Proc. Natl. Acad. Sci. USA
Vol. 81, pp. 1272–1275, February 1984
Mathematics

Indefinite intertwining operators

(unitary representations/semisimple Lie groups)

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Communicated by H. Kesten, October 24, 1983

ABSTRACT **For a wide class of linear connected semisimple Lie groups, one obtains formulas limiting the Langlands parameters of irreducible unitary representations obtained from maximal parabolic subgroups. The formulas relate unitarity to the number of roots satisfying certain conditions. Some evidence is presented that the formulas are sharp. The results confirm aspects of conjectures that relate unitary parameters to cohomological induction.**

For a linear connected semisimple Lie group G it is known that the problem of classifying the irreducible unitary representations comes down to deciding which ones of certain standard representations in Hilbert spaces admit new inner products with respect to which the representations become unitary. These standard representations are obtained as the unique irreducible quotients of representations of a particular kind induced from parabolic subgroups. As such, they are parameterized, roughly speaking, by triples (S,σ,ν), where $S = MAN$ runs through finitely many parabolic subgroups of G, σ runs through a discrete family of representations of M, and ν is a continuous parameter on A.

We shall assume in this paper that G has a compact Cartan subgroup, that all noncompact roots are short, and that MAN is a maximal parabolic subgroup. Under these conditions we obtain some broad theorems limiting the values of ν that can correspond to unitary representations. Our theorems tie in with conjectures of Zuckerman and Vogan (ref. 1, p. 408, and ref. 2) concerning what parameters should lead to unitary representations. The extent to which we expect our theorems to be sharp is the subject of the last section. Some additional results valid when MAN is not maximal will be the subject of a separate paper. Proofs of all results will appear elsewhere.

Our method is to study the Hermitian operator that relates two Hermitian forms—the candidate for the new inner product and the naturally given inner product. This operator is a standard intertwining operator in the sense of ref. 3, and for certain values of ν we find finite-dimensional subspaces on which this operator is indefinite. Our method for calculating the operator is an old one and has been used extensively by Klimyk (e.g., ref. 4) for particular classical groups.

1. Setting

Let G be a connected linear semisimple Lie group, and let K be a maximal compact subgroup. We shall assume that a maximal torus B in K is maximal abelian in G and that every noncompact root is short. We use corresponding lowercase German letters to denote the Lie algebras of the corresponding Lie groups.

Let $S = MAN$ be the Langlands decomposition of a parabolic subgroup of G relative to K. By a *standard induced representation* we mean

$$U(S,\sigma,\nu) = \mathrm{ind}_S^G(\sigma \otimes e^{\nu} \otimes 1),$$

where σ is a discrete series or limit of discrete series representation of M acting in a space V^{σ}, ν is a complex-valued linear functional on $\mathfrak{a}$, and the induction has G acting on the left and is normalized so that imaginary ν yields a unitary U. [The existence of such a σ implies rank M = rank $(K \cap M)$; cf. ref. 5.] When Re ν is in the open positive Weyl chamber relative to N (or when Re ν is on the edge of the chamber and an additional condition listed in ref. 6 is satisfied), $U(S,\sigma,\nu)$ has a unique irreducible quotient $J(S,\sigma,\nu)$, the *Langlands quotient*. The Langlands quotients are the standard representations alluded to in the Introduction. It is known (6, 7) that these representations $J(S,\sigma,\nu)$ exhaust the candidates for unitary representations and that it is enough to decide which of them with ν real-valued can be made unitary.

Under our assumptions so far on G and S there exists an element w in K normalizing A such that $Ad(w)$ acts as -1 on $\mathfrak{a}$. This element has $w\sigma \cong \sigma$ for all σ and $w\nu = -\bar{\nu}$ for ν real-valued (8, 9). By ref. 10 the conditions $w\sigma \cong \sigma$, $w\nu = -\bar{\nu}$, and $Ad(w^2)|_{\mathfrak{a}} = 1$ imply there exists a nonzero invariant Hermitian form on the K-finite vectors of $J(S,\sigma,\nu)$. Under the additional condition on ν that makes J exist uniquely, this form is unique up to a scalar and can be lifted to $U(S,\sigma,\nu)$. Apart from one difficulty on the edge of the Weyl chamber, this form is given on K-finite vectors by the integral operator $\sigma(w)A_S(w,\sigma,\nu)$ defined in equations 0.1–0.2 of ref. 3. The difficulty is that this operator can have poles when Re ν is on the edge of the Weyl chamber, and the operator requires a normalizing factor in order to be well defined. After it is so normalized, it intertwines the representations $U(S,\sigma,\nu)$ and $U(S,\sigma,-\nu)$ and depends holomorphically on ν. For a real-valued ν, the result is that $J(S,\sigma,\nu)$ can be made unitary if and only if this normalized operator is semidefinite.

