INDEFINITE INTERTWINING OPERATORS II *

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For an irreducible admissible representation of a semisimple Lie group, there is at most one invariant Hermitian form (up to scalar multiples), hence only one way the representation has a chance of being unitary. When such a representation is realized concretely by means of the Langlands classification [14, 13], this Hermitian form is given by an explicit intertwining operator [8]. Showing this operator is indefinite proves the representation cannot be unitary.

In [1] we introduced a technique for showing this operator is indefinite without actually computing the operator. The technique is based on an old idea that has been used extensively by Klimyk, often in collaboration with Gavrilik, for particular classical groups (see, e.g., [5]). It takes advantage of the intertwining property of the operator to relate the behavior on one subspace to that on another.

The scope of [1] was repeatedly limited by a certain multiplicity-one assumption. We are now able to drop this assumption, and consequently we can obtain significant generalizations of our earlier theorems.

The plan of the paper is as follows. In §1 we recall the technique of [1] and show how to modify it to eliminate the multiplicity-one assumption. In §2 we announce some general theorems that apply the modified technique. While all the results of [1] concerned representations constructed from maximal parabolic subgroups of semisimple groups, the new results give information about representations constructed from many nonmaximal parabolic subgroups. The proofs of these results are too long to include now and will be given elsewhere.

To illustrate the power of our theorems, we state in §3 and prove in

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§§4 - 6 a complete classification of the irreducible unitary representations of the groups $SU(p, 2)$ of complex matrices of determinant one that preserve a Hermitian form with p plus signs and 2 minus signs. In addition to our theorems, this classification makes use of the techniques and results of [9], irreducibility theorems of Speh and Vogan [16], results of Jakobsen [4] and Enright-Howe-Wallach [2] on unitary representations that have highest weight vectors, and a powerful theorem of Vogan [19] on preservation of unitarity under cohomological induction. For the most part, the application of our theorems works equally well for $SU(p, q)$, as we shall see during the argument.

We shall assume throughout the paper that our semisimple group has a compact Cartan subgroup and that all noncompact roots are short. We have preliminary results to indicate that both these assumptions are unimportant, and we shall report on this matter on a later occazion.

1. BACKGROUND AND TECHNIQUE

Let G be a linear connected semisimple Lie group, let K be a maximal compact subgroup, and let $S = MAN$ be a parabolic subgroup whose subgroup M possesses discrete series representations. We denote by $U(S, \sigma, \nu) = U(S, \sigma, \nu, \cdot)$ the induced representation

$$U(S, \sigma, \nu) = \mathrm{ind}_S^G(\sigma \otimes e^{\nu} \otimes 1), \tag{1.1}$$

where σ is a discrete series or limit of discrete series representation of M on a space V^{σ} and ν is a complex-valued linear functional on the Lie algebra $\mathfrak{a}$ of A. In (1.1.), the induction is normalized or «unitary» induction and G is to act on the left. The space in which these representations act may be regarded as a space of V^{σ}-valued square integrable functions on K that is independent of ν.

When $Re\,\nu$ is in the open positive Weyl chamber relative to N (or when $Re\,\nu$ is on the edge of the chamber and an additional condition listed in [12] is satisfied), $U(S, \sigma, \nu)$ has a unique irreducible quotient $J(S, \sigma, \nu)$, the *Langlands quotient*. It is known [14, 13, 8] that these representations $J(S, \sigma, \nu)$ exhaust the candidates for irreducible unitary representations, even when σ is limited to be nondegenerate in the sense of [13]. Moreover, it is enough to decide which of them with ν real-valued can be made unitary.

We assume that G has a compact Cartan subgroup B (with Lie algebra $\mathfrak{b}$) and that all noncompact roots are short. Under just the first of these assumptions, there exists an element w_0 in K normalizing A such that Ad (w_0) acts as -1 on $\mathfrak{a}$. This element has $w_0\sigma \cong \sigma$ for all σ and $w_0\nu = -\overline{\nu}$ for all real-

-valued ν. When ν satisfies the conditions that make $J(S, \sigma, \nu)$ exist uniquely, then [13] shows that the existence of w in K normalizing A such that $\mathrm{Ad}(w^2)|_{\mathfrak{a}} = 1$, $w\,\sigma \cong \sigma$, and $w\nu = -\overline{\nu}$ is equivalent with the existence of a nonzero invariant Hermitian form on the K-finite vectors of $J(S, \sigma, \nu)$. This form lifts to $U(S, \sigma, \nu)$. Apart from one difficulty when $Re\,\nu$ is on the edge of the positive Weyl chamber, this form is necessarily given on K-finite vectors by a multiple of the form

$$(1.2) \qquad \langle f, g\rangle = (\sigma(w)A_S(w, \sigma, \nu)f, g)_{L^2(K)},$$

where $\sigma(w)A_S(w, \sigma, \nu)$ is the convergent integral intertwining operator defined explicitly in equations (0.1) and (0.2) of [10]. The difficulty is that the integral operator can have poles when $Re\,\nu$ is on the edge of the Weyl chamber, and the operator requires normalization to be well defined. After it is so normalized, it intertwines $U(S, \sigma, \nu)$ and $U(S, \sigma, w\nu)$ and depends holomorphically on ν for $Re\,\nu$ in the closure of the positive Weyl chamber. For ν satisfying $w\nu = -\overline{\nu}$, the result is that $J(S, \sigma, \nu)$ can be made unitary if and only if the normalized version of (1.2) is semidefinite, if and only if the normalized operator is semidefinite.

As we have said, it is enough to consider real-valued ν. Then $w_0\nu = -\overline{\nu}$, and the above considerations apply. We seek conditions on the real-valued parameter ν so that the normalized operator corresponding to w_0 is indefinite.

We begin as in [1]. To normalize the operator, we first fix a minimal K-type τ_Λ of $U(\nu) = U(S, \sigma, \nu)$ with highest weight Λ in $(i\mathfrak{b})'$, i.e., an irreducible representation of K that occurs in $U(\nu)|_K$ and is minimal in the sense of Vogan [17]. The intertwining operator is scalar on the τ_Λ subspace since τ_Λ is known to have multiplicity one in $U(\nu)$ and since K acts by translations, and we normalize the intertwining operator so as to be the identity on this K-type for all ν. Let $T(\nu)$ be the normalized operator. Then $T(\nu)$ is known to be real analytic for ν in the closure of the positive Weyl chamber. Since $T(\nu)$ is positive definite on the τ_Λ subspace, it will follow that $J(S, \sigma, \nu)$ is not unitary whenever we can produce a K-type $\tau_{\Lambda'}$ such that $T(\nu)$ fails to be positive semidefinite on the $\tau_{\Lambda'}$ subspace.

We recall the technique of [1]. The intertwining identity satisfied by $T(\nu)$ is

$$(1.3) \qquad U(-\nu, X)\,T(\nu) = T(\nu)\,U(\nu, X) \quad \text{for} \quad X \in \mathfrak{g}^{\mathbb{C}},$$

where $\mathfrak{g}$ is the Lie algebra of G. Let τ_{Λ_1} be an irreducible representation of K and define P_{Λ_1} to be the projection of the induced space to the τ_{Λ_1} subspace:

$$(1.4)\qquad P_{\Lambda_1} f(k_0) = \mathrm{d}_{\Lambda_1}\int_K \overline{\chi_{\Lambda_1}(k)}\, f(k^{-1}k_0)\,\mathrm{d}k.$$

Here d_{Λ_1} is the degree of τ_{Λ_1}, and χ_{Λ_1} is the character. Also if h is any scalar-valued function on K and ω is an integral form on $\mathfrak{b}$, we let h_ω be the $-\omega$ Fourier component of h under the action of B on the right:

$$h(k)_\omega = \int_B h(kb)\,\xi_\omega(b)\,\mathrm{d}b,$$

where ξ_ω is the character of B corresponding to ω.

Fix f_0 in the induced space to be a nonzero highest weight vector for the minimal K-type τ_Λ, fix ω integral on $\mathfrak{b}$, and let u be in the representation space V^σ of σ. Let $\tau_{\Lambda_1}, \dots, \tau_{\Lambda_n}$ be representations of K, let $X_1, \dots, X_n$ be in $\mathfrak{g}^{\mathbb{C}}$, and form

$$(1.5)\qquad a(\nu, k) = \langle (P_{\Lambda_n} U(\nu, X_n) P_{\Lambda_{n-1}} \dots P_{\Lambda_1} U(\nu, X_1) f_0(k), u\rangle_\omega,$$

the inner product being taken in V^σ. If τ_{Λ_n} has multiplicity one in $U(\nu)$, then $T(\nu)$ acts as a scalar, say $c(\nu)$, on the τ_{Λ_n} subspace. Since $T(\nu)$ commutes with each P_{Λ_i}, it follows from (1.3) that

$$c(\nu) P_{\Lambda_n} U(\nu, X_n) P_{\Lambda_{n-1}} \dots P_{\Lambda_1} U(\nu, X_1) f_0 = P_{\Lambda_n} U(-\nu, X_n) P_{\Lambda_{n-1}} \dots P_{\Lambda_1} U(-\nu, X_1) f_0.$$

Evaluating at k, taking the inner product with u, and projecting by ω, we obtain

$$(1.6)\qquad c(\nu) = a(-\nu, k)/a(\nu, k),$$

provided the denominator is not identically zero. The expression $a(\nu, k)$ does not involve the intertwining operator, and we can conclude that $J(S, \sigma, \nu)$ is not unitary whenever the right side of (1.6) can be shown to be negative.

This was the technique in [1], and we now modify it. Without assuming τ_{Λ_n} is of multiplicity one, define

$$F(\nu) = (T(\nu) P_{\Lambda_n} U(\nu, X_n) P_{\Lambda_{n-1}} \dots P_{\Lambda_1} U(\nu, X_1) f_0,\ P_{\Lambda_n} U(\nu, X_n) P_{\Lambda_{n-1}} \dots P_{\Lambda_1} U(\nu, X_1) f_0)_{L^2(K)}.$$

Certainly $J(S, \sigma, \nu)$ fails to be unitary if $F(\nu)$ is negative. We use (1.3) to commute $T(\nu)$ in so as to act on f_0 and then go away. Next we use the adjoint relation

$$U(\nu, X)^* = U(-\nu, -\overline{X})$$

to move all the operators to the left member of the inner product. Then we have

$$\begin{aligned} F(\nu) = (P_\Lambda U(-\nu, -\overline{X}_1)P_{\Lambda_1} \dots P_{\Lambda_{n-1}} U(-\nu, -\overline{X}_n)P_{\Lambda_n} \\ U(-\nu, X_n)P_{\Lambda_{n-1}} \dots P_{\Lambda_1} U(-\nu, X_1) f_0, f_0)_{L^2(K)}. \end{aligned}$$

Let f be the left member of this inner product. Then f is in the τ_Λ subspace of the induced space. Moreover, if we define Pf to be the projection of f according to the weight Λ, namely

$$Pf(k) = \int_B f(b^{-1}k)\,\overline{\xi_\Lambda(b)}\,\mathrm{d}b,$$

then Pf is a weight vector of weight Λ in the τ_Λ subspace. Since the minimal K-type τ_Λ has multiplicity one, Pf must be a multiple of f_0. Evaluating the multiple by means of an inner product, we obtain

$$Pf = \|f_0\|^{-2} F(\nu) f_0.$$

If v_0 denotes a nonzero highest weight vector in an abstract representation space V^Λ of K of type τ_Λ, then f_0 is necessarily of the form

$$f_0(k) = A\tau_\Lambda(k)^{-1} v_0 \tag{1.7}$$

for a unique operator A in $\mathrm{Hom}_{K \cap M}(V^\Lambda, V^\sigma)$. Under natural conditions that we shall impose on MAN and the positive system of roots, there exists a special element u_0 in V^σ with $A^* u_0 = v_0$.[1] Define

$$\begin{aligned} b(\nu, k) = \langle P_\Lambda U(\nu, \overline{X}_1)P_{\Lambda_1} \dots P_{\Lambda_{n-1}} U(\nu, \overline{X}_n)P_{\Lambda_n} \\ U(\nu, X_n)P_{\Lambda_{n-1}} \dots P_{\Lambda_1} U(\nu, X_1) f_0(k), u_0\rangle_\Lambda. \end{aligned} \tag{1.8}$$

(Notice that $b(\nu, k)$ is a special case of the expression $a(\nu, k)$ as defined in (1.5).) On the one hand,

$$b(\nu, b_0) = \overline{\xi_\Lambda(b_0)}\, b(\nu, 1)$$

for b_0 in B because we are taking the $-\Lambda$ component in (1.8). Hence the equality

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[1] To be quite precise we must work with $\sigma^\#$ and $M^\#$ as defined in §2 below and take A in $\mathrm{Hom}_{K \cap M^\#}(V^\Lambda, V^{\sigma\#})$. This point is a small one for now, and we shall ignore it until §2.

$$\left\langle \int_B f(b_0^{-1}k)\,\overline{\xi_\Lambda(b_0)}\,\mathrm{d}b_0, u_0 \right\rangle_\Lambda = (-1)^n \int_B b(-\nu, b_0^{-1}k)\,\overline{\xi_\Lambda(b_0)}\,\mathrm{d}b_0$$

reduces at $k = 1$ to

$$\langle Pf(1), u_0\rangle_\Lambda = (-1)^n b(-\nu, 1).$$

On the other hand, $Pf = \|f_0\|^{-2} F(\nu) f_0$ says

$$\langle Pf(1), u_0\rangle_\Lambda = \|f_0\|^{-2}F(\nu)\langle f_0(1), u_0\rangle = \|f_0\|^{-2}F(\nu)\langle Av_0, u_0\rangle = \|f_0\|^{-2}F(\nu)\,|\,v_0|^2.$$

Thus $F(\nu)$ is a positive multiple of $(-1)^n b(-\nu, 1)$. We conclude that $J(S, \sigma, \nu)$ cannot be made unitary if $(-1)^n b(-\nu, 1)$ is negative.

2. GENERAL THEOREMS

In this section we shall give a lemma and four theorems for calculating $b(\nu, k)$ in (1.8) in a number of situations. The first two theorems are intended for use in an inductive calculation, proceeding one step and two steps at a time, respectively. The third theorem could perhaps be stated in an inductive framework as well, but we prefer to state it more narrowly now. These three results together are what are needed from our technique, apart from theorems in [1], to classify the irreducible unitary representations of $SU(p, 2)$.

The final result in this section is of a different nature; it gives identities for simplifying the formulas produced by the first three theorems. Its significance will be explained in remarks at the end of the section.

We begin by fixing the orderings that we shall use. Let $\Delta = \Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{b}^{\mathbb{C}})$ be the set of roots of $\mathfrak{g}$, and let Δ_K and Δ_n be the subsets of compact and noncompact roots. If α is a root, we normalize H_α in $\mathfrak{b}^{\mathbb{C}}$ and the root vectors X_α and $X_{-\alpha}$ as in [11]; for α noncompact, this normalization is such that the α Cayley transform $\tilde{\alpha}$ has $\tilde{\alpha}(X_\alpha + X_{-\alpha}) = 2$.

Fix a nonempty ordered set $\alpha_1, \ldots, \alpha_l$ of noncompact roots that are *superorthogonal* in the sense that no nontrivial linear combination of the α_j is a root. Define

$$\mathfrak{a} = \sum_{j=1}^{l} \mathbb{R}(X_{\alpha_j} + X_{-\alpha_j}),$$

and use the lexicographic ordering from the ordered basis

$$X_{\alpha_1} + X_{-\alpha_1}, \ldots, X_{\alpha_l} + X_{-\alpha_l}$$

to define a notion of positivity. Using this $\mathfrak{a}$ and this notion of positivity, we can construct a parabolic subgroup MAN in the usual way, and MAN will be *cuspidal* in the sense that $\operatorname{rank} M = \operatorname{rank}(K \cap M)$. This kind of parabolic subgroup will not be the most general cuspidal parabolic subgroup in G, even after account is taken of the usual equivalences. For example, a minimal parabolic subgroup of $SO(4, 4)$ is not of this kind. The most general cuspidal parabolic subgroup would arise if the noncompact roots $\alpha_1, \ldots, \alpha_l$ were assumed merely to be strongly orthogonal (no $\alpha_i \pm \alpha_j$ in Δ).

