that $\lambda - \rho$ be integral ant that $e^{\lambda - \rho}$ agree with η on $T \cap Z_M$, and two pairs (λ, η) and (λ', η') of parameters lead to equivalent discrete series if and only if $\eta = \eta'$ and λ is equivalent to λ' under the Weyl group $W(T:M)$.)
We can write $\Theta^M(\lambda, C, \eta)$ for the character, where C is the (unique) Weyl chamber of $i\mathfrak{t}$ with respect to which λ is dominant. For ν imaginary on $\mathfrak{a}$, set

$$\Theta^{MA}(\lambda, C, \eta, \nu) = \Theta^M(\lambda, C, \eta) \otimes e^{\nu} \quad .$$

Then

$$\Theta = \operatorname{ind}_{MAN}^{G} \Theta^{MA}(\lambda, C, \eta, \nu)$$

is tempered and is the character of a unitary representation, which we say is induced from discrete series. This representation is quasisimple but is not necessarily irreducible.

Theorem 2 (Trombi [16] , Langlands [14] , Harish-Chandra) .

Every irreducible tempered representation is infinitesimally equivalent with a constituent of some representation induced from discrete series.

More examples. In the definition of $\Theta^M(\lambda, C, \eta)$ it is possible to allow λ to become singular but still dominant with respect to C , and the result is still a unitary character. The formula for the character is of the same general nature as for discrete series characters except that λ has become a singular parameter. C is now nonunique, and distinct C's may give different characters. The more general kind of representation of M, with λ regular or singular, is called a limit of discrete series. See [22]. If ν is imaginary, we can again form $\Theta^{MA}(\lambda, C, \eta, \nu)$ and $\Theta = \mathrm{ind}_{MAN}^{G} \Theta^{MA}(\lambda, C, \eta, \nu)$.

Again Θ is tempered and is the character of a unitary quasisimple representation, which we call a basic representation.

The same basic character may arise from completely different sets of data, or it may even be 0. The ambiguity arises already when we consider $SL(2,\mathbb{R})$ and the group $SL^{\pm}(2,\mathbb{R})$ of real 2-by-2 matrices of determinant ± 1. In $G = SL(2,\mathbb{R})$, consider the principal series character with M-parameter $\begin{pmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{pmatrix} \to \epsilon_1$ and with A-parameter 0. This decomposes as the sum of limits of discrete series characters

$$\Theta^G(0,+,\mathrm{sgn}) + \Theta^G(0,-,\mathrm{sgn}) \quad ,$$

and there is no ambiguity. However, if we pass to $G = SL^{\pm}(2,\mathbb{R})$, we find that the principal series character for the same M and A parameters is equal to $\Theta^G(0,+,\mathrm{sgn})$ and also to $\Theta^G(0,-,\mathrm{sgn})$. So in $SL^{\pm}(2,\mathbb{R})$ a basic character can arise from data attached to two totally different parabolic subgroups. This degeneracy arises because of the existence of the element $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ in $SL^{\pm}(2,\mathbb{R})$; this element is a representative of the nontrivial element of the Weyl group of the compact Cartan subgroup $\begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix}$.

Another degeneracy occurs if a basic character is 0. In fact, $\Theta^M(\lambda, C, \eta)$ is 0 if and only if λ is singular with respect to a C-simple compact root α of $(\mathfrak{m}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$. (See [4] for a proof of the "if" direction.) Again the degeneracy arises because reflection in the root α exists as a member of the Weyl group $W(T:M)$.

We say that the data $\Theta^{MA}(\lambda, C, \eta, \nu)$ for a basic character are nondegenerate if, for each root α of $(\mathfrak{m}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$ with $\langle \lambda, \alpha \rangle = 0$, the reflection p_α is not in the Weyl group $W(T:M)$.

As noted in [11], any degeneracy allows us to rewrite a nonzero basic character in terms of a more noncompact Cartan subgroup of G. Consequently each nonzero basic character can be given in terms of nondegenerate data.

Our new classification of irreducible tempered representations will be given in terms of basic characters with nondegenerate data. The classification results from our having an irreducibility criterion, an equivalence criterion, and a completeness theorem.

Irreducibility is given in terms of the R-group, which is described fully in [11] . We give the flavor of the definition here without recalling the details.