We shall use the notion of minimal K-type ["lowest K-type," in the sense of Vogan (11)] as a benchmark for normalizing the operators. The members of the induced space are determined by their restrictions to K, and we transfer the action of $U(S,\sigma,\nu)$ to the space of restrictions. In this realization the space of $U(S,\sigma,\nu)$ is independent of ν and K acts by the left-regular representation.

Fix a positive system of roots for K and form highest weights. Let $2\rho_K$ be the sum of the positive roots. Decomposing $U(S,\sigma,\nu)|_K$ into irreducible subspaces, write

$$U(S,\sigma,\nu)|_K = \sum n_{\Lambda'}\tau_{\Lambda'},$$

where $\tau_{\Lambda'}$ is the representation of K with highest weight Λ'. We say Λ (or really τ_Λ) is a *minimal K-type* of $U(S,\sigma,\nu)$ if $|\Lambda' + 2\rho_K|^2$ is minimized among Λ' with $n_{\Lambda'} \neq 0$ by $\Lambda' = \Lambda$. Minimal K-types exist, and the representations of K so defined are independent of the choice of positive system. From the work of Vogan (1, 11) one knows that

(*i*) τ_Λ has multiplicity one in $U(S,\sigma,\nu)$

(*ii*) τ_Λ occurs in $J(S,\sigma,\nu)$.

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Consequently, the intertwining operator from $U(S,\sigma,\nu)$ to $U(S,\sigma,-\nu)$ is nonvanishing and scalar on the τ_Λ space and hence can be normalized to be 1 on the τ_Λ space; in this case the normalized operator is pole-free for Re ν in the closed positive Weyl chamber. Fixing a minimal K-type Λ for $U(S,\sigma,\nu)$, we normalize the interwining operator in this fashion and call the result $T(\nu)$. To disprove unitarity we seek a single $\tau_{\Lambda'}$ so that $T(\nu)$ is not positive semidefinite on the $\tau_{\Lambda'}$ subspace. Then $T(\nu)$ will be indefinite on the sum of the τ_Λ and $\tau_{\Lambda'}$ subspaces.

Klimyk's approach to this problem is to use the intertwining property

$$T(\nu)U(S,\sigma,\nu) = U(S,\sigma,-\nu)T(\nu) \qquad [1]$$

to get recursion relations for $T(\nu)$ on the various K-types. In our case we shall use K-types that are suitably close to our minimal K-type Λ, and we illustrate the style of argument by the case of a K-type Λ' that is "one step away from Λ." Let f be a member of the τ_Λ subspace with weight Λ, let $P^K_{\Lambda'}$ be the projection to the K-type $\tau_{\Lambda'}$, and let P^B_ω be the projection to the weight ω under right translation. For X a root vector in $\mathfrak{g}^{\mathbf{C}}$, we apply Eq. **1** to $P^K_{\Lambda'}U(S,\sigma,\nu,X)f$. If $\tau_{\Lambda'}$ occurs in $U(S,\sigma,\nu)$ with multiplicity one, then $T(\nu)$ acts as a scalar $c(\nu)$ on the $\tau_{\Lambda'}$ space and Eq. **1** gives

$$c(\nu)P^K_{\Lambda'}U(S,\sigma,\nu,X)f = P^K_{\Lambda'}U(S,\sigma,-\nu,X)f.$$

For any ν' in V^σ and any weight ω, we therefore have

$$c(\nu)P^B_\omega\langle(P^K_{\Lambda'}U(S,\sigma,\nu,X)f)(k),\nu'\rangle_{V^\sigma} = P^B_\omega\langle(P^K_{\Lambda'}U(S,\sigma,-\nu,X)f(k),\nu'\rangle_{V^\sigma}. \qquad [2]$$

For a particular choice of Λ' and X, we shall choose ν' and ω so that

$$P^B_\omega\langle(P^K_{\Lambda'}U(S,\sigma,\nu,X)f)(k),\nu'\rangle_{V^\sigma} = a(\nu)I(k) \qquad [3]$$

with $a(\nu)$ and $I(k)$ not identically 0. Under the assumption that $\tau_{\Lambda'}$ has multiplicity one, Eq. **2** gives

$$c(\nu) = a(-\nu)/a(\nu),$$

and it is a simple matter to read off values of ν for which $c(\nu)$ is negative and hence $T(\nu)$ is indefinite.