Let $\overline{\rho}$ be half the sum, with multiplicities counted, of the roots of $(\mathfrak{g}, \mathfrak{a})$ that are positive relative to N.

Let $\mathfrak{b}_-$ be the common kernel of the α_j's in $\mathfrak{b}$. Then $\mathfrak{b}_-$ is a compact Cartan subalgebra of the Lie algebra $\mathfrak{m}$ of M, and

$$\Delta_- = \{\gamma \in \Delta \mid \gamma \perp \alpha_j \quad \text{for all } j\}$$

may be regarded as the root system of $(\mathfrak{m}^{\mathbb{C}}, \mathfrak{b}_-^{\mathbb{C}})$. It is known that our given discrete series or limit σ of M is induced from a discrete series or limit $\sigma^{\#}$ of the subgroup $M^{\#} = M_0 Z_M$, the product of the identity component and the center of M. Moreover, Lemma 2.1c of [13] implies that $\sigma^{\#}$ is determined by its Harish-Chandra parameter $(\lambda_0, (\Delta_-)^+)$ and its scalar value on each element $\gamma_{\alpha_j} = \exp \pi i H_{\alpha_j}$ of Z_M. (Here we use the superorthogonality of the α_j's).

Let λ be the minimal $(K \cap M^{\#})$-type of $\sigma^{\#}$ given on $\mathfrak{b}^-$ by

$$\lambda = \lambda_0 - \rho_{-,c} + \rho_{-,n},$$

where $\rho_{-,c}$ and $\rho_{-,n}$ are the respective half sums of the positive M-compact and M-noncompact roots of Δ_-. Following the procedure of [6], we introduce a positive system Δ^+ containing $(\Delta_-)^+$ such that each α_j is simple for Δ^+. (Again we use the superorthogonality). If we set $\Delta_K^+ = \Delta^+ \cap \Delta_K$, then [6] says that highest weights of the minimal K-types of $U(\nu) = U(S, \sigma, \nu)$ are given by all Δ_K^+ dominant expressions of the form

$$\Lambda = \lambda - \sum_{j=1}^{l} \frac{\langle 2\rho_K, \alpha_j \rangle}{|\alpha_j|^2} \alpha_j + \mu. \tag{2.1}$$

Here ρ_K is half the sum of the members of Δ_K^+, and μ is given by $\Sigma_{j=1}^{l} s_j \alpha_j$ with each s_j equal to $\pm \dfrac{1}{2} \alpha_j$ or 0, depending on the value of $\sigma^{\#}(\gamma_{\alpha_j})$. Moreover, at least one choice of the system of signs in μ gives a Δ_K^+ dominant Λ. We fix such a choice of μ and hence Λ.

It is clear that $\Lambda|_{\mathfrak{b}_-} = \lambda$, and the proof of the minimal K-type formula shows that a highest weight vector for τ_Λ is highest of type τ_λ for $K \cap M_0$ and that the value of $\tau_\Lambda(\gamma_{\alpha_j})$ on a highest weight vector is the same as the scalar value of $\sigma^\#(\gamma_{\alpha_j})$. If v_0 is a highest weight vector of τ_Λ and A is a member of $\mathrm{Hom}_{K \cap M^\#}(V^\Lambda, V^{\sigma^\#})$, then it follows that Av_0 is a multiple of a λ highest weight vector u_0 in $V^{\sigma^\#}$ and that A^*u_0 is a multiple of v_0.

By double induction we identify $U(\nu)$ with

$$\text{(2.2)} \qquad \mathrm{ind}_{M^\# AN}^{G}(\sigma^\# \otimes e^\nu \otimes 1),$$

and then we can identify the function f_0 of § 1 with a function whose values are in $V^{\sigma^\#}$, rather than V^σ. We define A in $\mathrm{Hom}_{K \cap M^\#}(V^\Lambda, V^{\sigma^\#})$ by (1.7), and we normalize u_0 by the requirement

$$A^*u_0 = v_0 .$$

If μ' is any integral form on $\mathfrak{b}$, we denote by $(\mu')\check{}$ the dominant integral form on $\mathfrak{b}$ to which μ' is conjugate by the Weyl group of Δ_K. Let $\mathfrak{p}$ be the -1 eigenspace of the Cartan involution of $\mathfrak{g}$, and let «$\perp\perp$» refer to strong orthogonality of roots.

LEMMA 2.1. Let μ' be an integral form on $\mathfrak{b}$, and let β be a noncompact root. Let $\Lambda' = (\mu')\check{}$ and $\Lambda'' = (\mu' + \beta)\check{}$. If v' is a nonzero vector of weight μ' in $\tau_{\Lambda'}$, then the projection

$$v'' = E_{\Lambda''}(v' \otimes X_\beta)$$

of $v' \otimes X_\beta$ in $\tau_{\Lambda'} \otimes \mathfrak{p}^{\mathbb{C}}$ to the $\tau_{\Lambda''}$ subspace is nonzero.

THEOREM 2.2. Fix an index r with $1 \leqslant r \leqslant l$, an integral form μ' on $\mathfrak{b}$, and a choice of a sign ±. Let $\Lambda' = (\mu')\check{}$ and $\Lambda'' = (\mu' \pm \alpha_r)\check{}$. Fix a nonzero vector v' of weight μ' in $\tau_{\Lambda'}$, and for each ν, let $B(\nu)$ be a member of $\mathrm{Hom}_{K \cap M^\#}(V^{\Lambda'}, V^{\sigma^\#})$. Let f_1 be the member of the induced space given by

$$f_1(k) = B(\nu)\, \tau_{\Lambda'}(k)^{-1} v'.$$

Suppose that

(a) the only weight in $\tau_{\Lambda'}$ of the form $\mu' \pm \alpha_r + \alpha_j$ or $\mu' \pm \alpha_r - \alpha_j$ is μ' itself,

(b) there exists a system $\Delta_{L'} \subseteq \Delta$ generated by Δ^+ simple roots such that
  (i) $\alpha_1, \ldots, \alpha_l$ are in $\Delta_{L'}$, and $\Delta_{L'}$ has real rank exactly l.
  (ii) $\Lambda' - \Lambda$ is an integral linear combination of roots in $\Delta_{L'}$.

Let v'' be the nonzero vector $E_{\Lambda''}(v' \otimes X_{\pm\alpha_r})$. Then

$$\frac{\langle P_{\Lambda''}U(\nu, X_{\pm\alpha_r})f_1(k), u_0\rangle_{\mu'\pm\alpha_r}}{\langle\tau_{\Lambda''}(k)^{-1}v'', v''\rangle} = \frac{|\alpha_r|^2}{4}\, \mathrm{d}(\nu)\, \frac{\langle f_1(k), u_0\rangle_{\mu'}}{\langle\tau_{\Lambda'}(k)^{-1}v', v'\rangle}\ ,$$

where

$$\mathrm{d}(\nu) = (\nu+\bar{\rho})(X_{\alpha_r} + X_{-\alpha_r}) + \frac{2\langle\mu', \pm\alpha_r\rangle}{|\alpha_r|^2} -$$
$$-\,2\#\{\beta\in\Delta_n \,|\, \beta\perp\alpha_1, \ldots, \alpha_{r-1}; \beta-(\pm\alpha_r)\in\Delta; \langle\mu', \beta-(\pm\alpha_r)\rangle > 0\}.$$

REMARKS. Assumption (b) is satisfied with $\Delta_{L'} = \Delta$ if MAN is a minimal parabolic subgroup. The assumption should be regarded as an extension to the current setting of the condition in Theorem 1 of [1] that the δ^+ or δ^- subgroup have real rank one. In fact, Theorem 1 of [1] is a special case of the present theorem if we substitute for $\mathrm{d}(\nu)$ from the equality (a) = (c) in Theorem 2.5 below.

Theorem 2.2 says that the ν dependence in $b(\nu, k)$ as defined in (1.8) under suitable circumstances is a product of linear factors $\mathrm{d}(\nu)$, each coming from a single step of the action of $\mathfrak{g}^{\mathbb{C}}$ on the representation space. A simple way in which these circumstances can fail is when the theorem is to be applied twice, first to pass from $(\mu')^{\vee}$ to $(\mu'+\alpha_r)^{\vee}$ and then to pass from $(\mu'+\alpha_r)^{\vee}$ to $(\mu'+\alpha_r+\alpha_s)^{\vee}$; assumption (a) will fail at the second step if $\mu'+\alpha_r$ is conjugate to $\mu'+\alpha_s$ by the Weyl group of Δ_K. Theorem 2.3 addresses this situation, giving a formula for the combined effect of the two steps.

THEOREM 2.3. Fix roots $\pm\alpha_r$ and $\pm\alpha_s$ with $r\neq s$ and with the two choices of sign not necessarily the same, and fix an integral form μ' on $\mathfrak{b}$. Suppose that $\mu'\pm\alpha_r$ and $\mu'\pm\alpha_s$ are conjugate by the Weyl group of Δ_K. Let $\Lambda' = (\mu')^{\vee}$, $\Lambda'' = (\mu'\pm\alpha_r)^{\vee} = (\mu'\pm\alpha_s)^{\vee}$, and $\Lambda''' = (\mu'\pm\alpha_r\pm\alpha_s)^{\vee}$. Fix a nonzero vector v' of weight μ' in $\tau_{\Lambda'}$, and for each ν, let $B(\nu)$ be a member of $\mathrm{Hom}_{K\cap M^{\#}}(V^{\Lambda'}, V^{\sigma^{\#}})$. Let f_1 be the member of the induced space given by

$$f_1(k) = B(\nu)\tau_{\Lambda'}(k)^{-1}v'.$$

Suppose that

(a) the only weight in $\tau_{\Lambda'}$ obtainable by adding or subtracting some α_j from $\mu'\pm\alpha_r$ or $\mu'\pm\alpha_s$ is μ' itself,

(b) the only weights in $\tau_{\Lambda''}$ obtainable from $\mu'\pm\alpha_r\pm\alpha_s$ by adding or subtracting some α_j are $\mu'\pm\alpha_r$ and $\mu'\pm\alpha_s$,

(c) there exists a system $\Delta_{L'} \subseteq \Delta$ generated by Δ^+ simple roots such that

(i) $\alpha_1, \ldots, \alpha_l$ are in $\Delta_{L'}$, and $\Delta_{L'}$ has real rank exactly l

(ii) $\Lambda' - \Lambda$ is an integral linear combination of roots in $\Delta_{L'}$

(iii) $\Lambda'' - \Lambda$ is an integral linear combination of roots in $\Delta_{L'}$.

Let v''' be the nonzero vector $E_{\Lambda'''}((E_{\Lambda''}(v' \otimes X_{\pm\alpha_r}) \otimes X_{\pm\alpha_s})$. Then

$$\frac{\langle P_{\Lambda'''} U(\nu, X_{\pm\alpha_s}) P_{\Lambda''} U(\nu, X_{\pm\alpha_r}) f_1(k), u_0 \rangle_{\mu' \pm \alpha_r \pm \alpha_s}}{\langle \tau_{\Lambda'''}(k)^{-1} v''', v''' \rangle}$$

$$= \frac{|\alpha_r|^4}{16} \left(\mathrm{d}_1(\nu)\, \mathrm{d}_3(\nu) + \mathrm{d}_2(\nu)\, \mathrm{d}_4(\nu)\right) \frac{\langle f_1(k), u_0 \rangle_{\mu'}}{\langle \tau_{\Lambda'}(k)^{-1} v', v' \rangle},$$

where

$$\mathrm{d}_1(\nu) = (\nu + \overline{\rho})(X_{\alpha_s} + X_{-\alpha_s}) + \frac{2\langle \mu', \pm\alpha_s \rangle}{|\alpha_s|^2}$$
$$- 2\#\{\beta \in \Delta_n \mid \beta \perp\!\!\perp \alpha_1, \ldots, \alpha_{s-1}; \beta - (\pm\alpha_s) \in \Delta; \langle \mu' \pm \alpha_r, \beta - (\pm\alpha_s) \rangle > 0\}$$

$$\mathrm{d}_2(\nu) = (\nu + \overline{\rho})(X_{\alpha_r} + X_{-\alpha_r}) + \frac{2\langle \mu', \pm\alpha_r \rangle}{|\alpha_r|^2}$$
$$- 2\#\{\beta \in \Delta_n \mid \beta \perp\!\!\perp \alpha_1, \ldots, \alpha_{r-1}; \beta - (\pm\alpha_r) \in \Delta; \langle \mu' \pm \alpha_s, \beta - (\pm\alpha_r) \rangle > 0\}$$

$$\mathrm{d}_3(\nu) = (\nu + \overline{\rho})(X_{\alpha_r} + X_{-\alpha_r}) + \frac{2\langle \mu', \pm\alpha_r \rangle}{|\alpha_r|^2}$$
$$- 2\#\{\beta \in \Delta_n \mid \beta \perp\!\!\perp \alpha_1, \ldots, \alpha_{r-1}; \beta - (\pm\alpha_r) \in \Delta; \langle \mu', \beta - (\pm\alpha_r) \rangle > 0\}$$

$$\mathrm{d}_4(\nu) = (\nu + \overline{\rho})(X_{\alpha_s} + X_{-\alpha_s}) + \frac{2\langle \mu', \pm\alpha_s \rangle}{|\alpha_s|^2}$$
$$- 2\#\{\beta \in \Delta_n \mid \beta \perp\!\!\perp \alpha_1, \ldots, \alpha_{s-1}; \beta - (\pm\alpha_s) \in \Delta; \langle \mu', \beta - (\pm\alpha_s) \rangle > 0\}.$$

Presumably Theorems 2.2 and 2.3 are the special cases $m = 1$ and $m = 2$ of some single theorem about taking m steps by m distinct α_j's, with all m of the $(\mu' + \alpha_j)$'s conjugate. Such a result would be helpful in handling $SU(p, q)$ with $q > 2$, but for $SU(p, 2)$ the two theorems above are sufficient.

Our third theorem deals with the effect of failing to take the second step in Theorem 2.3 but instead returning to the starting point. We allow a situation somewhat more general than just conjugacy of $\mu' \pm \alpha_r$ with $\mu' \pm \alpha_s$, because the more general situation arises in $SU(p, 2)$. We do not see well enough how this kind of result might be useful in an inductive calculation and therefore make only the calculation required for (1.8), with f_0 as starting function.

THEOREM 2.4. Fix roots $\pm\alpha_r$ and $\pm\alpha_s$ with $r \neq s$ and with the two choices of sign not necessarily the same. Let $\Lambda' = (\Lambda \pm \alpha_r)^{\check{}}$, and suppose that $\Lambda \pm \alpha_s$ is a weight of multiplicity one in $\tau_{\Lambda'}$. Suppose that

(a) the only weights in $\tau_{\Lambda'}$ of the form $\Lambda + \alpha_j$ or $\Lambda - \alpha_j$ are $\Lambda \pm \alpha_r$ and $\Lambda \pm \alpha_s$,

(b) there exists $C > 0$ such that the nonzero vector

$$v'' = E_\Lambda(E_{\Lambda'}(v_0 \otimes X_{\pm\alpha_r}) \otimes X_{-(\pm\alpha_r)})$$

satisfies

$$v'' = CE_\Lambda(E_{\Lambda'}(v_0 \otimes X_{\pm\alpha_s}) \otimes X_{-(\pm\alpha_s)}),$$

(c) whenever β in Δ_n is such that $\beta \perp\!\!\!\perp \alpha_1, \ldots, \alpha_{s-1}$, $\beta + (\pm\alpha_s) \in \Delta$, and $\Lambda - \beta$ is a weight of $\tau_{\Lambda'}$, then $\langle \Lambda, \beta + (\pm\alpha_s)\rangle \geqslant 0$,

(d) there exists a system $\Delta_{L'} \subseteq \Delta$ generated by Δ^+ simple roots such that
    (i) $\alpha_1, \ldots, \alpha_l$ are in $\Delta_{L'}$, and $\Delta_{L'}$ has real rank exactly l
    (ii) $\Lambda' - \Lambda$ is an integral linear combination of roots in $\Delta_{L'}$.