Let $\Theta = \mathrm{ind}_{MAN}^{G} \Theta^{MA}(\lambda, C, \eta, \nu)$ be induced from nondegenerate data, and let

$$W_{\Theta^{MA}} = \{ w \in W(A:G) \mid (\Theta^{MA})^w = \Theta^{MA} \}$$

be the stability subgroup within the Weyl group of A . Let

$$\Delta' = \{\alpha \text{ useful root of } (\mathfrak{J}, \mathfrak{a}) \mid \mu_{\Theta^M, \alpha}(\nu) = 0\} .$$

Here "useful" is defined in [6], and μ is the Plancherel factor described in [11] and built from a maximal parabolic subgroup within a subgroup of G that is defined in terms of α . Then Δ' is a root system, $W_{\Theta^{MA}}$ leaves Δ' stable, and the Weyl group $W(\Delta')$ of Δ' is contained in $W_{\Theta^{MA}}$. It follows that if we define R by

$$R = \{ w \in W_{\Theta^{MA}} \mid w\alpha > 0 \text{ for } \alpha > 0 \text{ in } \Delta' \} ,$$

then $W_{\Theta^{MA}}$ splits as a semidirect product $W_{\Theta^{MA}} = W(\Delta')R$ with $W(\Delta')$ normal.

From the results of [11], we can read off an irreducibility criterion and completeness theorem.

<u>Theorem 3</u> . <u>Let $\Theta = \mathrm{ind}_{MAN}^{G} \Theta^{MA}(\lambda, C, \eta, \nu)$ be induced from nondegenerate data. Then Θ is the sum of exactly $|R|$ irreducible basic characters, and these are distinct . Moreover, the R-group tells how to write Θ as the sum of irreducible basic characters with nondegenerate data. Consequently</u>

<u>(i) Θ is irreducible if and only if $|R| = 1$, and</u>

<u>(ii) every irreducible tempered character is basic (and can be written with nondegenerate data)</u> .

The classification results by combining Theorem 3 with the following equivalence criterion.

Theorem 4. For two basic characters with nondegenerate data, an equality

$$\operatorname{ind}_{MAN}^{G} \Theta^{MA}(\lambda, C, \eta, \nu) = \operatorname{ind}_{M'A'N'}^{G} \Theta^{M'A'}(\lambda', C', \eta', \nu')$$

holds if and only if there is an element w in G carrying M to M', A to A', t to t', and (λ, C, η, ν) to $(\lambda', C', \eta', \nu')$.

3. Irreducible quasisimple representations.

We can now insert the information provided in § 2 in the Langlands result, Theorem 1, to obtain a new listing of irreducible quasisimple representations. After all, Theorem 1 gives a classification in terms of induction from tempered representations, and § 2 shows that a tempered representation is itself induced. By the double induction formula, one expects a classification of irreducible quasisimple representations in terms of induction from a cuspidal parabolic subgroup MAN, with a limit of discrete series on M and a parameter ν on $\mathfrak{a}$ with $\operatorname{Re}\nu$ in the closure of the positive Weyl chamber.

We shall formulate such a result more precisely as Theorem 5. It has two features worth noting: (1) Under the isomorphism given by the double induction formula, the kernels of appropriate intertwining operators correspond, so that the Langlands quotient representation can be defined without reference to the intermediate parabolic subgroup. (2) The equivalences in Theorem 4 with tempered representations come from mapping MA to $M'A'$, whereas the equivalences in Theorem 1 come from mapping MAN to $M'A'N'$; when the two stages of induction are combined, the equivalence condition can be expected to become messy. In fact, we shall not write down a combined equivalence theorem in general, contenting ourselves with completeness and irreducibility in the general case and an equivalence theorem in a special case.

In order to formulate these results precisely, we need notation that corresponds to the decomposition of the induction in stages into the two individual stages. Let ν be a parameter on $\mathfrak{a}$ with $\operatorname{Re}\nu$ dominant, and let

ξ be a limit of discrete series on M with nondegenerate data . Put

$$\mathfrak{a}_* = \sum_{\beta \perp \operatorname{Re}\nu} \mathbb{R} H_\beta \subseteq \mathfrak{a}$$

$$\mathfrak{a}_1 = \mathfrak{a}_*^\perp \subseteq \mathfrak{a} \tag{3.1}$$

$$\mathfrak{n}_* = Z_{\mathfrak{n}}(\mathfrak{a})$$

$$\mathfrak{n}_1 = \mathfrak{n}_*^\perp \subseteq \mathfrak{n}$$

$$\mathfrak{m}_1 = \mathfrak{m} \oplus \mathfrak{a}_* \oplus \mathfrak{n}_* \oplus \overline{\mathfrak{n}}_* \ .$$

Then $\mathfrak{m}_1 \oplus \mathfrak{a}_1 \oplus \mathfrak{n}_1$ is a parabolic subalgebra with corresponding parabolic subgroup $M_1 A_1 N_1$, say. The Langlands parameters are the group $M_1 A_1 N_1$, the tempered representation

$$\pi = \operatorname{ind}_{MA_* N_*}^{M_1} (\xi \oplus \exp(\nu \,|\, \mathfrak{a}_*))$$

provided π is irreducible) , and the linear functional $\nu \,|\, \mathfrak{a}_1$.