In the remainder of this paper, we shall assume that $S = MAN$ is maximal parabolic—i.e., dim $\mathfrak{a} = 1$. In this context let us pin down the orderings on roots that we shall use. Let $\Delta = \Delta(\mathfrak{g}^{\mathbf{C}}, \mathfrak{b}^{\mathbf{C}})$ be the set of roots of G, $\Delta_K = \Delta(\mathfrak{k}^{\mathbf{C}}, \mathfrak{b}^{\mathbf{C}})$ the subset of compact roots, and $\Delta_n = \Delta - \Delta_K$ the set of noncompact roots. If α is a root, we normalize H_α in $\mathfrak{b}^{\mathbf{C}}$ and the root vector X_α as in ref. 12: in particular, if α is noncompact, the α Cayley transform $\tilde{\alpha}$ has $\tilde{\alpha}(X_\alpha + X_{-\alpha}) = 2$.

Under our assumption that MAN is maximal, we may assume that $\mathfrak{a} = \mathbf{R}(X_\alpha + X_{-\alpha})$ for some noncompact root α. Let $\mathfrak{b}_-$ be the compact Cartan subalgebra of $\mathfrak{m}$ defined by the orthogonal decomposition $\mathfrak{b} = \mathfrak{b}_- \oplus \mathbf{R}iH_\alpha$. It is known (13) that σ is induced from a representation $\sigma^\sharp$ of the subgroup $M^\sharp = M_0Z_M$, the product of the identity component of M and the center of M. Let λ_0 be the Harish-Chandra parameter (5) of $\sigma^\sharp$ on $\mathfrak{b}_-$; this parameter determines a positive system $(\Delta_-)^+$ for the roots

$$\Delta_- = \{\gamma \in \Delta | \gamma \perp \alpha\}$$

of M. Let λ be the minimal $(K \cap M^\sharp)$-type of $\sigma^\sharp$ given on $\mathfrak{b}_-$ by

$$\lambda = \lambda_0 - \rho_{-,c} + \rho_{-,n}.$$

(Here $\rho_{-,c}$ and $\rho_{-,n}$ refer to half-sums of positive compact and noncompact roots of M, and the formula for λ is proved in ref. 14.) Ref. 15 describes a procedure for introducing a particular positive system Δ^+ containing $(\Delta_-)^+$ such that α is simple for Δ^+. With $\Delta_K^+ = \Delta^+ \cap \Delta_K$, the minimal K-types of $U(S,\sigma,\nu)$ are given by

$$\Lambda = \lambda - \frac{\langle 2\rho_K,\alpha\rangle}{|\alpha|^2}\,\alpha + \mu, \qquad [4]$$

according to ref. 15. Here μ is $\pm\frac{1}{2}\alpha$ or 0, depending on a parity condition for σ. In the first instance (called the "cotangent case") at least one of $+\frac{1}{2}\alpha$ and $-\frac{1}{2}\alpha$ gives a Δ_K^+ dominant Λ, and the minimal K-types are the dominant Λ's. In the second instance (called the "tangent case") $\mu = 0$ gives the unique minimal K-type. In every case $\Lambda|_{\mathfrak{b}_-}$ equals λ, and a highest weight for τ_Λ is highest of type λ for $K \cap M^\sharp$.

2. Formulas

Continuing with the notation of §1, we let α be the noncompact Δ^+ simple root defining $\mathfrak{a}$. Let Λ be a minimal K-type of $U(S,\sigma,\nu)$, and let μ be the last term of Λ in Eq. **4**. With $(\cdot)^\vee$ meaning "$(\cdot)$ made dominant relative to Δ_K^+," we define $\Lambda_1^+ = (\Lambda + \alpha)^\vee$ and $\Lambda_1^- = (\Lambda - \alpha)^\vee$. It follows from our assumption on root lengths for G that $\Lambda_1^+ = \Lambda + \delta^+$ and $\Lambda_1^- = \Lambda + \delta^-$ for noncompact roots δ^+ and δ^- that are positive for Δ^+ and $s_\alpha\Delta^+$, respectively. (Here s_α is the reflection in α.)