Then

$$\langle P_\Lambda U(\nu, X_{-(\pm\alpha_r)}) P_{\Lambda'} U(\nu, X_{\pm\alpha_r}) f_0(k), u_0\rangle_\Lambda$$

$$= \frac{|\alpha_r|^4}{16}\, \{(\nu(X_{\alpha_r} + X_{-\alpha_r})^2 - c_r^2) + C^{-1}(\nu(X_{\alpha_s} + X_{-\alpha_s})^2 - c_s^2)\} \langle \tau_\Lambda(k)^{-1} v'', v''\rangle,$$

where

$$c_r = \overline{\rho}(X_{\alpha_r} + X_{-\alpha_r}) + \frac{2\langle \Lambda, \pm\alpha_r\rangle}{|\alpha_r|^2}$$

$$-2\#\{\beta \in \Delta_n \mid \beta \perp\!\!\!\perp \alpha_1, \ldots, \alpha_{r-1}; \beta - (\pm\alpha_r) \in \Delta; \langle \Lambda, \beta - (\pm\alpha_r)\rangle > 0\}$$

and where c_s is defined similarly.

THEOREM 2.5. For any integral form μ', the following two expressions are equal:

(a) $\overline{\rho}(X_{\alpha_j} + X_{-\alpha_j}) + \dfrac{2\langle \mu', \pm\alpha_j\rangle}{|\alpha_j|^2}$

$$-2\#\{\beta\in\Delta_n \mid \beta\perp\perp\alpha_1,\dots,\alpha_{j-1};\beta-(\pm\alpha_j)\in\Delta;\langle\mu',\beta-(\pm\alpha_j)\rangle>0\}$$

$$\text{(b)}\ -[\overline{p}(X_{\alpha_j}+X_{-\alpha_j})-\frac{2\langle\mu'+(\pm\alpha_j),\pm\alpha_j\rangle}{|\alpha_j|^2}$$

$$-2\#\{\beta\in\Delta_n \mid \beta\perp\perp\alpha_1,\dots,\alpha_{j-1};\beta+(\pm\alpha_j)\in\Delta;\langle\mu'+(\pm\alpha_j),\beta+(\pm\alpha_j)\rangle>0\}].$$

Moreover, if μ' is the parameter Λ of the minimal K-type, then both these expressions are equal to

$$\text{(c)}\ 1+\frac{2\langle\mu,\pm\alpha_j\rangle}{|\alpha_j|^2}$$

$$+2\#\{\beta\in\Delta_n^+ \mid \beta\perp\perp\alpha_1,\dots,\alpha_{j-1};\beta-(\pm\alpha_j)\in\Delta;\langle\Lambda,\beta-(\pm\alpha_j)\rangle=0\}.$$

REMARKS. The equality of (a) and (b) is a simple matter, but the equality of these two expressions with (c) uses the minimal K-type formula and various identities to relate ρ's. The significance of the result is as follows: When (1.8) is calculated by iteration, one should expect each pair X_i and $\overline{X}_i$ to lead to one occurrence of (a) and one occurrence of (b). With care these expressions can then be related to (c), which is a local expression in the sense of involving only simple roots that are close to $\alpha_1,\dots,\alpha_l$ in the Dynkin diagram of Δ^+. Expression (c) allows the possibility of matching estimates for nonunitarity of representations related by cohomological induction.

3. CLASSIFICATION FOR SU(p, 2)

In this section we shall state results that, in the light of [8], give a classification of the irreducible unitary representations of $SU(p,2)$ for $p\geqslant 3$. A number of the intermediate results are valid for all $SU(p,q), p\geqslant q$, and we begin by establishing notation in this more general context.

In $SU(p,q)$ with $p\geqslant q$, K consists of matrices with nonzero entries only in the upper left p-by-p block and lower right q-by-q block. We take B to be the diagonal subgroup and let e_i denote evaluation of the i^{th} diagonal entry of a member of $\mathfrak{b}$. For $1\leqslant i\leqslant q$, define

$$\alpha_i=e_{q+i}-e_{p+q+1-i}. \tag{3.1}$$

For each l with $1\leqslant l\leqslant q$, we can construct as in §2 a parabolic subgroup MAN from the superorthogonal set $\alpha_1,\dots,\alpha_l$ of noncompact roots. Together with G itself, these q parabolic subgroups are the only ones needed for classification. According to the results collected in [8], the classification is reduced to a routine bookkeeping question if one knows which representations $J(MAN, \sigma, \nu)$ can be made unitary when MAN is as above, σ is a discrete series or nondegenerate limit of discrete series, and ν is a real-valued parameter in the positive Weyl chamber such that $J(MAN, \sigma, \nu)$ is defined. All the representations attached to G itself are unitary, and we are left with the proper parabolic subgroups.

In the case of the maximal parabolic subgroup ($l = 1$), we can answer the unitarity question for all the groups $SU(p, q)$.

THEOREM 3.1. In $SU(p, q)$ let $S = MAN$ be the maximal parabolic subgroup built from $\{\alpha_1\}$. Fix a discrete series or nondegenerate limit of discrete series σ on M, and let notation and orderings be as in §2. Define

$$\nu_0^+ = 1 + \frac{2\langle\mu, \alpha_1\rangle}{|\alpha_1|^2} + 2\#\{\beta \in \Delta_n^+ \mid \beta - \alpha_1 \in \Delta \text{ and } \langle\Lambda, \beta - \alpha_1\rangle = 0\}$$

$$\nu_0^- = 1 - \frac{2\langle\mu, \alpha_1\rangle}{|\alpha_1|^2} + 2\#\{\beta \in \Delta_n^+ \mid \beta + \alpha_1 \in \Delta \text{ and } \langle\Lambda, \beta + \alpha_1\rangle = 0\}.$$

If ν is real-valued, then $J(S, \sigma, \nu)$ is unitary for

$$0 < \nu(X_{\alpha_1} + X_{-\alpha_1}) \leqslant \min\{\nu_0^+, \nu_0^-\}$$

and is not unitary for

$$\min\{\nu_0^+, \nu_0^-\} < \nu(X_{\alpha_1} + X_{-\alpha_1}).$$

REMARK. There is nothing special about α_1 in Theorem 3.1. This root may be replaced everywhere by another noncompact root (such as another α_j) as long as the positive system Δ^+ satisfies the conditions required by [6].

Let us now specialize to $q = 2$. The remaining case is that $l = 2$ and MAN is minimal parabolic. To state the result concisely, we use the notion of «basic case» as defined in [9]. Fix the representation σ of M, and let λ_0 be its infinitesimal character. Let $(\{\alpha_1, \alpha_2\}, \Delta^+, \chi, \mu)$ be a compatible format for λ_0, in the sense of [9], and let $\lambda_{b,0}$ be the basic case for this format. Then §4 of [9] associates to λ_0 a subgroup L of $G = SU(p, 2)$ with complexified Lie algebra $\mathfrak{l}^{\mathbb{C}}$ built from $\mathfrak{h}^{\mathbb{C}}$ and all $\beta \in \Delta$ with $\langle\lambda_0 - \lambda_{b,0}, \beta\rangle = 0$, as well as a format for L and a parameter λ_0^L that is the basic case for this format of L. Let σ^L be the corresponding representation of the M of L. The roor system

$$(3.2) \qquad \Delta^L = \{\beta \in \Delta \mid \langle \lambda_0 - \lambda_{b,0}, \beta \rangle = 0\}$$

of L contains $\{\alpha_1, \alpha_2\}$ automatically, and thus ν makes sense on the $\mathfrak{a}$ subalgebra of $\mathfrak{l}$.

THEOREM 3.2. In $G = SU(p, 2)$ let $S = MAN$ be the minimal parabolic subgroup built from $\{\alpha_1, \alpha_2\}$. Fix an irreducible representation σ of the compact group M, and let notation and orderings be as in §2. Choose a compatible format and construct the subgroup L. For real-valued ν the Langlands quotient $J(S, \sigma, \nu)$ for G is unitary if and only if the Langlands quotient $J(S \cap L, \sigma^L, \nu)$ for L is unitary.

Theorem 3.2 reduces matters to the basic cases that can arise from $SU(p, 2)$. The subalgebra $\mathfrak{l}$, apart from abelian and compact factors (which play a trivial role), is necessarily of one of the forms $\mathfrak{su}(p', 1) \oplus \mathfrak{su}(p'', 1)$ or $\mathfrak{su}(p', 2)$. The irreducible unitary representations of $SU(p', 1)$ are well known; for the basic cases, the results are assembled on p. 128 of [9]. For $SU(p', 2)$, there are a number of basic cases given in §6 below that lead to no unitarity. The remaining basic cases [2] are given by

$$\begin{array}{cc} & \begin{array}{cc} p'-2 & \qquad 4 \end{array} \\ \sigma & \left(\begin{array}{c|cccc} \omega & & & & \\ \hline & e^{i\theta} & & & \\ & & e^{i\varphi} & & \\ & & & e^{i\varphi} & \\ & & & & e^{i\theta} \end{array}\right) = e^{i(m\theta + n\varphi)} \end{array}$$

with $|m| \leqslant p' - 1$ and $|n| \leqslant p' - 1$. Since complex conjugation is an outer automorphism of $SU(p', 2)$ fixing A and sending (m, n) to $(-m, -n)$, it is enough to understand $m \geqslant n$. Theorem 2.1 of [9] lists the unitary points exactly, except for one ambiguous ν for each σ having $m = n$ and $|m| \leqslant p' - 2$ or having $0 > m > n \geqslant -(p' - 2)$. For these ambiguous ν, it is remarked that preliminary calculations by Vogan and Wallach indicated that the

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[2] The paper [8] inaccurately gives the impression that the only basic cases are these interesting ones.

corresponding representations are unitary. In fact, the calculations can be carried through with the help of the work of Jakobsen [4] or Enright-Howe-Wallach [2], and the representations in question are indeed unitary.

4. PROOF OF THEOREM 3.1

In this section we let $S = MAN$ be the maximal parabolic subgroup of $SU(p, q)$ built from one noncompact root, and we shall prove the formula for unitary points that is asserted in Theorem 3.1. Except when $p = q = 1$, the group M is connected, and thus the nondegeneracy condition on σ is the assumption that the parameter λ_0 is not orthogonal to any compact root of Δ_-.

The assumption of nondegeneracy is vital in the theorem, as the following example shows. When $G = SU(3, 2)$, there is a degenerate σ that is essentially the 0th spherical principal series of $SU(2, 1)$. The smaller of ν_0^+ and ν_0^- is 4, whereas the correct cut-off for unitarity is 2. What is happening is that the line of ν parameters for this case imbeds as the x-axis in the two-dimensional picture for the minimal parabolic subgroup of G and the trivial representation of the corresponding M. There are no unitary points on the x-axis beyond the point 2, but there is a unitary point in the plane (corresponding to the trivial representation of G) whose x coordinate is 4.

Theorem 3.1 was known already in some cases. When σ is a discrete series representation of M, the result is given as Theorem 7 of [1]. If also $q = 2$, then an equivalent form of the result was given as Proposition 9.1 of [9]. The idea of the proof of Theorem 3.1 is to obtain the smaller of ν_0^+ and ν_0^- as a cut-off for unitarity by applying Theorem 2.2 to the passage $\Lambda \to (\Lambda \pm \alpha_1)^\vee \to \Lambda$ (or equivalently by using Theorems 1, 3, and 4 of [1]). To prove unitarity out as far as $\min\{\nu_0^+, \nu_0^-\}$, we use the results of Speh-Vogan [16][3] to prove irreducibility of $U(S, \sigma, \nu)$ for ν in the open interval in question, and then the Hermitian form must have constant signature in the open interval; irreducibility at $\nu = 0$ forces the form to be definite in the open interval and therefore semidefinite in the closed interval.

For any single σ, the above style of proof amounts to a routine computation. The difficulty is to do the bookkeeping necessary to handle all σ simultaneously without an uncontrolled proliferation into special situations. For this purpose we shall use the notion of basic cases introduced in [9]

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[3] Occasionally we need a slight generalization of the results of [16], such as when σ is not actually a discrete series representation. For such generalizations, see pages 408 and 545 of [18] and §4 of [19].

and reviewed above in §3. The passage from G to L in the theory of basic cases preserves ν_0^+ and ν_0^-, as well as the conditions needed to apply Theorem 2.2 to this situation, and the first part of the proof (the nonunitarity) is thereby reduced to the group L and a basic case σ^L on the M group of L. But even with this reduction, the bookkeeping is still complicated, because there are many basic cases. Thus we shall single out some basic cases as «special», and we shall prove that we can pass successfully between L and a special basic case of a subgroup L'. The possible situations in L' are quite limited, and we can handle them directly. For the second part of the proof (the unitarity, as a consequence of irreducibility), we handle L' directly and then use the Speh-Vogan theory to pass directly to G without L as an intermediate step.

We turn to the details. We begin with two general lemmas that are valid without assuming $G = SU(p, q)$. In the context of §4 of [9], fix a format $(\{\alpha_j\}, \Delta^+, \chi, \mu)$ and a compatible λ_0. Construct Δ^L as in (3.2) above (or as in §4 of [9]), and let $\mathfrak{u}$, $\rho(\mathfrak{u})$, χ^L, and $\rho(\mathfrak{u} \cap \mathfrak{p}^{\mathbb{C}})$ be defined as in §4 of [9].

LEMMA 4.1. Let $\Delta^{L'}$ be a subsystem of Δ^L that is generated by simple roots and contains the roots α_j, form the corresponding reductive subalgebra $\mathfrak{l}'$ of $\mathfrak{g}$, and let L' be the associated analytic subgroup. Define $\mathfrak{u}'$ to be the sum of the root spaces of $\mathfrak{g}^{\mathbb{C}}$ for the positive roots not in $\Delta^{L'}$. Let $\chi^{L'}$ be defined from $\mathfrak{u}'$ the way χ^L is defined from $\mathfrak{u}$. Put

$$\lambda_0^{L'} = \lambda_0 - \rho(\mathfrak{u}') \tag{4.1}$$

and

$$\Lambda^{L'} = \Lambda - 2\rho(\mathfrak{u}' \cap \mathfrak{p}^{\mathbb{C}}). \tag{4.2}$$

Then $\chi^{L'}$ and $\lambda_0^{L'}$ consistently define $\sigma^{L'}$, and $(\{\alpha_j\}, \Delta^+ \cap \Delta^{L'}, \chi^{L'}, \mu)$ is a compatible format. The minimal $(K \cap L')$-type parameter associated to $\sigma^{L'}$ and this format is $\Lambda^{L'}$. The parameter $\lambda_0^{L'}$ is a basic case for this format.

REMARKS. The lemma is proved in the same way as for Proposition 4.1 and 4.2 of [9]. Only notional changes are required.

To prove the nonunitarity in Theorem 3.1, we shall use Theorems 1, 3, and 4 of [1]. For the next lemma, we work in the context of a maximal parabolic subgroup obtained from a general G satisfying our hypotheses. Let the parabolic subgroup be built from the noncompact root α. Let $\Delta_{K,\perp}$ be the subsystem of Δ_K of compact roots orthogonal to Λ, and let $W_{K,\perp}$ be its Weyl group. Choose s^+ and s^- in $W_{K,\perp}$ so that $\delta^+ = s^+\alpha$ and

$\delta^- = s^-(-\alpha)$ are dominant for $\Delta^+_{K,\perp}$. Our assumption that all noncompact roots are short implies that $(\Lambda + \alpha)^\smile = \Lambda + \delta^+$ and $(\Lambda - \alpha)^\smile = \Lambda + \delta^-$. The δ^+ *group of G* is the semisimple subgroup corresponding to the subsystem of Δ generated by all simple roots needed for the expansion of δ^+ and α. The δ^- *group of G* is defined similarly.

LEMMA 4.2. (a) If β is in Δ^+_n and $\beta - \alpha$ is in Δ and $\langle \beta - \alpha, \Lambda \rangle = 0$, then β is in the δ^+ subgroup.

(b) If β is in Δ^+_n and $\beta + \alpha$ is in Δ and $\langle \beta + \alpha, \Lambda \rangle = 0$, then β is in the δ^- subgroup.