A member of the representation space of $\operatorname{ind}_{M_1 A_1 N_1}^{G} (\pi \oplus \exp(\nu \,|\, \mathfrak{a}_1))$ is a function on G whose value at x in G is a certain kind of function on M_1 . The map sending F to $F(\,.\,)(1)$ exhibits the equivalence of

$$\operatorname{ind}_{M_1 A_1 N_1}^{G} (\pi \oplus \exp(\nu \,|\, \mathfrak{a}_1))$$

with

$$\operatorname{ind}_{MAN}^{G} (\xi \oplus \exp \nu)$$

and carries the intertwining operator $A(M_1 A_1 \overline{N}_1 : M_1 A_1 N_1 : \pi : \nu \,|\, \mathfrak{a}_1)$ to

$$A(MAN_* \overline{N}_1 : MAN_* N_1 : \xi : \nu) \tag{3.2}$$

Finally we must ensure that π is irreducible . The condition translates as follows : If

$$\Delta' = \{\alpha \text{ useful root of } (\mathfrak{g}, \mathfrak{a}) \mid p_\alpha \nu = \nu \text{ and } \mu_{\xi,\alpha}(\nu) = 0\} \ ,$$

then $W(\Delta')$ is contained in

$$W_{\xi,\nu} = \{ w \in W(A:G) \mid w\xi \cong \xi \text{ and } w\nu = \nu \} .$$

The condition for the irreducibility of π is that $|W_{\xi,\nu}/W(\Delta')| = 1$. See the discussion that predeces Theorem 3 .

Fix a minimal parabolic subgroup P_0 . A <u>data point</u> is a triple (P, ξ, ν) such that

(i) $P = MAN$ is a cuspidal parabolic subgroup containing P_0

(ii) ξ is a limit of discrete series on M with nondegenerate data

(iii) ν is a linear functional on the Lie algebra $\mathfrak{a}$ of A with $\operatorname{Re} \nu$ in the closure of the positive Weyl chamber

(iv) $|W_{\xi,\nu}/W(\Delta')| = 1$.

<u>The representation associated</u> to the data point (P, ξ, ν) is the quotient of $\operatorname{ind}_{MAN}^{G}(\xi \oplus \exp \nu)$ by the kernel of the operator (3.2), where η_* and η_1 are defined in (3.1) .

<u>Theorem 5</u>. <u>The representations associated with data points (P, ξ, ν) are all irreducible, and they exhaust the irreducible quasisimple representations.</u>

To get an equivalence theorem, we investigate conditions under which two data points lead to the same Langlands parameters. The result will have a simple formulation only under an additional assumption. To indicate the problem, we make no special assumption yet. With P_0 fixed and $P_1 = M_1 A_1 N_1$ containing P_0 , let π be tempered on M_1 . According to Theorem 4 , π determines a Cartan subgroup of M_1 up to conjugacy. Choose a parabolic subgroup MA_*N_* of M_1 containing the minimal parabolic $P_0 \cap M_1$ and associated to the Cartan subgroup determined by π . It would be nice if any two choises of MA_*N_* were conjugate, but this need not be so . (See Example 3 at the end of the section) . <u>Let us assume that the Cartan subgroup of M_1 is as noncompact as possible</u>. Then it follows that MA_*N_* is minimal and $MA_*N_* = P_0 \cap M_1$. That is , MA_* and N_* are unique . Tracking down the ambiguity from Theorem 4 when $MA = M'A'$, we arrive at the following equivalence theorem.

Theorem 6 . If $P_0 = M_0A_0N_0$ is a fixed minimal parabolic subgroup, then the representations associated with data points (P_0, ξ, ν) and (P_0, ξ', ν') are infinitesimally equivalent if and only if there is an element w in $W(A_0 : G)$ such that $w\xi \cong \xi'$ and $w\nu = \nu'$.

Remark. If G has only one conjugacy class of Cartan subgroups, then only P_0 can occur as the first item in a data point, and Theorem 6 therefore gives all infinitesimal equivalences for the representations associated with data points. Moreover, condition (iv) in the definition of data point is redundant, as was first shown by Wallach in unpublished work (cf. [19]) .

Examples:

(1) G complex semisimple. Theorems 5 and 6 , interpreted in the light of the remark, in this case are due to Želobenko [20 , 21] . An exposition of these results is given by Duflo [1] .

(2) G of real-rank one. The parabolic in Theorem 5 can always be P_0 , and it can be G itself if rank G = rank K . In the latter case we are led to limits of discrete series with nondegenerate data, with equivalences given as in Theorem 4 . When the parabolic is $P_0 = M_0A_0N_0$, ξ is a finite-dimensional representation of the compact group M_0 and ν is $z\rho$ with $\mathrm{Re}\, z \geq 0$. The irreducibility condition, given as (iv) in the definition of data point, excludes points with $w\xi \cong \xi$ and $z = 0$, where w is the nontrivial element of $W(A_0 : G)$, if $U_{P_0}(\xi : 0)$ is reducible. All other points with $\mathrm{Re}\, z \geq 0$ are retained, and Theorem 6 says that (ξ, iy) and $(w\xi, -iy)$ lead to infinitesimally equivalent representations and there are no other equivalences.