By the δ^+ *subgroup of G* we mean the semisimple group built from all simple roots of Δ^+ needed for the expansion of δ^+ in terms of Δ^+ simple roots. The δ^- *subgroup of G* is defined similarly in terms of δ^- and $s_\alpha\Delta^+$.

Let $\Delta_{K,\perp}$ be the subsystem of Δ_K orthogonal to Λ, and let $W_{K,\perp}$ be its Weyl group.

THEOREM 1. (a) *If Λ_1^+ has multiplicity one in* U(S,σ,ν), *then* J(S,σ, ν) *is not unitary for real-valued ν satisfying*

$$a < \nu(X_\alpha + X_{-\alpha}),$$

where

$$a = 1 + \frac{2\langle\mu,\alpha\rangle}{|\alpha|^2} + 2\#\{\beta \in \Delta_n^+ | \beta - \alpha \in \Delta \text{ and } \beta - \alpha \perp \Lambda\}. \qquad [5]$$

The K-*type Λ_1^+ is of multiplicity one when the δ^+ subgroup of* G *has real rank one and $\langle\Lambda,\alpha\rangle \le 0$.*

(b) *If Λ_1^- has multiplicity one in* U(S,σ,ν), *then* J(S,σ,ν) *is not unitary for real-valued ν satisfying*

$$a' < \nu(X_\alpha + X_{-\alpha}),$$

where

$$a' = 1 - \frac{2\langle\mu,\alpha\rangle}{|\alpha|^2} + 2\#\{\beta \in \Delta_n^+ | \beta + \alpha \in \Delta \text{ and } \beta + \alpha \perp \Lambda\}. \qquad [6]$$

The K-*type Λ_1^- is of multiplicity one when the δ^- subgroup of* G *has real rank one and $\langle\Lambda,\alpha\rangle \ge 0$.*

Further circumstances under which Λ_1^+ and Λ_1^- have multiplicity one are given in §3.

The second theorem gives extra information in the special case that $\Lambda_1^+ = \Lambda_1^-$. We write Λ_1 for their common value and let δ be the common value of δ^+ and δ^-.

THEOREM 2. *Suppose α and $-\alpha$ are conjugate via* $W_{K,\perp}$. *If the three conditions*

(i) $\langle\delta,\alpha\rangle < 0$,
(ii) *the semisimple group built from all simple roots of Δ^+ needed for the expansion of $\delta_1 + \delta$ has real rank one, where δ_1 is the root in Δ_n^+ for which $(\Lambda_1 + \alpha)^\vee = \Lambda_1 + \delta_1$, and*
(iii) $\Lambda \not\perp \beta$ *for every β in Δ_n strongly orthogonal to α,*

are satisfied, then the K*-types Λ_1 and $(\Lambda_1 + \alpha)^\vee$ have multiplicity one in* $U(S,\sigma,\nu)$, *and* $J(S,\sigma,\nu)$ *is not unitary for real-valued ν satisfying*

$$a < \nu(X_\alpha + X_{-\alpha})$$

or

$$b < \nu(X_\alpha + X_{-\alpha}) < a,$$

where

$$a = 1 + 2\#\{\beta \in \Delta_n^+ | \beta + \alpha \in \Delta \quad \textit{and} \quad \Lambda \perp \beta\} \qquad [7]$$

and

$$b = a - 2\#\{\beta \in \Delta_n^+ | \beta \perp \alpha, \beta \textit{ not strongly } \perp \alpha, \textit{and } \Lambda \perp \beta\}. \qquad [8]$$

Some remarks may be helpful in connection with *Theorem 2*. The conjugacy of α and $-\alpha$ implies that $\mu = 0$ and $\Lambda \perp \alpha$. Moreover, the constants a and a' in Eqs. **5** and **6** are equal and their common value is that in Eq. **7**. As long as $\langle\delta,\alpha\rangle \neq 0$, condition *i* may be viewed as a normalization, since we can replace Δ^+ by $s_\alpha\Delta^+$ if necessary. *Theorem 2* suggests that the unitary parameters may have a gap with an isolated unitary representation where $\nu(X_\alpha + X_{-\alpha}) = a$. Eq. **8** gives a root-theoretic interpretation for the width of the gap.