PROOF. For (a), let $\gamma_0 = \beta - \alpha$. Then γ_0 is in $\Delta_{K,\perp}$ and the reflection s_{γ_0} carries α to β. Hence $\Lambda + \beta$ is conjugate to $\Lambda + \alpha$, and it follows that $\Lambda + \beta$ is a weight of $\tau_{\Lambda + \delta^+}$. Hence

$$\delta^+ - \beta = (\Lambda + \delta^+) - (\Lambda + \beta) = \sum_{\gamma \in \Delta^+_k} n_\gamma \gamma.$$

Then the equation $\delta^+ = \beta + \Sigma\, n_\gamma \gamma$ forces β to be in the span of the simple roots needed for the expansion of δ^+. This proves (a), and (b) is proved similarly.

Let us return to $G = SU(p, q)$ and write α for α_1. Twice a noncompact root in $SU(p, q)$ is not in the span of the compact roots. Thus we can conclude from Theorems 3, 4b, and 4d of [1] that $(\Lambda + \alpha)$ has multicity one in the induced representation whenever the δ^+ subgroup has real rank one. Similarly $(\Lambda - \alpha)^\smile$ has multiplicity one whenever the δ^- subgroup has real rank one. Bringing in Theorem 1 of [1], we see that there are no unitary points beyond ν_0^+ whenever the δ^+ subgroup has real rank one, and there are no unitary points beyond ν_0^- whenever the δ^- subgroup has real rank one. To complete the proof of nonunitarity, it is therefore enough to prove the following lemma.

LEMMA 4.3. In $G = SU(p, q)$,

(a) $\nu_0^+ < \nu_0^-$ implies the δ^+ subgroup has real rank one,

(b) $\nu_0^- < \nu_0^+$ implies the δ^- subgroup has real rank one,

(c) $\nu_0^+ = \nu_0^-$ implies that either the δ^+ subgroup or the δ^- subgroup has real rank one.

To begin the proof of the lemma, we construct the standard L and Δ^L as above. According to §4 of [1] (or Lemma 6.2 below), $\Delta_{K,\perp}$ is contained in Δ^L, and thus δ^+, δ^-, and all the roots that contribute to the formulas for ν_0^+ and ν_0^- are in Δ^L. By (4.2) if we compute ν_0^+ and ν_0^- and the $\delta^\pm$ subgroups within Δ^L for the parameter λ_0^L, we get the same answer as within Δ for the parameter λ_0. Moreover λ_0^L is a basic case. Since λ_0 is a nondegenerate parameter, (3.2) shows λ_0^L is nondegenerate. Changing notation and discarding irrelevant simple factors, we see that we may assume in Lemma 4.3 that λ_0 is a basic case, still nondegenerate.

To continue, we require tools for calculating δ^+ and δ^- for basic cases. First we determine λ_0 itself. If β is a simple root, then $2\langle\lambda_0, \beta\rangle / |\beta|^2$ is given by one of the following, according to Corollary 2.3 of [7]:

$$\begin{array}{ll} 0 & \text{for } \beta = \alpha \\ 1 & \text{for } \beta \perp \alpha \text{ and } \beta \text{ compact} \\ 0 & \text{for } \beta \perp \alpha \text{ and } \beta \text{ noncompact} \\ 0, \frac{1}{2}, 1 & \text{for } \beta \text{ adjacent to } \alpha \text{ and compact, when } \mu = -\frac{1}{2}\alpha, 0, \frac{1}{2}\alpha \\ 1, \frac{1}{2}, 0 & \text{for } \beta \text{ adjacent to } \alpha \text{ and noncompact, when } \mu = -\frac{1}{2}\alpha, 0, \frac{1}{2}\alpha. \end{array}$$

Next, we give a different formula for Λ by specializing (4.13) of [9]:

$$\Lambda = \lambda_0 + \rho - 2\rho_K + \mu - \frac{1}{2}\alpha, \tag{4.3}$$

where ρ is half the sum of the members of Δ^+. Using this formula, we can assemble some conditions for a compact root to be in $\Delta_{K,\perp}$. The proofs are routine computations, and we omit them:

1) If γ is compact and Δ^+ simple, then $\Lambda \perp \gamma$.
2) If β is noncompact, Δ^+ simple, and adjacent to α in the Dynkin diagram, then $\Lambda \perp \beta + \alpha$.
3) If $\beta_1, \gamma_1, \ldots, \gamma_r, \alpha$ are consecutive Δ^+ simple roots in the Dynkin diagram, if β_1 is noncompact, if each γ_i is compact, and if $r \geqslant 1$, then $\Lambda \perp (\beta_1 + \gamma_1 + \ldots + \gamma_r + \alpha)$ if and only if $r = 1$ and $\mu = -\frac{1}{2}\alpha$.
4) If $\beta_1, \gamma_1, \ldots, \gamma_r, \beta, \alpha$ are consecutive Δ^+ simple roots in the Dynkin diagram, if β_1 and β are noncomact, if each γ_i is compact, and if $r \geqslant 0$, then $\Lambda \perp (\beta_1 + \gamma_1 + \ldots + \gamma_r + \beta)$ if and only if $r = 0$ and $\mu = \frac{1}{2}\alpha$.
5) If $\beta_1, \gamma_1, \ldots, \gamma_r, \beta$ are consecutive Δ^+ simple roots in the Dynkin

diagram, if β_1 and β are noncompact and not adjacent to α, if each γ_i is compact, and if $r \geqslant 1$, then $\beta_1 + \gamma_1 + \ldots + \gamma_r + \beta$ is not orthogonal to Λ.

Finally since λ_0 is nondegenerate, we have the following additional condition:

6) If β_1 and β are noncompact Δ^+ simple roots that are not adjacent to α, then β_1 and β are not adjacent. [In fact, otherwise $\beta_1 + \beta$ would be a compact root orthogonal to λ_0.]

We shall say that a basic case is *special* if the only noncompact simple roots are α and possibly some roots adjacent to α in the Dynkin diagram.

LEMMA 4.4. In $SU(p, q)$ in a special basic case, $\nu_0^+ \leqslant \nu_0^-$ implies the δ^+ subgroup has real rank one, and $\nu_0^- \leqslant \nu_0^+$ implies the δ^- subgroup has real rank one. Moreover, $U(\nu)$ is irreducible for $\nu(X_\alpha + X_{-\alpha})$ in the half-open interval $[0, \min\{\nu_0^+, \nu_0^-\})$.

PROOF. First we prove the statements about the δ^+ and δ^- subgroups. If α is at one end of the Dynkin diagram or if α is away from both ends and its two adjacent simple roots are of opposite type (compact or noncompact), then the whole group is of real rank one. Hence so are the δ^+ and δ^- subgroups. Thus suppose α is away from both ends and the two adjacent simple roots are of the same type. Possibly by reflecting everything in α, we may assume that the adjacent simple roots are both noncompact. Let us relabel the consecutive simple roots as $e_1 - e_2$, $e_2 - e_3, \ldots$, putting $\alpha = e_i - e_{i+1}$. Then i is the first index in one component of Δ_K^+ and $i + 1$ is the last index in the other component. Hence $\delta^+ = \alpha$. On the other hand, to compute δ^-, we reflect matters in α. Then $-\alpha$ becomes a simple root, and its neighbors are compact. By condition (1) above, δ^- is the largest root of this system, hence the largest root of Δ^+. The δ^+ group is thus of real rank one, and the δ^- group is not (since α has noncompact neighbors on both sides). To compute ν_0^+ and ν_0^-, let β_1 and β_2 be the noncompact neighbors of α. Then we find

$$\nu_0^+ = 1 + \frac{2\langle \mu, \alpha \rangle}{|\alpha|^2} + 2 \cdot 0 \leqslant 2$$

$$\nu_0^- \geqslant 1 - \frac{2\langle \mu, \alpha \rangle}{|\alpha|^2} + 2\#\{\beta_1, \beta_2\} \geqslant 4.$$

Hence $\nu_0^+ < \nu_0^-$, as required. This completes the proof of the first statement of the lemma.

For the second statement of the lemma, we may assume that Δ^+ is not of real rank one, since the case of $SU(n, 1)$ can be handled by routine comparison of the formulas for ν_0^+ and ν_0^- with the known length of the complementary series. Again, possibly by reflecting in α, we may assume that there are two noncompact simple roots adjacent to α. In view of our computations above, we are to check that the induced representation is irreducible for $0 \leqslant \nu(X_\alpha + X_{-\alpha}) < 1 + 2\langle \mu, \alpha \rangle / |\alpha|^2$. By Proposition 6.1 of Speh-Vogan [16], the only possible difficulty is at 0 when $\mu = \dfrac{1}{2}\alpha$.

We check this case separately. Let $\Delta^{L''}$ be the $SU(2, 1)$ system generated by α and one of the adjacent (noncompact) simple roots. Form $\lambda_0^{L''}$ by Lemma 4.1. This is the basic case for $\mu = \dfrac{1}{2}\alpha$, by that lemma. According to our table of basic cases, $\lambda_0^{L''}$ is orthogonal to the simple roots of $\Delta^{L''}$. Since the 0th spherical principal series of $SU(2, 1)$ is irreducible, the Speh-Vogan theory says we have irreducibility in G if $\lambda_0 + \nu$ has inner product $\geqslant 0$ with the roots of $\mathfrak{u}$ for our value of ν, namely $\nu = 0$. Our positive system Δ^+ was chosen with λ_0 dominant, and hence the condition is satisfied. Thus we have the required irreducibility in G, and Lemma 4.4 is completely proved.

Remembering that we have reduced Lemma 4.3 to nondegenerate basic cases, we now associate to our format for $L = G$ a subsystem $\Delta^{L'}$ that is a special basic case. Namely the simple roots of $\Delta^{L'}$ are the compact simple roots of Δ, the root α, and the simple roots adjacent to α. (Only the component of α in $\Delta^{L'}$ will play a role in our argument, and this corresponds to the group $SU(p', q')$ to which we shall apply Lemma 4.4). We have δ^+ and δ^- subgroups of G and corresponding $\delta^+_{L'}$ and $\delta^-_{L'}$ subgroups of L'. Similarly we have parameters ν_0^+ and ν_0^- for G and corresponding parameters $\nu^+_{L'}$ and $\nu^-_{L'}$ for L'. Equation (4.2) implies that the $\delta^+_{L'}$ subgroup of L' is contained in the δ^+ subgroup of G and that $\nu^+_{L'} \leqslant \nu_0^+$. Similar statements apply to δ^- and ν_0^-.

LEMMA 4.5.

(a) Suppose that $\nu^+_{L'} < \nu^-_{L'}$. Then the δ^+ subgroup of G equals the $\delta^+_{L'}$ subgroup of L' and is of real rank one. Moreover $\nu_0^+ = \nu^+_{L'}$.

(b) Suppose that $\nu^-_{L'} < \nu^+_{L'}$. Then the δ^- subgroup of G equals the $\delta^-_{L'}$ subgroup of L' and is of real rank one. Moreover, $\nu_0^- = \nu^-_{L'}$.

(c) Suppose that $\nu^+_{L'} = \nu^-_{L'}$. Then at least one of the δ^+ and δ^- subgroups is unchanged in passing from L' to G and is of real rank one. The corresponding ν parameter also is unchanged in passing from L' to G.

PROOF. Taking into account the effect of reflection in α, we may assume that $\nu_{L'}^+ \leqslant \nu_{L'}^-$. We divide matters into cases depending on the nature of the simple roots adjacent to α in $\Delta^{L'}$.

First suppose that α is at one end of the Dynkin diagram of $\Delta^{L'}$ and the adjacent root is compact. Then the Dynkin diagram of Δ still has α at the end. By condition (1), $\delta_{L'}^+$ is the largest root in $\Delta^{L'}$. By conditions (5) and (6), $\delta_{L'}^+ = \delta^+$. Thus the δ^+ subgroup of G equals the $\delta_{L'}^+$ subgroup of L' and is of real rank one. Moreover, $\nu_0^+ = \nu_{L'}^+$ by Lemma 4.2a.

Next suppose that α is at one end of the Dynkin diagram of $\Delta^{L'}$ and the adjacent root is noncompact, say β. Then the Dynkin diagram of Δ still has α at one end, and if there are additional roots at the other end, the first new one is noncompact, say β_1. The root $\delta_{L'}^+$ is just α. For δ^+ to fail to be α, the compact root $\beta + \ldots + \beta_1$ must be orthogonal to Λ. By condition (4) this can happen only if there are no simple roots between β and β_1 and also $\mu = \frac{1}{2}\alpha$. In this case $\Delta^{L'}$ has only the simple roots α and β, and we readily compute that $\nu_{L'}^+ = \nu_{L'}^- = 2$. Thus we are in the situation of (c) in the present lemma. Using conditions (5) and (6), we see that $\delta_{L'}^- = \delta^- = \beta$. In either case, we obtain the conclusion of (a) or (c) in the lemma by applying Lemma 4.2a.

Next suppose that α has two simple roots adjacent to it, a noncompact one β and a compact one γ. Then $\delta_{L'}^+ = \alpha + \gamma + \ldots + \gamma_r$, where γ_r is the last root in the Dynkin diagram of $\Delta^{L'}$, by condition (1). Moreover, δ^+ cannot involve further roots on the same side of α as γ, by conditions (5) and (6). Thus the only way that $\delta_{L'}^+ = \delta^+$ can fail is for α to be conjugated by a member of $\Delta_{K,\perp}$ on the β side of α in the diagram. In this case, condition (4) implies that $\mu = \frac{1}{2}\alpha$ and that there is a noncompact simple root β_1 (in Δ) adjacent to β on the opposite side from α. Then

$$\nu_{L'}^- = 1 - \frac{2\langle\mu, \alpha\rangle}{|\alpha|^2} + 2\#\{\beta\} = 2,$$

while

$$\nu_{L'}^+ \geqslant 1 + \frac{2\langle\mu, \alpha\rangle}{|\alpha|^2} + 2\#\{\alpha + \gamma\} = 4.$$

Hence we have $\nu_{L'}^- < \nu_{L'}^+$, in contradiction to assumption.

Finally suppose that α has two simple roots adjacent to it, both of the same kind, compact or noncompact. Since we are assuming $\nu_{L'}^+ \leqslant \nu_{L'}^-$, Lemma 4.4 says that the $\delta_{L'}^+$ subgroup is of real rank one. By condition (1) the two simple roots adjacent to α must be noncompact. Let us call them β and β_2. We have $\delta_{L'}^+ = \alpha$. For δ^+ to fail to be α, we must have $\mu = \frac{1}{2}\alpha$, and there must be a noncompact simple root $\beta_1 \neq \alpha$ adjacent to β or β_2, say to β. So we have consecutive simple roots β_1, β, α, β_2 in the Dynkin diagram of Δ, and $\mu = \frac{1}{2}\alpha$. Put $\gamma = \beta_1 + \beta + \alpha + \beta_2$. Then γ is a compact root of $\mathfrak{m}$, and $\langle \lambda_0, \gamma \rangle = 0$. Thus λ_0 is degenerate, in contradiction to assumption. This proves Lemma 4.5.

PROOF OF LEMMA 4.3. We can match the cases of Lemma 4.5 with the resulting inequalities for ν_0^+ and ν_0^-. In part (a) of Lemma 4.5, we have $\nu_0^+ < \nu_0^-$. In part (b) we have $\nu_0^- < \nu_0^+$. In part (c) we have either $\nu_0^+ \leqslant \nu_0^-$ with the δ^+ subgroup of real rank one or $\nu_0^- \leqslant \nu_0^+$ with the δ^- subgroup of real rank one.

Now suppose we are in (a) of Lemma 4.3. Since $\nu_0^+ < \nu_0^-$, the corresponding case of Lemma 4.5 is either (a) or the first half of (c). In either situation, the δ^+ subgroup is asserted to be of real rank one. The other conclusions of Lemma 4.3 follow similarly.