(3) $G = SL(4, \mathbb{R})$ as an example with a complicated equivalence between data points. Let P_0 be the upper triangular subgroup, and let

$$P_1 = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot \end{pmatrix} \quad .$$

In the description preceding Theorem 6, M_1 is $SL^{\pm}(3,\mathbb{R})$. We can arrange that a tempered representation π on M_1 leads to a Cartan subgroup with one compact dimension and one noncompact dimension. In this case, MA_*N_* can be chosen as either

$$\begin{pmatrix} \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & 0 \\ 0 & 0 & \cdot & 0 \\ 0 & 0 & 0 & \cdot \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & 0 \\ 0 & 0 & 0 & \cdot \end{pmatrix}$$

and the corresponding groups $P = MAN$ are

$$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot \\ 0 & 0 & 0 & \cdot \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot \end{pmatrix} ,$$

which are not conjugate in G. Data points corresponding to these two choices of P can lead to infinitesimally equivalent representations, and the corresponding mapping on the data will partly conjugate only the (ξ,ν) and partly affect the whole triple (P,ξ,ν).

4. Irreducible unitary representations.

Classification of the irreducible unitary representations amounts to deciding which Langlands representations are infinitesimally unitary.

Theorem 7. $J_P(\pi:\nu_A)$ is infinitesimally unitary if and only if

(i) the formal symmetry conditions hold: there exists w in K normalizing $\mathfrak{a}$ with $wPw^{-1} = \bar{P}$, $w\pi \cong \pi$, and $w\nu_A = -\bar{\nu}_A$, and

(ii) the Hermitian intertwining operator

$$B = \pi(w)R(w)A(\bar{P}:P:\pi:\nu_A) , \tag{4.1}$$

where $R(w)$ is right translation by w, is positive or negative semidefinite.

Remark. For Theorem 7 in the case of complex G, see Duflo [2].

Proof. We shall show below that the representations $J_P(\pi:\nu_A:x^{-1})^*$ and $J_{\bar{P}}(\pi:-\bar{\nu}_A:x)$ are infinitesimally equivalent. (Take the adjoint here to be defined just on K-finite vectors). If $J_P(\pi:\nu_A)$ is infinitesimally unitary, then $J_P(\pi:\nu_A:x^{-1})^*$ and $J_P(\pi:\nu_A:x)$ are infinitesimally equivalent. Hence

$$J_P(\pi:\nu_A:x) \quad \text{and} \quad J_{\bar{P}}(\pi:-\bar{\nu}_A:x)$$

are infinitesimally equivalent. By Theorem 1, condition (i) must hold. Since w exists and $w\pi \cong \pi$, the operator B given in (4.1) is defined. B is Hermitian and satisfies

$$U_P(\pi:-\bar{\nu}_A)B = BU_P(\pi:\nu_A) \; . \tag{4.2}$$

(Cf. [10] and [8], Lemma 62.) Define a Hermitian form on the space of $U_P(\pi:\nu_A)$ by

$$<u,v> = (Bu,v)_{L^2(K)} \; . \tag{4.3}$$

A simple computation that is indicated in [7] shows that

$$U_P(\pi:\nu_A:X)^* = U_P(\pi:-\bar{\nu}_A:-X) \tag{4.4}$$

on the Lie algebra level, and (4.2) and (4.4) imply that

$$<U_P(\pi:\nu_A:X)u,v> + <u,U_P(\pi:\nu_A:X)v> = 0 \tag{4.5}$$

Since the kernel of $A(\bar{P}:P:\pi:\nu_A)$ is equal to the kernel of B, $<.,.>$ descends to a Hermitian form on the space for $J_P(\pi:\nu_A)$, and (4.5) holds for $J_P(\pi:\nu_A)$. Now $J_P(\pi:\nu_A)$ is assumed infinitesimally unitary, and we let $<<.,.>>$ be an invariant Hermitian inner product. Then

$$<u,v> = <<Lu,v>>$$

for a Hermitian operator L that is a self-intertwining operator for $J_P(\pi:\nu_A)$, by (4.5). Since $J_P(\pi:\nu_A)$ is irreducible, L is scalar. Say $L = cI$ with c real and $\neq 0$. Then

$$<<u,v>> = c^{-1}<u,v> = c^{-1}(Bu,v) \quad ,$$

and $c^{-1}B$ must be positive semidefinite.

Conversely if (i) and (ii) hold with B positive semidefinite, we define an inner product by (4.3), and $J_P(\pi:\nu_A)$ is infinitesimally unitary by (4.5) for $J_P(\pi:\nu_A)$.