While *Theorem 1* uses a "one-step" formula like that in Eq. **3**, *Theorem 2* is more complicated. It comes about by examining the effect of moving two steps by a quadratic element of the universal enveloping algebra of $\mathfrak{g}$, from Λ through Λ_1 to $(\Lambda_1 + \alpha)^\vee$.

When G has real rank one, the hypotheses of one or the other of *Theorems 1* and *2* are always satisfied, and the combined results are sharp (cf. ref. 16). In applying *Theorem 1* when Λ_1^+ and Λ_1^- both have multiplicity one, the correct bound on unitarity is the smaller of a and a'. For further commentary on the sharpness of the theorems, see §4.

3. Multiplicity Results

In this section we give more detailed information about the multiplicities that enter the hypotheses of *Theorem 1*. We use the notation $[\tau_\omega|_H:\tau_{\omega'}]$ for the multiplicity of the representation $\tau_{\omega'}$ in the restriction of τ_ω to the subgroup H, and we write $[\tau_\omega:\varepsilon]$ for the multiplicity of the weight ε in the representation τ_ω.

Let us write

$$\sigma^\sharp|_{K\cap M^\sharp} = \sum_{\lambda'} n_{\lambda'}\tau_{\lambda'}$$

as the decomposition into $(K \cap M^\sharp)$-types. By the double induction formula, $U(S,\sigma,\nu)$ is induced from $\sigma^\sharp \otimes e^\nu \otimes 1$ on $M^\sharp AN$. Therefore, Frobenius reciprocity gives

$$[U(S,\sigma,\nu)|_K\colon \tau_{\Lambda'}] = \sum_{\lambda'} n_{\lambda'}[\tau_{\Lambda'}|_{K\cap M^\sharp}\colon \tau_{\lambda'}]$$

for any K-type $\tau_{\Lambda'}$.

THEOREM 3. *If the δ^+ subgroup of* G *has real rank one, then $[\tau_{\Lambda_1^+}|_{K\cap M^\sharp}\colon \tau_{\lambda'}] = 0$ for all $\lambda' \neq \lambda$; in this case*

$$[U(S,\sigma,\nu)|_K\colon \tau_{\Lambda_1^+}] = [\tau_{\Lambda_1^+}|_{K\cap M^\sharp}\colon \tau_\lambda].$$

Similarly, if the δ^- subgroup of G *has real rank one, then $[\tau_{\Lambda_1^-}|_{K\cap M^\sharp}\colon \tau_{\lambda'}] = 0$ for all $\lambda' \neq \lambda$; in this case*

$$[U(S,\sigma,\nu)|_K\colon \tau_{\Lambda_1^-}] = [\tau_{\Lambda_1^-}|_{K\cap M^\sharp}\colon \tau_\lambda].$$

THEOREM 4. *Suppose α and $-\alpha$ are not conjugate via* $W_{K,\perp}$. *Then*

(a) $[\tau_{\Lambda_1^+}\colon \Lambda + \alpha] = 1$

(b) $1 \leq [\tau_{\Lambda_1^+}|_{K\cap M^\sharp}\colon \tau_\lambda] \leq 1 + \sum_{n\geq 0} [\tau_{\Lambda_1^+}\colon \Lambda - (2n + 1)\alpha]$

(c) $[\tau_{\Lambda_1^+}|_{K\cap M^\sharp}\colon \tau_\lambda] = 1$ *if* $\langle\Lambda,\alpha\rangle \leq 0$

(d) *the* nth *term in the sum on the right side of* b *is* 0 *for* n > 0 *if the δ^+ subgroup of* G *is classical. If, in addition, $2\delta^+$ is not a sum of members of Δ_K^+, then the* 0th *term is* 0.

(e) *results analogous to* a–d *are valid for* Λ_1^-.