This proves the nonunitarity half of Theorem 3.1. We come now to the unitarity half. Our parameter λ_0 for G is no longer assumed to be a basic case. However, we have defined $\Delta^L \subseteq \Delta$ and an associated basic case λ_0^L for it, and we have defined $\Delta^{L'} \subseteq \Delta^L$ and a special basic case $\lambda_0^{L'} = (\lambda_0^L)^{L'}$ for that. It is enough to prove that the induced representation $U(\nu)$ of G is irreducible for $0 \leqslant \nu(X_\alpha + X_{-\alpha}) < \min\{\nu_0^+, \nu_0^-\}$. By Proposition 6.1 of [16], it is enough to prove the irreducibility for

$$0 \leqslant \nu(X_\alpha + X_{-\alpha}) \leqslant \min\{\nu_0^+, \nu_0^-\} - 2.$$

By Lemma 4.5, it is enough to prove the irreducibility for

$$(4.4) \qquad 0 \leqslant \nu(X_\alpha + X_{-\alpha}) \leqslant \min\{\nu_{L'}^+, \nu_{L'}^-\} - 2.$$

We shall apply the Speh-Vogan theory [4] to the passage from L' to G. We

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[4] See footnote 3.

have to check irreducibility on the L' level, and we have to check that $\lambda_0 + \nu$ has inner product $\geqslant 0$ with the roots defining $\mathfrak{u}'$.

The irreducibility on the L' level is established in Lemma 4.4. Thus we have only to prove that

$$\langle \lambda_0 + \nu, \beta \rangle \geqslant 0 \quad \text{for} \quad \beta \in \Delta^+ - \Delta^{L'} \tag{4.5}$$

whenever ν satisfies (4.4). When L' has real rank > 1, we have seen that the right side of (4.4) is at most 0. Then (4.5) follows (for $\nu = 0$) from the Δ^+ dominance of λ_0. Thus we may assume that L' has real rank one.

Label the simple roots of Δ^+ as $e_1 - e_2, e_2 - e_3, \ldots$. Let i and j be the first and last indices corresponding to the component of α in $\Delta^{L'}$, and let α be $e_k - e_{k+1}$. In (4.5), λ_0 is Δ^+ dominant, and we can regard ν as $\frac{1}{2}\nu(X_\alpha + X_{-\alpha})\alpha$ if we suppress notation that refers to the Cayley transform. Thus we have only to check that

$$\langle \lambda_0, \beta \rangle \geqslant \frac{1}{2} \min\{\nu_{L'}^+, \nu_{L'}^-\} - 1$$

for $\beta = e_r - e_k$ with $r < i$ and for $\beta = e_{k+1} - e_s$ with $j < s$.

The worst possible cases are $\beta = e_{i-1} - e_k$ (if index $i - 1$ exists) and $\beta = e_{k+1} - e_{j+1}$ (if index $j + 1$ exists). Any simple roots in Δ^L that are adjacent to α are included in $\Delta^{L'}$. Hence if $i - 1$ exists, then either $i < k$ or $\langle \lambda_0, e_{i-1} - e_k \rangle > \langle \lambda_{b,0}, e_{i-1} - e_k \rangle$. Similarly if $j + 1$ exists, then either $k + 1 < j$ or $\langle \lambda_0, e_{k+1} - e_{j+1} \rangle > \langle \lambda_{b,0}, e_{k+1} - e_{j+1} \rangle$.

Suppose $i - 1$ exists. If $i < k$, then $\beta = e_{i-1} - e_k$ satisfies

$$\frac{2\langle \lambda_0, \beta \rangle}{|\beta|^2} \geqslant \frac{2\langle \lambda_0, e_i - e_k \rangle}{|\beta|^2} \geqslant \frac{2\langle \lambda_{b,0}, e_i - e_k \rangle}{|\beta|^2} = \frac{2\langle \lambda_0^{L'}, e_i - e_k \rangle}{|\beta|^2}$$

by (4.1). If $e_{k-1} - e_k$ is compact, then the right side of this expression is

$$= (k - i - 1) + \frac{1}{2}\left(1 + \frac{2\langle \mu, \alpha \rangle}{|\alpha|^2}\right)$$

$$= \frac{1}{2}\left(1 + \frac{2\langle \mu, \alpha \rangle}{|\alpha|^2} + 2(k - i)\right) - 1$$

$$\geqslant \frac{1}{2}\nu_{L'}^+ - 1$$

$$\geqslant \frac{1}{2} \min\{\nu_{L'}^+, \nu_{L'}^-\} - 1,$$

as required. We argue similarly with $\nu_{L'}^-$ if $e_{k-1} - e_k$ is noncompact.

If $i = k$, then we have said that $\langle \lambda_0, e_{i-1} - e_k \rangle > \langle \lambda_{b,0}, e_{i-1} - e_k \rangle$. Thus $\beta = e_{i-1} - e_k$ satisfies

$$\frac{2\langle \lambda_0, \beta \rangle}{|\beta|^2} \geqslant \frac{2\langle \lambda_{b,0}, \beta \rangle}{|\beta|^2} + 1 \geqslant 1.$$

Meanwhile one of $\nu_{L'}^+$ and $\nu_{L'}^-$ is at most 2, depending on whether $e_{k+1} - e_{k+2}$ is noncompact or compact. So again we have

$$\frac{2\langle \lambda_0, \beta \rangle}{|\beta|^2} \geqslant \frac{1}{2} \min \{\nu_{L'}^+, \nu_{L'}^-\} - 1.$$

This handles $\beta = e_{i-1} - e_k$, and $\beta = e_{k+1} - e_{j+1}$ is handled similarly. This completes the second half of the proof of Theorem 3.1.

5. PREPARATION FOR THEOREM 3.2

Before beginning the proof of Theorem 3.2, we shall interpret Theorems 2.2, 2.3 and 2.4 for $G = SU(p, q)$ in the way that we shall want to apply them. We continue to assume $p \geqslant q$ and to allow general q, because this level of generality more clearly shows the role of the key hypotheses in the theorems of §2. We shall work with the minimal parabolic subgroup of G, but this is not an essential point. Thus our parabolic subgroup is to be built from the superorthogonal set $\alpha_1, \ldots, \alpha_q$ with α_i defined as in (3.1). All the roots have the same length, which we shall denote simply by $|\alpha|$. Fix the representation σ of M, determine Δ^+ as usual, and fix a parameter μ and the corresponding minimal K-type τ_Λ for the induced representations $U(S, \sigma, \nu)$.

If we expand Λ in terms of the linear functionals e_j, we can regard Λ as a $(p+q)$-tuple of integers, unique up to addition of a constant. The forms $\Lambda + \alpha_r$ and $\Lambda + \alpha_s$ are conjugate if and only if the entries of Λ corresponding to α_r and α_s in the upper left p-by-p block are equal and the entries corresponding to α_r and α_s in the lower right q-by-q block are equal. In this case we say α_r and α_s are *conjugate modulo* Λ. If at least one of these two equalities holds, we say α_r and α_s are *partially conjugate modulo* Λ. (Beware: Partial conjugacy is not necessarily transitive.)

We are using a norm $|\cdot|$ on $i\mathfrak{b}'$ and shall want to use two parts of such a norm in proofs in this section. For this purpose we (temporarily) adjust matters so that $|\alpha|^2 = 2$ for all roots α. Now let μ' in $i\mathfrak{b}'$ correspond to $(a_1, \ldots, a_p, a_{p+1}, \ldots, a_{p+q})$. In our normalization

$$|\mu'|^2=\sum_{j=1}^{p+q} a_j^2-(p+q)^{-1}\left(\sum_{j=1}^{p+q} a_j\right)^2.$$

We define $|\mu'|_1^2$ and $|\mu'|_2^2$ by

$$|\mu'|_1^2=\sum_{j=1}^{p} a_j^2-p^{-1}\left(\sum_{j=1}^{p} a_j\right)^2$$

$$|\mu'|_2^2=\sum_{j=p+1}^{p+q} a_j^2-q^{-1}\left(\sum_{j=p+1}^{p+q} a_j\right)^2.$$

Let $\langle\cdot,\cdot\rangle$, $\langle\cdot,\cdot\rangle_1$, and $\langle\cdot,\cdot\rangle_2$ be the corresponding inner products. The relevance of these definitions is as follows: If μ' is a weight of $\tau_{\Lambda'}$, then $|\mu'|\leqslant|\Lambda'|$, $|\mu'|_1\leqslant|\Lambda'|_1$, and $|\mu'|_2\leqslant|\Lambda'|_2$. Moreover, all three norms are preserved by the action of the Weyl group of Δ_K.

LEMMA 5.1. Let S be a subset of $\{1,\ldots,q\}$, and let r and s be two distinct indices not in S. Then $\Lambda+\sum_{i\in S}\alpha_i+\alpha_r-\alpha_s$ is not a weight of $\left(\Lambda+\sum_{i\in S}\alpha_i\right)^{\sim}$.

PROOF. Let $j(i)=q+i$ and $k(i)=p+q+1-i$ be the two indices attached to α_i in (3.1), and let $\Lambda=(a_1,\ldots,a_{p+q})$. Then

$$\left|\Lambda+\sum_{i\in S}\alpha_i+\alpha_r-\alpha_s\right|_1^2-\left|\Lambda+\sum_{i\in S}\alpha_i\right|_1^2=2(a_{j(r)}-a_{j(s)})+|\alpha_r-\alpha_s|_1^2 \tag{5.1}$$

and

$$\left|\Lambda+\sum_{i\in S}\alpha_i+\alpha_r-\alpha_s\right|_2^2-\left|\Lambda+\sum_{i\in S}\alpha_i\right|_2^2=-2(a_{k(r)}-a_{k(s)})+|\alpha_r-\alpha_s|^2. \tag{5.2}$$

If the form in question is a weight, then both (5.1) and (5.2) are $\leqslant 0$. From the first of these inequalities, we obtain $a_{j(s)}>a_{j(r)}$. Thus $e_{j(s)}-e_{j(r)}>0$ since Λ is Δ_K^+ dominant. From the second of these inequalities, we similarly obtain $e_{k(r)}-e_{k(s)}>0$. Then

$$\alpha_s=(e_{j(s)}-e_{j(r)})+\alpha_r+(e_{k(r)}-e_{k(s)}) \tag{5.3}$$

exhibits α_s as a nontrivial sum of positive roots, in contradiction to the simplicity of α_s.

LEMMA 5.2. If α_r and α_s are not partially conjugate modulo Λ, then $\langle \Lambda, \alpha_r - \alpha_s \rangle_1$ and $\langle \Lambda, \alpha_r - \alpha_s \rangle_2$ are nonzero and of opposite sign.

PROOF. Let $j(i) = q + i$, $k(i) = p + q + 1 - i$, and $\Lambda = (a_1, \ldots, a_{p+q})$. Then $\langle \Lambda, \alpha_r - \alpha_s \rangle_1 = a_{j(r)} - a_{j(s)}$ and $\langle \Lambda, \alpha_r - \alpha_s \rangle_2 = a_{k(s)} - a_{k(r)}$. These two expressions are nonzero because of the assumed failure of partial conjugacy. Suppose they are both negative. Then the Δ_K^+ dominance of Λ implies $e_{j(s)} - e_{j(r)}$ and $e_{k(r)} - e_{k(s)}$ are both positive roots, and (5.3) contradicts the simplicity of α_s. Similarly if both expressions are positive, we obtain a contradiction to the simplicity of α_r.

THEOREM 5.3. In $SU(p, q)$ for the minimal parabolic subgroup, let Λ be the parameter of the minimal K-type of the induced representation. Suppose that $\{\alpha_{m_1}, \ldots, \alpha_{m_n}\}$ is an ordered subset of $\{\alpha_1, \ldots, \alpha_q\}$ such that

(a) each pair α_{m_i} and α_{m_j} is partially conjugate modulo Λ
(b) $\langle \Lambda, \alpha_{m_1} \rangle < \langle \Lambda, \alpha_{m_2} \rangle < \ldots < \langle \Lambda, \alpha_{m_n} \rangle$
(c) for each i with $1 \leqslant i \leqslant n$ and each j not in $\{m_1, \ldots, m_n\}$, either
  (i) $\langle \Lambda, \alpha_{m_i} \rangle < \langle \Lambda, \alpha_j \rangle$ or
  (ii) α_{m_i} and α_j are not partially conjugate modulo Λ.

Define

$$\mu_k = \begin{cases} \Lambda + \alpha_{m_1} + \ldots + \alpha_{m_k} & \text{for} \quad 0 \leqslant k \leqslant n \\ \\ \Lambda + \alpha_{m_1} + \ldots + \alpha_{m_{2n-k}} & \text{for} \quad n \leqslant k \leqslant 2n. \end{cases}$$

Then, for a certain nonzero constant c,

$$\begin{aligned} &\langle P_\Lambda U(\nu, X_{-\alpha_{m_1}}) P_{\check{\mu}_1} \ldots P_{\check{\mu}_{n-1}} U(\nu, X_{-\alpha_{m_n}}) P_{\check{\mu}_n} \\ &\quad \times U(\nu, X_{\alpha_{m_n}}) \ldots P_{\check{\mu}_1} U(\nu, X_{\alpha_{m_1}}) f_0(k), u_0 \rangle_\Lambda \\ &= c \langle \tau_\Lambda(k)^{-1} v_0, v_0 \rangle \times \\ &\quad \times \prod_{i=1}^{n} \Bigg[(\nu + \overline{\rho})(X_{\alpha_{m_i}} + X_{-\alpha_{m_i}}) + \frac{2\langle \mu_{i-1}, \alpha_{m_i} \rangle}{|\alpha|^2} \\ &\qquad - 2\#\{\beta \in \Delta_n \,|\, \beta \perp \alpha_1, \ldots, \alpha_{m_i - 1}; \beta - \alpha_{m_i} \in \Delta; \langle \mu_{i-1}, \beta - \alpha_{m_i} \rangle > 0\} \Bigg] \\ &\quad \times \prod_{i=n+1}^{2n} \Bigg[(\nu + \overline{\rho})(X_{\alpha_{m_{2n+1-i}}} + X_{-\alpha_{m_{2n+1-i}}}) - \frac{2\langle \mu_{i-1}, \alpha_{m_{2n+1-i}} \rangle}{|\alpha|^2} \end{aligned}$$

$$-2\#\{\beta\in\Delta_n\,|\,\beta\perp\alpha_1,\ldots,\alpha_{m_{2n+1-i}}-1;\beta+\alpha_{m_{2n+1-i}}\in\Delta;$$
$$\langle\mu_{i-1},\beta+\alpha_{m_{2n+1-i}}\rangle>0\}\Big].$$

PROOF. We apply Theorem 2.2 recursively, using

$$f_i(k)=\begin{cases}P_{\tilde{\mu}_i}U(\nu,X_{\alpha_{m_i}})f_{i-1}(k) & \text{for}\quad 0<i\leqslant n\\[2ex] P_{\tilde{\mu}_i}U(\nu,X_{-\alpha_{m_{2n+1-i}}})f_{i-1}(k) & \text{for}\quad n<i\leqslant 2n\end{cases}$$

and

$$v_i=\begin{cases}E_{\tilde{\mu}_i}(v_{i-1}\otimes X_{\alpha_{m_i}}) & \text{for}\quad 0<i\leqslant n\\[2ex] E_{\tilde{\mu}_i}(v_{i-1}\otimes X_{-\alpha_{m_{2n+1-i}}}) & \text{for}\quad n<i\leqslant 2n.\end{cases}$$

Lemma 2.1 guarantees that the vectors v_i are nonzero, and hypothesis (b) in Theorem 2.2 is trivially satisfied. The only thing that needs checking is that hypothesis (a) is satisfied in Theorem 2.2.