To complete the proof, we are to show that $J_P(\pi:\nu_A:-X)^*$ and $J_{\bar{P}}(\pi:-\bar{\nu}_A:X)$ are infinitesimally equivalent. Let

$$V = (\ker A(\bar{P}:P:\pi:\nu_A))^{\perp} = \text{image } A(\bar{P}:P:\pi:\nu_A)^* = \text{image } A(P:\bar{P}:\pi:-\bar{\nu}_A).$$

(The last equality is given in [9].) Let E be the orthogonal projection (relative to $L^2(K)$) on V. Then $J_P(\pi:\nu_A:X)$ acts in V as $EU_P(\pi:\nu_A:X)E$. By (4.4), $J_P(\pi:\nu_A:-X)^*$ acts in V as

$$EU_P(\pi:\nu_A:-X)^*E = EU_P(\pi:-\bar{\nu}_A:X)E .$$

Now $A(P:\bar{P}:\pi:-\bar{\nu}_A)$ is a linear isomorphism from $\widetilde{V} = (\ker A(P:\bar{P}:\pi:-\bar{\nu}_A))^{\perp}$ onto $V = \text{image } A(P:\bar{P}:\pi:-\bar{\nu}_A)$. Therefore $J_P(\pi:\nu_A:-X)^*$ pulls back from V to a unique operator $S(X)$ on $\widetilde{V}$ satisfying

$$J_P(\pi:\nu_A:-X)^* A(P:\bar{P}:\pi:-\bar{\nu}_A) = A(P:\bar{P}:\pi:-\bar{\nu}_A)S(X). \tag{4.6}$$

The left side of (4.6) is

$$\begin{aligned}
&= EU_P(\pi:-\bar{\nu}_A:X)EA(P:\bar{P}:\pi:-\bar{\nu}_A) \\
&= EU_P(\pi:-\bar{\nu}_A:X)A(P:\bar{P}:\pi:-\bar{\nu}_A)\widetilde{E} \qquad (\widetilde{E} = \text{projection on } \widetilde{V}) \\
&= EA(P:\bar{P}:\pi:-\bar{\nu}_A)U_{\bar{P}}(\pi:-\bar{\nu}_A:X)\widetilde{E} \qquad \text{by [9]} \\
&= A(P:\bar{P}:\pi:-\bar{\nu}_A)\widetilde{E}U_{\bar{P}}(\pi:-\bar{\nu}_A:X)\widetilde{E} .
\end{aligned}$$

We conclude that

$$\begin{aligned}
S(X) &= \widetilde{E}U_{\bar{P}}(\pi:-\bar{\nu}_A:X)\widetilde{E} \\
&= J_{\bar{P}}(\pi:-\bar{\nu}_A:X) .
\end{aligned}$$

Thus the linear isomorphism $A(P:\bar{P}:\pi:-\bar{\nu}_A)$ from $\widetilde{V}$ to V exhibits the required infinitesimal equivalence. The proof of Theorem 7 is complete.

Corollary. Let P be minimal and let ρ_A be half the sum of the positive $\mathfrak{a}$-roots, repeated with multiplicities. If $\operatorname{Re}\nu_A - \rho_A$ is dominant, then $J_P(\pi:\nu_A)$ cannot be infinitesimally unitary unless π is trivial and $\nu_A = \rho_A$.

Proof. Let B be as in (4.1). The same computation as in Lemma 56 of [8] shows that

$$Bf(k_0) = \int_K e^{(\nu_A - \rho_A)\log a(w^{-1}k)} \pi(w)\pi(m(w^{-1}k))f(k_0k)\,dk ,$$

apart from a positive constant depending on normalizations of Haar measures. Here $\bar{N}NAM$ is dense in G and we are decomposing elements in the dense set as $g = \bar{n}na(g)m(g)$.

If $J_P(\pi:\nu_A)$ is unitary, then Theorem 7 and the considerations in the proof of Proposition 45 of [8] show that one of

$$\pm e^{(\nu_A - \rho_A)\log a(w^{-1}k)} \pi(w)\pi(m(w^{-1}k)) \tag{4.6}$$

is a positive definite function on K .

If λ is the highest $\mathfrak{a}$-weight of a finite-dimensional irreducible representation π_λ of G , if φ_λ is a unit highest weight vector, and if P_λ is the projection on the λ weight space, then

$$e^{\lambda \log a(g)} = |P_\lambda \pi_\lambda(g)\varphi_\lambda| , \tag{4.7}$$

and the left side thereby extends to a continuous function on all of G .

Hence it follows from our hypothesis on $\operatorname{Re}\nu_A$ that the function (4.6) is bounded, as well as positive definite. Therefore it is continuous and its absolute value attains its maximum at the identity of K . However, the value at the identity will be 0 by (4.7) unless $\nu_A - \rho_A$ is imaginary. In this case, the function will be discontinuous at the identity unless $\nu_A = \rho_A$ and $\pi = 1$.

Examples.