THEOREM 5. *Suppose α and $-\alpha$ are conjugate via* $W_{K,\perp}$. *Then*

(a) $[\tau_{\Lambda_1}\colon \Lambda + \alpha] = [\tau_{\Lambda_1}\colon \Lambda - \alpha] = 1$

(b) $1 \leq [\tau_{\Lambda_1}|_{K\cap M^\sharp}\colon \tau_\lambda] \leq 2$

(c) $[\tau_{\Lambda_1}|_{K\cap M^\sharp}\colon \tau_\lambda] = 1$ *if the δ subgroup of* G *has real rank one.*

4. Cohomological Induction and Sharpness of Results

We conclude by addressing the extent to which our formulas are sharp. We can attach to $\sigma^\sharp$ the tuple $(\{\alpha\},\Delta^+,\sigma^\sharp|_{Z_M},\mu)$ with definitions as in §1. Such a tuple is called a *format* in ref. 2. That paper examines all parameters λ_0 that can lead to a particular format when G^C is simply connected and shows there is a unique smallest one $\lambda_{0,b}$, which is called the *basic case* for the format. (A formula for $\lambda_{0,b}$ is given in ref. 17.) The difference $\lambda_0 - \lambda_{0,b}$ is dominant for Δ^+, and ref. 2 defines a reductive subgroup L of G depending on λ_0 and having root system

$$\Delta^L = \{\beta \in \Delta | \langle\lambda_0 - \lambda_{0,b},\beta\rangle = 0\}.$$

The system Δ^L is spanned by Δ^+ simple roots.

The relevance of L is its connection with the theory of cohomological induction presented in ref. 1. One expects cohomological induction to carry unitary representations to unitary representations, and this L is thought to be large enough so that a suitable converse is valid. However, cohomological induction plays no role in the statements of our results, which simply deal with parameters. The starting point for using L is the following lemma.

LEMMA. *The set $\Delta_{K,\perp}$ is contained in Δ^L. Therefore, the δ^+ and δ^- subgroups of* G *are contained in* L, *and all the roots β in Eqs.* **5** *and* **6** *or* **7** *and* **8** *lie in Δ^L.*

If L has real rank one and α is conjugate to $-\alpha$ via $W_{K,\perp}$ and condition *i* holds in *Theorem 2*, then the subgroup in condition *ii* of that theorem (built from $\delta + \delta_1$) also is contained in L. Bringing in the Speh–Vogan theory (18), we obtain the following result.

THEOREM 6. *Suppose* G *is classical,* σ *is a genuine discrete series representation of* M, *and* L *is of real rank one.*

(a) *If* α *and* $-\alpha$ *are not conjugate via* $W_{K,\perp}$ *or if* $\langle\delta,\alpha\rangle = 0$, *then* $J(S,\sigma,\nu)$ *is unitary for*

$$0 < \nu(X_\alpha + X_{-\alpha}) \leq \min\{a,a'\},$$

with a *and* a′ *as in Eqs.* **5** *and* **6**, *and is not unitary for*

$$\min\{a,a'\} < \nu(X_\alpha + X_{-\alpha}).$$

(b) *If* α *and* $-\alpha$ *are conjugate via* $W_{K,\perp}$ *and if conditions* i *and* iii *hold in Theorem 2, then* $J(S,\sigma,\nu)$ *is unitary for*

$$0 < \nu(X_\alpha + X_{-\alpha}) \leq b,$$

not unitary for

$$b < \nu(X_\alpha + X_{-\alpha}) < a,$$

and not unitary for

$$a < \nu(X_\alpha + X_{-\alpha}).$$

Here a *and* b *are as in Eqs.* **7** *and* **8**.

When G is exceptional (satisfying our conditions) and L is of real rank one, there are finitely many discrete series of M that we cannot handle as in *Theorem 6*.

For the group $SU(p,q)$, α is never conjugate to $-\alpha$, and Λ_1^+ and Λ_1^- have multiplicity one if the respective δ subgroups have real rank one. If σ is a discrete series representation and L is not of real rank one, then L is of a very special form that is easily handled. As a consequence, *Theorem 6* yields the following result.

THEOREM 7. *If* $G = SU(p,q)$ *and* σ *is a discrete series representation of* M, *then* $J(S,\sigma,\nu)$ *is unitary for*

$$0 < \nu(X_\alpha + X_{-\alpha}) \leq \min\{a,a'\},$$

with a *and* a′ *as in Eqs.* **5** *and* **6**, *and is not unitary for*

$$\min\{a,a'\} < \nu(X_\alpha + X_{-\alpha}).$$

M.W.B.-S. was partially supported by National Science Foundation Grant MCS81-08814 at the Institute for Advanced Study (Princeton, NJ). A.W.K. was supported by a Guggenheim Fellowship and National Science Foundation Grant MCS80-01854.

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