The proof of (a) consists of two parts. In the first part we are to show that the only weight in $\tau_{\tilde{\mu}_i}$, $0\leqslant i\leqslant n-1$, of the form $\mu_i+\alpha_{m_{i+1}}\pm\alpha_j$ is μ_i itself. We can rule out $+\alpha_j$ since $\alpha_{m_{i+1}}+\alpha_j$ is not a sum of compact roots. We can rule out $-\alpha_j$ for $j\notin\{m_1,\ldots,m_{i+1}\}$ by Lemma 5.1. Thus consider $j=m_k$ with $k<i+1$. Then we have

$$\begin{aligned}|\mu_i+\alpha_{m_{i+1}}-\alpha_{m_k}|^2-|\mu_i|^2&=2\langle\mu_i,\alpha_{m_{i+1}}-\alpha_{m_k}\rangle+2|\alpha|^2\\&=2\langle\Lambda+\alpha_{m_1}+\ldots+\alpha_{m_i},\alpha_{m_{i+1}}-\alpha_{m_k}\rangle+2|\alpha|^2\\&=2\langle\Lambda,\alpha_{m_{i+1}}-\alpha_{m_k}\rangle.\end{aligned}$$

By assumption (b) in the present theorem, this quantity is >0, and thus $\mu_i+\alpha_{m_{i+1}}-\alpha_{m_k}$ cannot be a weight for $k<i+1$. This handles the first part of (a) in Theorem 2.2.

In the second part of the proof of (a) in Theorem 2.2, we are to show that the only weight in $\tau_{\tilde{\mu}_i}$, $n\leqslant i<2n$, of the form $\mu_i-\alpha_{m_{2n-i}}\pm\alpha_j$ is μ_i itself. We can rule out $-\alpha_j$ since $-\alpha_{m_{2n-i}}-\alpha_j$ is not a sum of compact roots. For $+\alpha_j$ with $j=m_k$ and $k>2n-i$, we have

$$\begin{aligned}|\mu_i-\alpha_{m_{2n-i}}+\alpha_j|^2-|\mu_i|^2&=2\langle\mu_i,\alpha_j-\alpha_{m_{2n-i}}\rangle+2|\alpha|^2\\&=2\langle\Lambda,\alpha_j-\alpha_{m_{2n-i}}\rangle.\end{aligned}\tag{5.4}$$

By assumption (b) in the present theorem, this quantity is > 0, and thus $\mu_i - \alpha_{m_{2n-i}} + \alpha_{m_k}$ cannot be a weight for $k > 2n - i$.

For $+\alpha_j$ with $j \notin \{m_1, \ldots, m_n\}$, we apply assumption (c). If (i) holds for $\alpha_{m_{2n-i}}$ and α_j, then (5.4) shows $\mu_i - \alpha_{m_{2n-i}} + \alpha_j$ cannot be a weight. If (ii) holds, then we recalculate (5.4) using $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$. If $\mu_i - \alpha_{m_{2n-i}} + \alpha_j$ is a weitht, then the right side must be $\leqslant 0$ for both inner products, in contradiction to Lemma 5.2.

Finally for $+\alpha_j$ with $j = m_k$ and $k < 2n - i$, we shall use assumption (a). We have

$$\begin{aligned}(5.5)\qquad |\mu_i - \alpha_{m_{2n-i}} + \alpha_{m_k}|_1^2 - |\mu_i|_1^2 &= 2\langle \mu_i, \alpha_{m_k} - \alpha_{m_{2n-i}}\rangle_1 + 2|\alpha|_1^2 \\ &= 2\langle \Lambda, \alpha_{m_k} - \alpha_{m_{2n-i}}\rangle_1 + 2|\alpha|_1^2\end{aligned}$$

and similarly for $|\cdot|_2^2$. Assumption (a) implies that the first term on the right side of (5.5) vanishes either for $|\cdot|_1^2$ or for $|\cdot|_2^2$. The right side of (5.5) or the corresponding relation for $|\cdot|_2^2$ is then > 0 for this norm, and hence $\mu_i - \alpha_{m_{2n-i}} + \alpha_{m_k}$ cannot be a weight for $k < 2n - i$. The proof of Theorem 5.3 is complete.

REMARKS. A variant of Theorem 5.3 allows all α_j to be replaced everywhere in the statement and proof by $-\alpha_j$; the formulas in assumptions (b) and (c, i) are to read

$$\langle \Lambda, -\alpha_{m_1}\rangle < \langle \Lambda, -\alpha_{m_2}\rangle < \ldots < \langle \Lambda, -\alpha_{m_n}\rangle$$

and

$$\langle \Lambda, -\alpha_{m_i}\rangle < \langle \Lambda, -\alpha_j\rangle.$$

We shall make use of both the original statement and the variant.

COROLLARY 5.4. In $SU(p, 2)$ for the minimal parabolic, let Λ be the parameter of the minimal K-type. Suppose that α_1 and α_2 are not partially conjugate modulo Λ.

(a) For $\mu_1 = \Lambda + \alpha_1$ and for suitable nonzero c,

$$\langle P_\Lambda U(\nu, X_{-\alpha_1}) P_{\check{\mu_1}} U(\nu, X_{\alpha_1}) f_0(k), u_0\rangle_\Lambda = c\langle \tau_\Lambda(k)^{-1} v_0, v_0\rangle$$

$$\times \left\{ \nu(X_{\alpha_1} + X_{-\alpha_1})^2 - \left[1 + \frac{2\langle \mu, \alpha_1\rangle}{|\alpha|^2} + 2\#\{\beta \in \Delta_n^+ \mid \beta - \alpha_1 \in \Delta, \langle \Lambda, \beta - \alpha_1\rangle = 0\}\right]^2 \right\}.$$

Moreover, a similar conclusion is valid if α_1 is replaced by $-\alpha_1$.

(b) For $\mu_1 = \Lambda + \alpha_2$ and for suitable nonzero c,

$$\langle P_\Lambda U(\nu, X_{-\alpha_2}) P_{\tilde{\mu}_1} U(\nu, X_{\alpha_2}) f_0(k), u_0 \rangle_\Lambda = c \langle \tau_\Lambda(k)^{-1} v_0, v_0 \rangle$$

$$\times \left\{ \nu(X_{\alpha_2} + X_{-\alpha_2})^2 - \left[1 + \frac{2\langle \mu, \alpha_2 \rangle}{|\alpha|^2} + 2\#\{\beta \in \Delta_n^+ \mid \beta \perp \alpha_1, \beta - \alpha_2 \in \Delta, \langle \Lambda, \beta - \alpha_2 \rangle = 0\} \right]^2 \right\}.$$

Moreover, a similar conclusion is valid if α_2 is replaced by $-\alpha_2$.

PROOF. The corollary results by applying Theorem 5.3 or its variant to one of the subsets $\{\alpha_1\}$ and $\{\alpha_2\}$ and by substituting from Theorem 2.5.

COROLLARY 5.5. In $SU(p, 2)$ for the minimal parabolic, let Λ be the parameter of the minimal K-type. Suppose that α_1 and α_2 are partially conjugate modulo Λ, that $\langle \Lambda, e_{p-1} - e_p \rangle = 0$, and that $\langle \Lambda, \alpha_1 \rangle < \langle \Lambda, \alpha_2 \rangle$. Then

(a) for $\mu_1 = \Lambda + \alpha_1$ and for suitable nonzero c,

$$\langle P_\Lambda U(\nu, X_{\alpha_1}) P_{\mu_1} U(\nu, X_{\alpha_1}) f_0(k), u_0 \rangle_\Lambda = c \langle \tau_\Lambda(k)^{-1} v_0, v_0 \rangle$$

$$\times \left\{ \nu(X_{\alpha_1} + X_{-\alpha_1})^2 - \left[1 + \frac{2\langle \mu, \alpha_1 \rangle}{|\alpha|^2} + 2\#\{\beta \in \Delta_n^+ \mid \beta - \alpha_1 \in \Delta, \langle \Lambda, \beta - \alpha_1 \rangle = 0\} \right]^2 \right\}.$$

(b) for $\mu_1 = \Lambda - \alpha_2$ and for suitable nonzero c,

$$\langle P_\Lambda U(\nu, X_{\alpha_2}) P_{\tilde{\mu}_1} U(\nu, X_{-\alpha_2}) f_0(k), u_0 \rangle_\Lambda = c \langle \tau_\Lambda(k)^{-1} v_0, v_0 \rangle$$

$$\times \left\{ \nu(X_{\alpha_2} + X_{-\alpha_2})^2 - \left[1 - \frac{2\langle \mu, \alpha_2 \rangle}{|\alpha|^2} + 2\#\{\beta \in \Delta_n^+ \mid \beta \perp \alpha_1, \beta + \alpha_2 \in \Delta, \langle \Lambda, \beta + \alpha_2 \rangle = 0\} \right]^2 \right\}.$$

(c) for $\mu_1 = \Lambda - \alpha_2$, for $\mu_2 = \Lambda - \alpha_1 - \alpha_2$, and for suitable nonzero c,

$$\langle P_\Lambda U(\nu, X_{\alpha_2}) P_{\tilde{\mu}_1} U(\nu, X_{\alpha_1}) P_{\tilde{\mu}_2} U(\nu, X_{-\alpha_1}) P_{\tilde{\mu}_1} U(\nu, X_{-\alpha_2}) f_0(k), u_0 \rangle_\Lambda$$

$$= c \langle \tau_\Lambda(k)^{-1} v_0, v_0 \rangle$$

$$\times \left\{ \nu(X_{\alpha_2} + X_{-\alpha_2})^2 - \left[1 - \frac{2\langle \mu, \alpha_2 \rangle}{|\alpha|^2} + 2\#\{\beta \in \Delta_n^+ \mid \beta \perp \alpha_1, \beta + \alpha_2 \in \Delta, \langle \Lambda, \beta + \alpha_2 \rangle = 0\} \right]^2 \right\}$$

$$\times \left\{ \nu(X_{\alpha_1} + X_{-\alpha_1})^2 - \left[1 - \frac{2\langle \mu, \alpha_1 \rangle}{|\alpha|^2} + 2\#\{\beta \in \Delta_n^+ \mid \beta + \alpha_1 \in \Delta, \langle \Lambda, \beta + \alpha_1 \rangle = 0\} \right. \right.$$

$$\left. \left. - 2\#\{\beta \in \Delta_n \mid \beta + \alpha_1 \in \Delta, \beta \not\perp \alpha_2, \langle \Lambda, \beta + \alpha_1 \rangle = 0\} \right]^2 \right\}.$$

PROOF. For (a), we apply Theorem 5.3 to the subset $\{\alpha_1\}$ and then substitute from Theorem 2.5. For (b), we proceed similarly, using the variant of Theorem 5.3 and the subset $\{\alpha_2\}$. In (c), we apply the variant of Theorem 5.3 to the subset $\{\alpha_2, \alpha_1\}$. In this case the application of Theorem 2.5 is routine for the terms involving $\nu(X_{\alpha_2} + X_{-\alpha_2})$ but involves a few extra steps for the terms involving $\nu(X_{\alpha_1} + X_{-\alpha_1})$ and uses the hypothesis $\langle \Lambda, e_{p-1} - e_p \rangle = 0$. We omit the details.

THEOREM 5.6. In $SU(p, 2)$ for the minimal parabolic, let Λ be the parameter of the minimal K-type. Suppose that α_1 and α_2 are conjugate modulo Λ. For $\mu_1 = \Lambda - \alpha_2$, for $\mu_2 = \Lambda - \alpha_1 - \alpha_2$, and for suitable nonzero c,

$$\begin{aligned}
&\langle P_\Lambda U(\nu, X_{\alpha_2}) P_{\tilde{\mu}_1} U(\nu, X_{\alpha_1}) P_{\tilde{\mu}_2} U(\nu, X_{-\alpha_1}) P_{\tilde{\mu}_1} U(\nu, X_{-\alpha_2}) f_0(k), u_0 \rangle_\Lambda \\
&= c \langle \tau_\Lambda(k)^{-1} v_0, v_0 \rangle \\
&\times \left\{ \nu(X_{\alpha_2} + X_{-\alpha_2})^2 - \left[1 - \frac{2\langle \mu, \alpha_2 \rangle}{|\alpha|^2} + 2\#\{\beta \in \Delta_n^+ \mid \beta \perp \alpha_1, \beta + \alpha_2 \in \Delta, \langle \Lambda, \beta + \alpha_2 \rangle = 0\} \right]^2 \right\} \\
&\times \left\{ \nu(X_{\alpha_1} + X_{-\alpha_1})^2 - \left[1 - \frac{2\langle \mu, \alpha_1 \rangle}{|\alpha|^2} + 2\#\{\beta \in \Delta_n^+ \mid \beta + \alpha_1 \in \Delta, \langle \Lambda, \beta + \alpha_1 \rangle = 0\} \right.\right. \\
&\qquad\qquad \left.\left. - \#\{\beta \in \Delta_n \mid \beta + \alpha_1 \in \Delta, \beta \not\perp \alpha_2, \langle \Lambda, \beta + \alpha_1 \rangle = 0\} \right]^2 \right\}.
\end{aligned}$$

PROOF. We apply Theorem 2.3 twice. The first time we use $\mu' = \Lambda$, $\pm\alpha_r = -\alpha_1$, and $\pm\alpha_s = -\alpha_2$. The second time we use $\mu' = \Lambda - \alpha_1 - \alpha_2$, $\pm\alpha_r = \alpha_1$, and $\pm\alpha_s = \alpha_2$. The verifications of assumptions (a) and (b) are as in Theorem 5.3, and assumption (c) is trivially satisfied. In each application of Theorem 2.3, we have $\mathrm{d}_1(\nu) = \mathrm{d}_4(\nu)$. We factor these out and substitute for them from Theorem 2.5. However, $\mathrm{d}_2(\nu)$ and $\mathrm{d}_3(\nu)$ are not equal, and it is necessary to identify their difference in order to apply Theorem 2.5. This calculation is similar to the one we omitted at the end of the proof of Corollary 5.5.

THEOREM 5.7. In $SU(p, 2)$ for the minimal parabolic, let Λ be the parameter of the minimal K-type. Suppose that $\langle \Lambda, e_{p-1} - e_p \rangle = 0$. Let $\Lambda' = (\Lambda - \alpha_1)\check{}$, and suppose that $\Lambda - \alpha_2$ is a weight of $\tau_{\Lambda'}$. For a certain nonzero constant c,

$$\begin{aligned}
&\langle P_\Lambda U(\nu, X_{\alpha_1}) P_{\Lambda'} U(\nu, X_{-\alpha_1}) f_0(k), u_0 \rangle_\Lambda = c \langle \tau_\Lambda(k)^{-1} v_0, v_0 \rangle \\
&\quad \times \{ [\nu(X_{\alpha_1} + X_{-\alpha_1})^2 - c_1^2] + C^{-1} [\nu(X_{\alpha_2} + X_{-\alpha_2})^2 - c_2^2] \},
\end{aligned}$$

where

$$(5.6)\quad C=\frac{2\langle\Lambda, e_{p+2}-e_{p+1}\rangle}{|\alpha|^2}+1\geqslant 1,$$

$$c_1=1-\frac{2\langle\mu,\alpha_1\rangle}{|\alpha|^2}+2\#\{\beta\in\Delta_n^+\mid \beta+\alpha_1\in\Delta, \langle\Lambda,\beta+\alpha_1\rangle=0\},$$

$$c_2=1-\frac{2\langle\mu,\alpha_2\rangle}{|\alpha|^2}+2\#\{\beta\in\Delta_n^+\mid \beta\perp\alpha_1, \beta+\alpha_2\in\Delta, \langle\Lambda,\beta+\alpha_2\rangle=0\}.$$

PROOF. We shall apply Theorem 2.4 with $\pm\alpha_r=-\alpha_1$ and $\pm\alpha_s=-\alpha_2$. Assumptions (a) and (b) in that theorem are automatic. Let us verify (c). We are given $\beta\in\Delta_n$ with $\beta\perp\alpha_1$ and $\beta-\alpha_2\in\Delta$ such that $\Lambda-\beta$ is a weight of $\tau_{\Lambda'}$. Since $\beta\perp\alpha_1$ and $\beta-\alpha_2\in\Delta$, β is of the form e_j-e_{p+1} with $j<p-1$. Hence

$$(5.7)\qquad \langle\Lambda,\beta-\alpha_2\rangle=\langle\Lambda,\beta-\alpha_2\rangle_1.$$

Since $\Lambda-\beta$ is a weight of $\tau_{\Lambda'}$,

$$(5.8)\qquad 0\leqslant|\Lambda-\alpha_1|_1^2-|\Lambda-\beta|_1^2=2\langle\Lambda,\beta-\alpha_1\rangle_1.$$

The hypothesis $\langle\Lambda, e_{p-1}-e_p\rangle=0$ implies $2\langle\Lambda,\alpha_1\rangle_1=2\langle\Lambda,\alpha_2\rangle_1$. Substituting this fact into (5.8) and (5.7), we see that $\langle\Lambda,\beta-\alpha_2\rangle\geqslant 0$. This proves assumption (c).