(1) $G = SL(3, \mathbb{C})$. The classification in this case is due to Tsuchikawa [17]. To obtain it from the results here, we use Theorem 7. If $P = G$, we are led to the unitary principal series. If P is a maximal proper parabolic subgroup, condition (i) fails in Theorem 7 and we obtain no unitary representations. Thus the only interesting case is that P is minimal parabolic. Take P to be the upper triangular group. In order for (i) to hold, the character ξ of M must be fixed by the transposition (1 3) and ν must be of the form

$$\nu = a(e_1 - e_3) + bi(e_1 - 2e_2 + e_3) \qquad (4.2)$$

with a and b real. The condition that $\operatorname{Re} \nu$ be in the positive Weyl chamber implies $a > 0$. The values $0 < a \leq 1$ and b arbitrary give unitary representations (the complementary series if $0 < a < 1$) by Theorem 9 of [8], and $a \leq 2$ is necessary for a unitary representation by the corollary to Theorem 7. Since $A(\bar{P}:P:\xi:\nu)$ is easily seen to be nonsingular for $1 < a < 2$ and for $a = 2$ if $b \neq 0$, it follows that for given ξ there are only three possibilities on the interval $1 < a \leq 2$:

$$J_P(\xi:\nu) \text{ unitary} \qquad \begin{cases} \text{when } 1 < a \leq 2 \\ \text{when } a = 2 \text{ and } b = 0 \\ \text{for no values of } a \text{ and } b\,. \end{cases}$$

For $a = 2$, the only possibility for a unitary representation is the trivial representation, by the corollary to Theorem 7, and we are led to the following classification:

Unitary principal series

Complementary series : ξ fixed by (1 3) and
ν of the form (4.2) with $0 < a < 1$

End of complementary series : ξ fixed by (1 3) and
ν of the form (4.2) with $a = 1$

Trivial representation : $\xi = 1$ and ν of the form (4.2) with
$a = 2$, $b = 0$.

It will follow from the style of argument in § 5 that the end of the complementary series is also obtained by inducing with $MA = GL(2, \mathbb{C})$, using the trivial representation on M and a unitary character on A.

(2) $G = Sp(2, \mathbb{C})$ and G = complex G_2. Duflo [2] has given a classification in these cases starting from his version of Theorem 7 for complex groups.

(3) $G = SL(3, \mathbb{R})$. The classification in this case is due to Vahutinskii [18] It can be obtained also by computations similar to those in Example 1.

(4) $G = Spin(n, 1)$ and $G = SU(n, 1)$. The classification for $Spin(n, 1)$, the universal cover of $SO(n, 1)$, is due to Hirai [5]. The classification for $SU(n, 1)$ is substantially due to Kraljevič [13]. In view of Theorems 7 and 3, the Langlands representations for $P = G$ are the limits of discrete series and the irreducible unitary principal series. For P minimal, we are led by Theorem 7 to data points (ξ, ν) with $w\xi \cong \xi$ and $\nu = z\rho$ with $0 < z \leq 1$. The question of when (ii) holds is settled in [8] the answer being that $0 < z \leq z_c$, where z_c is the critical abscissa given in [8]. Thus the classification is

Limits of discrete series

Irreducible members of unitary principal series

Complementary series : $w\xi \cong \xi$ and $\nu = z\rho$, $0 < z < z_c$

End of complementary series : $w\xi \cong \xi$ and $\nu = z_c\rho$ if $z_c \neq 0$.

Note that $z_c = 0$ unless $(P, \xi, 0)$ satisfies the irreducibility condition (iv) in the definition of data point. Note also that the trivial representation is the end of the complementary series for ξ trivial.

5. Effect of induction on Langlands representations

A number of exceptional unitary representations arise as induced representations with a nontempered unitary representation on M. We shall give a theorem for locating some of these in the Langlands classification. Actually the proof is more useful than the statement of the theorem, since the proof will often apply when the statement does not. We shall illustrate the technique by locating in the Langlands classification the exceptional representations of $SL(4, \mathbb{C})$ produced by Stein [15].

Theorem 8. Let $P = MAN$ be a parabolic subgroup, and let ω be an irreducible representation of M with Langlands parameters $(M_* A_M N_M, \sigma, \lambda_M)$. Let λ be a linear functional on $\mathfrak{a}$ such that $<\mathrm{Re}\,\lambda, \alpha> > 0$ for all positive $\mathfrak{a}$-roots α. Choose an ordering on $\mathfrak{a} + \mathfrak{a}_M$ so that $\mathrm{Re}(\lambda + \lambda_M)$ is dominant, and let N_λ be the nilpotent group built from the positive roots. If λ is sufficiently small, then $\pi_{\lambda, N} = \mathrm{ind}_{MAN}^{G}(\omega \otimes \exp\lambda)$ is irreducible and its Langlands paremeters are $(M_*(A A_M) N_\lambda, \sigma, \lambda + \lambda_M)$.

Proof: In writing down intertwining operators and induced representations, we shall drop the reductive factors in the parabolic subgroups. The induced representation

$$U_{N_M}(\sigma : \lambda_M) = \mathrm{ind}_{M_* A_M N_M}^{M}(\sigma \otimes \exp \lambda_M)$$

maps onto

$$\omega \subseteq \mathrm{ind}_{M_* A_M \overline{N}_M}^{M}(\sigma \otimes \exp \lambda_M) \qquad (5.1)$$

under the Langlands map

$$f \to A(\overline{N}_M : N_M : \sigma : \lambda_M) f \quad .$$

Here f is a suitable kind of function on M with values in H^σ. the relevant induction in stages formula is

$$U_{\lambda, N_M N} = \mathrm{ind}_{M_*(A_M A)(N_M N)}^{G}(\sigma \otimes \exp(\lambda_M + \lambda)) \qquad (5.2a)$$

$$= \mathrm{ind}_{MAN}^{G}\left[\mathrm{ind}_{M_* A_M N_M}^{M}(\sigma \otimes \exp \lambda_M) \otimes \exp \lambda\right]. \qquad (5.2b)$$

A function in the representation space of (5.2b) carries G to the representation space of $\mathrm{ind}_{M_* A_M N_M}^{M}(\sigma \otimes \exp \lambda_M)$. To $F(g)$, regarded as a function on M, we can apply the operator $A(\overline{N}_M : N_M : \sigma : \lambda_M)$, obtaining a member of the space for ω, by (5.1). By examining the integral formulas in [9], we see that

$$[A(\bar{N}_M:N_M:\sigma:\lambda_M)(F(g))](m)$$

$$= [A(\bar{N}_M N:N_M N:\sigma:\lambda_M+\lambda)(F(.)(1))](gm). \quad (5.3)$$

Equations (5.3) and (5.1) allow us to interpret $A(\bar{N}_M:N_M:\sigma:\lambda_M)$ as a mapping that exhibits $\pi_{\lambda,N}$ as a quotient of $U_{\lambda,N_M N}$. Under this interpretation, the intertwining relations in [9] and [10] show that we have the following commutative diagram, apart form scalar factors (including poles !):

$$\begin{array}{ccccc}
U_{\lambda,N_\lambda} & \xrightarrow{A(N_M N:N_\lambda:\sigma:\lambda_M+\lambda)} & U_{\lambda,N_M N} & \xrightarrow{A(\bar{N}_M:N_M:\sigma:\lambda_M)} & \pi_{\lambda,N} \\
\Big\downarrow A(\bar{N}_\lambda:N_\lambda:\sigma:\lambda_M+\lambda) & & \Big\downarrow A(N_M\bar{N}:N_M N:\sigma:\lambda_M+\lambda) & & \Big\downarrow A(\bar{N}:N:\omega:\lambda) \\
 & & U_{\lambda,N_M\bar{N}} & \xrightarrow{A(\bar{N}_M:N_M:\sigma:\lambda_M)} & \pi_{\lambda,\bar{N}} \\
 & & \Big\downarrow A(\bar{N}_M\bar{N}:N_M\bar{N}:\sigma:\lambda_M+\lambda) & & \Big\downarrow I \\
U_{\lambda,\bar{N}_\lambda} & \xleftarrow{A(\bar{N}_\lambda:\bar{N}_M\bar{N}:\sigma:\lambda_M+\lambda)} & U_{\lambda,\bar{N}_M\bar{N}} & \xleftarrow{\text{Inclusion from (5.1)}} & \pi_{\lambda,\bar{N}}
\end{array}$$

We shall prove for small λ that

(i) $A(N_M N:N_\lambda:\sigma:\lambda_M+\lambda)$ and $A(\bar{N}_\lambda:\bar{N}_M\bar{N}:\sigma:\lambda_M+\lambda)$ are isomorphisms, and

(ii) $A(\bar{N}:N:\omega:\lambda)$ is an isomorphism.

First we show that (i) and (ii) prove the theorem. In fact, the vertical map at the left is a Langlands map, and the image is irreducible. By (i) the image in $U_{\lambda,\bar{N}_M\bar{N}}$ is irreducible, and hence the image in $\pi_{\lambda,\bar{N}}$ at the lower right is irreducible. However, as we go from top left to bottom right first along the top and then down the right, each map is onto – by (i), by (5.3) and (5.1) and by (ii). Thus the image in $\pi_{\lambda,\bar{N}}$ is all of $\pi_{\lambda,\bar{N}'}$ and $\pi_{\lambda,\bar{N}}$ is irreducible. By (ii), $\pi_{\lambda,N}$ is irreducible. Also $\pi_{\lambda,N}$ is isomorphic with $\pi_{\lambda,\bar{N}'}$ which we have seen is isomorphic with the Langlands quotient. Hence $\pi_{\lambda,N}$ has the required Langlands parameters.