Next let us see that $\Lambda-\alpha_2$ has multiplicity one in $\tau_{\Lambda'}$. In fact, the equality $\langle\Lambda, e_{p-1}-e_p\rangle=0$ means that $|\Lambda-\alpha_2|_1^2=|\Lambda-\alpha_1|_1^2$, so that the $SU(p)$ part of $\Lambda-\alpha_2$ is extreme in the $SU(p)$ part of $\tau_{\Lambda'}$. Hence only the $SU(2)$ part of $\Lambda-\alpha_2$ can contribute to the multiplicity of $\Lambda-\alpha_2$, and in $SU(2)$ the multiplicities are all one.

Finally we check assumption (b). We are to show that

$$(5.9)\qquad E_\Lambda(E_{\Lambda'}(v_0\otimes X_{-\alpha_1})\otimes X_{\alpha_1})=CE_\Lambda(E_{\Lambda'}(v_0\otimes X_{-\alpha_2})\otimes X_{\alpha_2})$$

with C as in (5.6). The fact that the constant in (5.6) is $\geqslant 1$ follows from the assumption that $\Lambda-\alpha_2$ is a weight of $\tau_{\Lambda'}$:

$$0\leqslant|\Lambda-\alpha_1|_2^2-|\Lambda-\alpha_2|_2^2=2\langle\Lambda,\alpha_2-\alpha_1\rangle_2=2\langle\Lambda,e_{p+2}-e_{p+1}\rangle.$$

Let us sketch the calculation of (5.9) under the assumption that $e_{p+2}-e_{p+1}$ is positive (which happens automatically unless $C=1$).

Define C by (5.6). Write X_{ij} as an abbreviation for $X_{e_i-e_j}$. And let s be a representative within the $SU(2)$ subgroup built from $e_{p-1}-e_p$ of the Weyl group reflection $s_{e_{p-1}-e_p}$. We write $s(-)$ for the group operation of s, dropping mention of the representation. The equality $\langle \Lambda, e_{p-1} - e_p \rangle = 0$ forces $sv_0 = v_0$. Thus

$$\begin{aligned} E_\Lambda(E_{\Lambda'}(v_0 \otimes X_{-\alpha_1}) \otimes X_{\alpha_1}) &= sE_\Lambda(E_{\Lambda'}(v_0 \otimes X_{-\alpha_1}) \otimes X_{\alpha_1}) \\ &= E_\Lambda(E_{\Lambda'}(v_0 \otimes sX_{-\alpha_1}) \otimes sX_{\alpha_1}). \end{aligned} \tag{5.10}$$

Meanwhile we take the identity

$$(C-1)E_\Lambda(E_{\Lambda'}(v_0 \otimes sX_{-\alpha_1}) \otimes X_{\alpha_1}) = X_{p+2,p+1} X_{p+1,p+2} E_\Lambda(E_{\Lambda'}(v_0 \otimes sX_{-\alpha_1}) \otimes X_{\alpha_1}),$$

expand the right side as a sum of four terms (two of which vanish), and rearrange the result, using (5.10), to obtain

$$\begin{aligned} &E_\Lambda(E_{\Lambda'}(v_0 \otimes X_{-\alpha_1}) \otimes X_{\alpha_1}) \\ &\quad = -E_\Lambda(E_{\Lambda'}(X_{p+1,p+2} v_0 \otimes sX_{-\alpha_1}) \otimes [X_{p+2,p+1}, sX_{\alpha_1}]) \\ &\qquad - E_\Lambda(E_{\Lambda'}(v_0 \otimes [X_{p+1,p+2}, sX_{-\alpha_1}]) \otimes [X_{p+2,p+1}, sX_{\alpha_1}]). \end{aligned} \tag{5.11}$$

Next we expand the right side of

$$2(C-1)E_{\Lambda'}(v_0 \otimes X_{-\alpha_2}) = X_{p+2,p+1} X_{p+1,p+2} E_{\Lambda'}(v_0 \otimes X_{-\alpha_2})$$

in the same way as the sum of four terms (two of which vanish), and we obtain

$$E_{\Lambda'}(X_{p+1,p+2} v_0 \otimes [X_{p+2,p+1}, X_{-\alpha_2}]) = (C-1)E_{\Lambda'}(v_0 \otimes X_{-\alpha_2}). \tag{5.12}$$

If we write

$$X_{-\alpha_2} = a[X_{p+1,p+2}, sX_{-\alpha_1}], \tag{5.13}$$

then we obtain

$$[X_{p+2,p+1}, X_{-\alpha_2}] = asX_{\alpha_1}$$

and (5.12) therefore becomes

$$E_{\Lambda'}(X_{p+1,p+2} v_0 \otimes sX_{\alpha_1}) = (C-1)E_{\Lambda'}(v_0 \otimes [X_{p+1,p+2}, sX_{-\alpha_1}]).$$

Hence (5.11) simplifies to

$$\begin{aligned} &E_\Lambda(E_{\Lambda'}(v_0 \otimes X_{-\alpha_1}) \otimes X_{\alpha_1}) \\ &\quad = -CE_\Lambda(E_{\Lambda'}(v_0 \otimes [X_{p+1,p+2}, sX_{-\alpha_1}]) \otimes [X_{p+2,p+1}, sX_{\alpha_1}]). \end{aligned} \tag{5.14}$$

The complex conjugate of (5.13) is

$$X_{\alpha_2} = -\bar{a}[X_{p+2,p+1}, sX_{\alpha_1}],$$

and it is easy to see that $|a|^2 = 1$. Substituting into (5.14), we therefore obtain (5.9).

Thus we can apply Theorem 2.4. Substituting from the formulas of Theorem 2.5, we obtain the conclusion of Theorem 5.7.

6. PROOF OF THEOREM 3.2

We turn now to the proof of Theorem 3.2. We work with $G = SU(p, 2)$, and we may assume $p \geqslant 3$. Before reintroducing the subgroup L of G, we define some parameters of special interest. Let us write

$$\sigma \begin{array}{c} \begin{array}{cc} p-2 & \quad 4 \end{array} \\ \left(\begin{array}{c|cccc} \omega & & & & \\ \hline & e^{i\theta} & & & \\ & & e^{i\varphi} & & \\ & & & e^{i\varphi} & \\ & & & & e^{i\theta} \end{array}\right) \end{array} = e^{i(m\theta + n\varphi)}\sigma_0(\omega),$$

where σ_0 is an irreducible representation of $U(p-2)$. This formula defines integers m and n. These integers are not unique since $e^{2i(\theta+\varphi)}$ can always be absorbed into σ_0; however, the difference $m-n$ is well defined.

Following §8 of [9], we introduce a *fundamental rectangle*

$$0 \leqslant \nu(X_{\alpha_1} + X_{-\alpha_1}) \leqslant a_0$$

$$0 \leqslant \nu(X_{\alpha_2} + X_{-\alpha_2}) \leqslant b_0$$

in $\mathbf{a}'$. Namely if we restrict σ to the subgroup of M where $\varphi = 0$, we obtain a representation σ_1 of the M for a subgroup $SU(p-1, 1)$ of G. The number a_0 is the endpoint of the complementary series of $SU(p-1, 1)$ associated to σ_1. It is described as the least value of $a \geqslant 0$ such that the infinitesimal character of the representation of $SU(p-1, 1)$ induced from σ_1 and the A parameter $\nu(X_{\alpha_1} + X_{-\alpha_1}) = a$ is integral and fails to be singular with respect to two linearly independent roots. The number b_0 is defined similarly in terms of a representation σ_2 obtained from the subgroup of M where $\theta = 0$.

As we introduce our Δ^+, we introduce also a normalization of σ. Complex conjugation is an (outer) automorphism of G. On the series induced from our MAN, this automorphism is implemented by sending σ to $\overline{\sigma}$ and leaving ν fixed. In the process, (m, n) is replaced by $(-m, -n)$, and all of our constructs behave nicely. As a consequence, we may assume without loss of generality that $m \geqslant n$. The contribution to the infinitesimal character λ_0 of σ from the last four indices is

$$\frac{1}{2}\, m(e_{p-1} + e_{p+2}) + \frac{1}{2}\, n(e_p + e_{p+1}).$$

When $m > n$, our choice of Δ^+ must therefore have $e_{p-1} - e_p$ and $e_{p+2} - e_{p+1}$ in Δ^+. When $m = n$, the roots $e_{p-1} - e_p$ and $e_{p+2} - e_{p+1}$ must anyway have the same sign, so that α_1 and α_2 can both be simple, and in our choice of Δ^+ we can insist that these roots be positive. Thus as part of our normalization, we shall insist that $e_{p-1} - e_p$ and $e_{p+2} - e_{p+1}$ are in Δ^+.

LEMMA 6.1. (a) The number a_0 is the smaller of

$$1 + \frac{2\langle \mu, \alpha_1 \rangle}{|\alpha|^2} + 2\#\{\beta \in \Delta_n^+ \mid \beta \perp \alpha_2, \beta - \alpha_1 \in \Delta, \langle \Lambda, \beta - \alpha_1 \rangle = 0\}$$

and

$$1 - \frac{2\langle \mu, \alpha_1 \rangle}{|\alpha|^2} + 2\#\{\beta \in \Delta_n^+ \mid \beta \perp \alpha_2, \beta + \alpha_1 \in \Delta, \langle \Lambda, \beta + \alpha_1 \rangle = 0\}.$$

(b) The number b_0 is the smaller of

$$1 + \frac{2\langle \mu, \alpha_2 \rangle}{|\alpha|^2} + 2\#\{\beta \in \Delta_n^+ \mid \beta \perp \alpha_1, \beta - \alpha_2 \in \Delta, \langle \Lambda, \beta - \alpha_2 \rangle = 0\}$$

and

$$1 - \frac{2\langle \mu, \alpha_2 \rangle}{|\alpha|^2} + 2\#\{\beta \in \Delta_n^+ \mid \beta \perp \alpha_1, \beta + \alpha_2 \in \Delta, \langle \Lambda, \beta + \alpha_2 \rangle = 0\}.$$

PROOF. Let G_1 be the subgroup of G built from all roots orthogonal to α_2, and use «G_1» as a superscript or subscript on constructs in G_1. The number a_0 is the length of the complementary series in G_1 corresponding to M parameter σ_1. By Theorem 3.1 (applied to the semisimple part of G_1, which is isomorphic to $SU(p-1, 1)$), a_0 is the smaller of

$$1 + \frac{2\langle \mu, \alpha_1 \rangle}{|\alpha|^2} + 2\#\{\beta \in \Delta_{n,G_1}^+ \mid \beta - \alpha_1 \in \Delta, \langle \Lambda^{G_1}, \beta - \alpha_1 \rangle = 0\}$$

and

$$1- \frac{2\langle \mu, \alpha_1\rangle}{|\alpha|^2} + 2\#\{\beta \in \Delta^+_{n,G_1} \mid \beta + \alpha_1 \in \Delta, \langle \Lambda^{G_1}, \beta + \alpha_1\rangle = 0\}.$$

So conclusion (a) follows if it is shown that $\langle \Lambda, \gamma\rangle = \langle \Lambda^{G_1}, \gamma\rangle$ for all compact roots γ orthogonal to α_2. From (2.1) we have

$$\Lambda = \lambda - E(2\rho_K) + \mu$$

$$\Lambda^{G_1} = \lambda - E(2\rho_K^{G_1}) + \mu$$

$$\langle E(2\rho_K), \gamma\rangle = \frac{\langle 2\rho_K, \alpha_1\rangle}{|\alpha|^2} \langle \alpha_1, \gamma\rangle + \frac{\langle 2\rho_K, \alpha_2\rangle}{|\alpha|^2} \langle \alpha_2, \gamma\rangle$$

$$\langle E(2\rho_K^{G_1}), \gamma\rangle = \frac{\langle 2\rho_K^{G_1}, \alpha_1\rangle}{|\alpha|^2} \langle \alpha_1, \gamma\rangle.$$

So matters reduce to showing that

$$(6.1) \qquad \langle 2\rho_K, \alpha_1\rangle = \langle 2\rho_K^{G_1}, \alpha_1\rangle.$$

The positive compact roots that contribute to the left side of (6.1) but not to the right side are $e_{p-1} - e_p$ and $e_{p+2} - e_{p+1}$, and their sum is orthogonal to α_1. Thus (6.1) is proved, and conclusion (a) follows. Conclusion (b) is proved similarly.

Now we fix a compatible format and introduce Δ^L, L, and σ^L as in §3. We can control the behavior of compact roots by the following lemma.

LEMMA 6.2. If γ is in Δ_K and $\langle \Lambda, \gamma\rangle = 0$, then γ is in Δ^L and $\langle \Lambda^L, \gamma\rangle = 0$.

PROOF. Form the basic case $\lambda_{b,0}$ for the given format, and let Λ_b be the corresponding minimal K-type parameter. Then we have

$$0 \leqslant \langle \lambda_0 - \lambda_{b,0}, \gamma\rangle = \langle \Lambda - \Lambda_b, \gamma\rangle = -\langle \Lambda_b, \gamma\rangle \leqslant 0.$$

Hence $\langle \lambda_0, \gamma\rangle = \langle \lambda_{b,0}, \gamma\rangle$, and γ is in Δ^L. By (4.2), $\langle \Lambda^L, \gamma\rangle = 0$.

The proof of Theorem 3.2 divides into two cases, depending on whether α_1 and α_2 are in the same simple component of Δ^L or not. We treat first the case where they are not in the same simple component. Then L is essentially a product of some $SU(p', 1)$ and some $SU(p'', 1)$. In view of Lemma 6.2 and the formulas of Lemma 6.1, the parameters ν that lead to unitary representations of L are the ones in the closed positive Weyl chamber that lie in the fundamental rectangle (of G) and that have $J^L(S \cap L, \sigma^L, \nu)$ defined. To get a match of the unitary parameters ν for L and those for G, we must therefore show:

(i) If $\nu \neq 0$, then $J^L(S \cap L, \sigma^L, \nu)$ is defined if and only if $J(S, \sigma, \nu)$ is defined.
(ii) $U(S, \sigma, \nu)$ is irreducible in the interior of the fundamental rectangle.
(iii) There are no unitary points for G outside the fundamental rectangle.

The key to (i) is the following lemma.

LEMMA 6.3. When α_1 and α_2 are in different components of Δ^L, the number

$$1 + \frac{2\langle \mu, \alpha_1\rangle}{|\alpha|^2} + 2\#\{\beta \in \Delta_n^+ \mid \beta \perp \alpha_2, \beta - \alpha_1 \in \Delta, \langle \Lambda, \beta - \alpha_1\rangle = 0\} \tag{6.2}$$

is zero if and only if $\Lambda + \alpha_1$ is a second minimal K-type parameter for $U(S, \sigma, \nu)$. Similar statements apply with α_1 replaced by $-\alpha_1$ and with α_1 and α_2 interchanged.

PROOF. Suppose (6.2) is zero. Then $2\langle \mu, \alpha_1\rangle / |\alpha|^2 = -1$ and every $\beta \in \Delta_n^+$ with $\beta \perp \alpha_2$ and $\beta - \alpha_1 \in \Delta$ has $2\langle \Lambda, \beta - \alpha_1\rangle / |\alpha|^2 \geqslant 1$. According to Theorem 1 of [6], the only way that $\Lambda + \alpha_1$ can fail to be a minimal K-type parameter is if $\Lambda + \alpha_1$ fails to be Δ_K^+ dominant. Thus suppose that there is some γ in Δ_K^+ with $\langle \Lambda + \alpha_1, \gamma\rangle < 0$. Since $\langle \Lambda, \gamma\rangle$ is $\geqslant 0$, we must have $\langle \Lambda, \gamma\rangle = 0$ and $\langle \alpha_1, \gamma\rangle < 0$. Thus $\beta = \gamma + \alpha_1$ is in Δ_n^+, has $\beta - \alpha_1 \in \Delta$, and has $\langle \Lambda, \beta - \alpha_1\rangle = 0$. From what we have assumed, we see that $\beta \not\perp \alpha_2$.