To prove that the appropriate operators are isomorphisms, we visualize decomposing each of them as a minimal product, as in §1 of [10]. Then for the operator $A(N_M N : N_\lambda : \sigma : \lambda_M + \lambda)$ it is enough to prove that $<\alpha, \lambda> \neq 0$ for all $(\mathfrak{a} + \mathfrak{a}_M)$ – roots α that are positive for N_λ and negative for $N_M N$. (The point is that this condition produces a nontrivial dependence on λ for each operator in the minimal product, and each operator depends only on one complex variable $<\alpha, \lambda>$. The poles of the operator are isolated, and the regularity follows for $A(N_M N : N_\lambda : \sigma : \lambda_M + \lambda)$. The same argument applies to $A(N_\lambda : N_M N : \sigma : \lambda_M + \lambda)$ to show it is regular. The product of these two operators is a scalar factor, and the scalar is not 0 for small λ by a third application of the minimal product decomposition. These facts prove $A(N_M N : N_\lambda : \sigma : \lambda_M + \lambda)$ is an isomorphism.)

We shall show that $<\alpha, \lambda> \neq 0$ if α is positive for N_λ and negative for $N_M N$. Write $\alpha = \alpha_R + \alpha_I$ with α_R defined on $\mathfrak{a}$ and α_I defined on $\mathfrak{a}_M$. The condition that α be positive for N_λ means that

$$<\alpha, \lambda + \lambda_M> \, > 0 \quad . \qquad (5.4)$$

The condition that α be negative for $N_M N$ means that $\alpha_R < 0$ or $\alpha_R = 0$ and $\alpha_I < 0$, i.e., that

$$<\alpha, \lambda> \, < 0 \qquad (5.5a)$$

or

$$<\alpha, \lambda> \, = 0 \quad \text{and} \quad <\alpha, \lambda_M> \, < 0 \quad . \qquad (5.5b)$$

If (5.5a) holds, then $<\alpha, \lambda> \neq 0$, as required. If (5.5b) holds, we obtain a contradiction to (5.4). Hence $<\alpha, \lambda> \neq 0$.

For the operator $A(\bar{N}_\lambda : \bar{N}_M \bar{N} : \sigma : \lambda_M + \lambda)$, it is enough to prove $<\alpha, \lambda> \neq 0$ when α is positive for $\bar{N}_M \bar{N}$ and negative for $\bar{N}_\lambda$. Then α is positive for N_λ and negative for $N_M N$, and we are reduced to the case of the previous paragraph.

Finally for $A(\bar{N} : N : \omega : \lambda)$, it is enough to prove $<\alpha, \lambda> \neq 0$ when α is an $\mathfrak{a}$ -root positive for N and negative for $\bar{N}$. This means that $<\alpha, \lambda> > 0$ holds, and so $<\alpha, \lambda> \neq 0$. The proof of Theorem 8 is complete.

<u>Example</u> : $SL(4, \mathbb{C})$. Let $P_0 = M_0 A_0 N_0$ be the minimal parabolic consisting of upper triangular matrices. We consider certain Langlands parameters $(P_0, 1, \nu)$.

In order for ν to be in the positive Weyl chamber and for (i) to hold in Theorem 7, we must have

$$\nu = u(e_1 - e_4) + v(e_2 - e_3) + wi(e_1 - e_2 - e_3 + e_4) \qquad (5.6)$$

with $u > v > 0$ and w real. The parameters that lead to ρ in the Corollary to Theorem 7 are $u = 3$, $v = 1$, $w = 0$. The complementary series occurs for $u < 1$, according to [12] or Theorem 9 of [8], and the parameters with $u = 1$ lead also to unitary representations (by a passage to the limit in (ii) of Theorem 7).

Let $P = MAN$ be given by

$$P = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot \end{pmatrix}$$

and consider the representations

$$\pi_t = \operatorname{ind}_{MAN}^{G} (1 \otimes \exp t(e_1 + e_2 - e_3 - e_4))$$

with $0 < t < 1$. In [15] Stein showed that these representations are infinitesimally unitary. Now the trivial representation of this M is not tempered but has Langlands parameters $(1, (e_1 - e_2) + (e_3 - e_4))$. One checks that the proof of Theorem 8 remains valid for $0 < t < 1$. Consequently π_t has Langlands parameters

$$(1,\ s(e_1 - e_2 + e_3 - e_4 + t(e_1 + e_2 - e_3 - e_4)))$$

with s a member of the Weyl group chosen to make $s(-)$ dominant. The relevant element s is the transposition $(2\,3)$, and the Langlands parameters are :

$$(1,\ (1+t)(e_1 - e_4) + (1-t)(e_2 - e_3)) \quad .$$

These parameters are the special case of (5.6) with $u = 1 + t$, $v = 1 - t$, and $w = 0$. In particular, these parameters have $u > 1$ and are outside the critical strip where the complementary series occur. Thus, Stein's exceptional unitary representations form a one-parameter family that extends from the edge of the three-parameter complementary series.

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