Let us see that $\beta \not\perp \alpha_2$ gives a contradiction. Lemma 6.2 and the equality $\langle \Lambda, \beta - \alpha_1\rangle = 0$ imply that $\beta - \alpha_1$ is in Δ^L. Hence β, α_1, and α_2 are in Δ^L. Since $\beta \not\perp \alpha_1$ and $\beta \not\perp \alpha_2$, α_1 and α_2 are in the same simple component of Δ^L. Thus we have the required contradiction. The first statement of the lemma follows, and the rest follows similarly.

To prove (i), we argue as follows. Suppose $J^L(S \cap L, \sigma^L, \nu)$ is not defined on the α_1 axis. This means that the induced representations of L have both Λ^L and $\Lambda^L \pm \alpha_2$ as minimal $(K \cap L)$- types. Lemmas 6.3 and 6.1b say that $b_0 = 0$. A second application of these lemmas allows us to conclude that the induced representations of G have both Λ and $\Lambda \pm \alpha_2$ as minimal K-types. Then $U(S, \sigma, \nu)$ is reducible for ν on the α_1 axis, and $J(S, \sigma, \nu)$ is not defined. For the converse, we reverse the steps.

Let us turn to (ii). From §8 of [9], the only reducibility in the interior of the fundamental rectangle occurs on the lines

$$(6.3)\qquad \begin{aligned} a+b &= m-n+2l, &\quad l \text{ an integer} \geqslant 1\\ a-b &= m-n+2k, &\quad k \text{ an integer} \geqslant 1, \end{aligned}$$

where

$$(6.4)\qquad a=\nu(X_{\alpha_1}+X_{-\alpha_1}) \quad\text{and}\quad b=\nu(X_{\alpha_2}+X_{-\alpha_2}).$$

(Recall we are assuming $m \geqslant n$). So it is enough to prove that

$$(6.5)\qquad a_0+b_0 \leqslant m-n+2.$$

Put

$$\nu_2^+ = 1+\frac{2\langle\mu,\alpha_2\rangle}{|\alpha|^2}+2\#\{\beta\in\Delta_n^+ \mid \beta\perp\alpha_1, \beta-\alpha_2\in\Delta, \langle\Lambda,\beta-\alpha_2\rangle=0\}$$

$$\nu_1^- = 1-\frac{2\langle\mu,\alpha_1\rangle}{|\alpha|^2}+2\#\{\beta\in\Delta_n^+ \mid \beta\perp\alpha_2, \beta+\alpha_1\in\Delta, \langle\Lambda,\beta+\alpha_1\rangle=0\}.$$

Let consecutive roots in the Dynkin diagram of Δ^+ be

$$e_{p-1}-e_{p+2}, e_{p+2}-e_i, \ldots, e_{j-1}-e_p, e_p-e_{p+1}.$$

Since α_1 and α_2 are not connected in the diagram for $(\Delta^L)^+$, the β's that contribute to ν_2^+ are at most

$$e_{j-1}-e_{p+1}, \ldots, e_i-e_{p+1},$$

while the β's that contribute to ν_1^- are at most

$$e_{p+2}-e_i, \ldots, e_{p+2}-e_{j-1}.$$

Moreover, the roots contributing to ν_2^+ are orthogonal to those contributing to ν_1^-. Thus

$$(6.6)\qquad \nu_2^+ + \nu_1^- \leqslant 2+\frac{2\langle\mu,\alpha_2-\alpha_1\rangle}{|\alpha|^2}+2(j-i).$$

On the other hand, we have

$$(6.7)\qquad \Lambda=\lambda_0+\rho-2\rho_K+\mu-\frac{1}{2}\alpha_1-\frac{1}{2}\alpha_2$$

from (4.13) of [9]. Since $e_{p-1}-e_p$ has nonzero inner product with both α_1 and α_2, it is not in Δ^L, and Lemma 6.2 gives $\langle\Lambda, e_{p-1}-e_p\rangle > 0$. Applying

$2\langle\cdot, e_{p-1}-e_p\rangle/|\alpha|^2$ to both sides of (6.7), we obtain

$$1 \leqslant \frac{1}{2}(m-n)+(j-i+3)-2(j-i+1)+\frac{2\langle\mu, e_{p-1}-e_p\rangle}{|\alpha|^2}+0.$$

Hence

$$j-i \leqslant \frac{1}{2}(m-n)+\frac{\langle\mu, \alpha_1-\alpha_2\rangle}{|\alpha|^2}. \tag{6.8}$$

Since $a_0+b_0 \leqslant \nu_2^+ + \nu_1^-$, (6.6) and (6.8) together prove (6.5) and hence (ii).

Finally we prove (iii). Lemma 6.2 implies that

$$\langle\Lambda, e_{p-1}-e_p\rangle > 0 \quad \text{and} \quad \langle\Lambda, e_{p+2}-e_{p+1}\rangle > 0.$$

Hence α_1 and α_2 are not partially conjugate modulo Λ, in the sense of §5. Thus we can apply the two estimates for $\nu(X_{\alpha_1}+X_{-\alpha_1})$ given in Corollary 4.4a and the two estimates for $\nu(X_{\alpha_2}+X_{-\alpha_2})$ given in Corollary 4.4b. The result is that there are no unitary points outside the fundamental rectangle. This proves (iii) and completes the proof of Theorem 3.2 in the case that α_1 and α_2 are in different simple components of Δ^L.

The second case for the proof of Theorem 3.2 is that α_1 and α_2 lie in the same simple component of Δ^L. This component yields some $SU(p', 2)$ with $p' \leqslant p$. Our procedure will be as follows: We compute $\langle\Lambda, e_{p-1}-e_p\rangle$ and $\langle\Lambda, e_{p+2}-e_{p+1}\rangle$ in order to see that α_1 and α_2 are partially conjugate modulo Λ. When $\langle\Lambda, e_{p-1}-e_p\rangle$ is nonzero, we shall see that there are no unitary points in L or in G. When $\langle\Lambda, e_{p-1}-e_p\rangle = 0$, the basic case σ^L will be one of those considered in §2 of [9], and we shall make a detailed comparison of the unitary points, partly with the help of Corollary 5.5 and Theorems 5.6 and 5.7.

LEMMA 6.4. When α_1 and α_2 are in the same simple component of Δ^L, then $\langle\Lambda, e_{p-1}-e_p\rangle = 0$ unless α_1 and α_2 have only one simple root (namely $e_{p+2}-e_p$) between them in the Dynkin diagram. If $\langle\Lambda, e_{p-1}-e_p\rangle > 0$, then there are no values of ν in L or G that correspond to unitary representations. If $\langle\Lambda, e_{p-1}-e_p\rangle = 0$, then $2\langle\Lambda, e_{p+2}-e_{p+1}\rangle/|\alpha|^2 = m-n$ and $\langle\Lambda, e_i-e_j\rangle = 0$ if e_i-e_j is in Δ^L and $i \leqslant p$ and $j \leqslant p$.

PROOF. Suppose that there is more than one simple root between α_1 and α_2. Let consecutive roots in the diagram be

$$e_{p-1}-e_{p+2}, e_{p+2}-e_i, \ldots, e_{j-1}-e_p, e_p-e_{p+1},$$

and let $\mu=\mu_1\alpha_1+\mu_2\alpha_2$. Plainly $e_{p-1}-e_p$ is the sum of $j-i+2$ simple roots of Δ^+ and $j-i+1$ simple roots of Δ_K^+, and all of these roots are in Δ^L. We calculate Λ from the identity (6.7) applied to Λ^L and from the formula for λ_0^L in Corollary 2.7 of [7]. Under the convention that $|\alpha|^2=2$, the result is

$$\begin{aligned}\langle\Lambda, e_{p-1}-e_p\rangle &= \langle\Lambda^L, e_{p-1}-e_p\rangle\\ &= \langle\lambda_0^L+\rho^L-2\rho_K^L+\mu-\frac{1}{2}\alpha_1-\frac{1}{2}\alpha_2, e_{p-1}-e_p\rangle\\ &= \left[\left(\frac{1}{2}-\mu_1\right)+(j-i-1)+\left(\frac{1}{2}+\mu_2\right)\right]+(j-i+2)\\ &\quad -2(j-i+1)+(\mu_1-\mu_2)=0.\end{aligned}$$

Also we have

$$\begin{aligned}&\langle\Lambda, (e_{p-1}-e_p)+(e_{p+2}-e_{p+1})\rangle\\ (6.9)\qquad &= \langle\lambda-E(2\rho_K)+\mu, (e_{p-1}+e_{p+2})-(e_p+e_{p+1})\rangle\\ &= \langle\lambda, (e_{p-1}+e_{p+2})-(e_p+e_{p+1})\rangle = m-n.\end{aligned}$$

Hence

$$\langle\Lambda, e_{p+2}-e_{p+1}\rangle = m-n.$$

We omit the easy calculation of $\langle\Lambda, e_i-e_j\rangle$ for the remaining compact roots of Δ^L.

The remaining case is that consecutive roots in the Dynkin diagram are

$$e_{p-1}-e_{p+2}, e_{p+2}-e_p, e_p-e_{p+1}.$$

We calculate Λ in the same way as above: From [7],

$$\langle\lambda_0^L, e_{p+2}-e_p\rangle = |\langle\mu, e_{p-1}-e_p\rangle| = |\mu_1-\mu_2|.$$

Since $e_{p-1}-e_p$ and $e_{p+2}-e_{p+1}$ are Δ_K^+ simple and are orthogonal to $\frac{1}{2}(\alpha_1+\alpha_2)$,

$$(6.10)\qquad \begin{aligned}\langle\Lambda, e_{p-1}-e_p\rangle &= \langle\lambda_0^L, e_{p-1}-e_p\rangle+\langle\mu, e_{p-1}-e_p\rangle = |\mu_1-\mu_2|+(\mu_1-\mu_2)\\ \langle\Lambda, e_{p+2}-e_{p+1}\rangle &= \langle\lambda_0^L, e_{p+2}-e_{p+1}\rangle+\langle\mu, e_{p+2}-e_{p+1}\rangle = |\mu_1-\mu_2|-(\mu_1-\mu_2).\end{aligned}$$

If $\langle \Lambda, e_{p-1} - e_p \rangle = 0$, then (6.9) shows that $\langle \Lambda, e_{p+2} - e_{p+1} \rangle = m - n$; the other calculations of $\langle \Lambda, e_i - e_j \rangle$ are easy and are omitted.

Suppose $\langle \Lambda, e_{p-1} - e_p \rangle > 0$. Then $\mu_1 - \mu_2 > 0$ by (6.10). So $\mu_1 \geqslant 0$ and $\mu_2 \leqslant 0$, with at least one of them nonzero. Under these circumstances, we shall see that $\Lambda - \alpha_1$ is another minimal K-type parameter if $\mu_1 > 0$, and $\Lambda + \alpha_2$ is another minimal K-type parameter if $\mu_2 < 0$. Similar conclusions apply also to L, and there are no unitary points in G or L, by Vogan's theory [17] of minimal K-types.

Say $\mu_2 < 0$. By Theorem 1 of [6], it is enough to show that $\Lambda + \alpha_2$ is Δ_K^+ dominant. We have

$$\begin{aligned} \langle \Lambda + \alpha_2, e_{p+2} - e_{p+1} \rangle &= \langle \alpha_2, e_{p+2} - e_{p+1} \rangle = 1 \\ \langle \Lambda + \alpha_2, e_i - e_p \rangle &= \langle \Lambda, e_i - e_{p-1} \rangle + \langle \Lambda, e_{p-1} - e_p \rangle - 1 \\ &\geqslant \langle \Lambda, e_i - e_{p-1} \rangle \geqslant 0 \quad \text{if } i \leqslant p-1 \text{ and } e_i - e_p > 0 \\ \langle \Lambda + \alpha_2, e_p - e_j \rangle &= \langle \Lambda, e_p - e_j \rangle + 1 \geqslant 1 \quad \text{if } j \leqslant p-1 \text{ and } e_p - e_j > 0. \end{aligned}$$

Hence $\Lambda + \alpha_2$ is Δ_K^+ dominant. The argument is similar when $\mu_1 > 0$, and the lemma follows.

In view of the lemma, we may henceforth assume that

$$\langle \Lambda^L, e_{p-1} - e_p \rangle = \langle \Lambda, e_{p-1} - e_p \rangle = 0 \tag{6.11a}$$

$$\frac{2\langle \Lambda^L, e_{p+2} - e_{p+1} \rangle}{|\alpha|^2} = \frac{2\langle \Lambda, e_{p+2} - e_{p+1} \rangle}{|\alpha|^2} = m - n. \tag{6.11b}$$

The part of L containing α_1 and α_2 is of the form $SU(p', 2)$ for some p' with $2 \leqslant p' \leqslant p$. The lemma says that τ_{Λ^L} is scalar on the $U(p')$ part of $L \cap K$. Since σ^L must occur in τ_{Λ^L}, the highest weight λ^L of σ^L must be of the form

$$\lambda^L = \frac{1}{2} m_L (e_{p-1} + e_{p+2}) + \frac{1}{2} n_L (e_p + e_{p+1}).$$

The integers m_L and n_L are uniquely determined if $p' > 2$ and satisfy $|m_L| \leqslant p' - 1$ and $|n_L| \leqslant p' - 1$. When $p' = 2$, we cover all cases by including $|m_L| \leqslant 1, |n_L| \leqslant 1$ [5]. The integers m_L and n_L are related to m and n by

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[5] When $p' = 2$, the M group of L is disconnected, and the behavior of σ^L on the second component normally needs to be taken into account. We leave this detail to the reader.

$$(6.12)\qquad \begin{aligned} m_L-n_L &= \frac{4\langle\lambda^L, e_{p-1}-e_p\rangle}{|\alpha|^2} = \frac{4\langle\lambda_0^L, e_{p-1}-e_p\rangle}{|\alpha|^2} \\ &= \frac{4\langle\lambda_0, e_{p-1}-e_p\rangle}{|\alpha|^2} = m-n. \end{aligned}$$

It follows from (6.11) and the inequality $m \geqslant n$ that α_1 and α_2 in G are partially conjugate modulo Λ, that $\langle\Lambda, \alpha_1\rangle \leqslant \langle\Lambda, \alpha_2\rangle$, and that $\Lambda-\alpha_2$ is a weight of $\tau_{(\Lambda-\alpha_1)^\sim}$. If $m<n$, then $\langle\Lambda, \alpha_1\rangle < \langle\Lambda, \alpha_2\rangle$. Moreover, the corresponding statements are true in L, and, by Lemmas 6.1 and 6.2, the numbers a_0 and b_0 for σ in G are the same as for σ^L in L. Thus the hypotheses of Corollary 5.5 and Theorems 5.6 and 5.7 are satisfied.

Before applying these results, however, we shall dispose of the cases where one or both of a_0 and b_0 is zero.

LEMMA 6.5. Let α_1 and α_2 be in the same simple component of Δ^L. Suppose that $\langle\Lambda, e_{p-1}-e_p\rangle = 0$ and that a_0 or b_0 is 0.

(a) If $m>n$, then there are no values of ν in L or G that correspond to unitary representations.

(b) If $m=n$, then the values of ν in L or G that correspond to unitary representations are given by

$$a+b \leqslant 2, \quad a>0, \quad b>0,$$

with a and b as in (6.4).

PROOF. (a) Let $m>n$. We start as in Lemma 6.3. Suppose $a_0=0$. If (6.2) is 0, we try to prove $\Lambda+\alpha_1$ is Δ_K^+ dominant. Problems can come only from $\gamma = e_{p-1}-e_p$ and $\gamma = e_{p+2}-e_{p+1}$, as in that proof. But

$$2\langle\Lambda+\alpha_1, e_{p-1}-e_p\rangle/|\alpha|^2 = 2\langle\alpha_1, e_{p-1}-e_p\rangle = 1$$

and

$$2\langle\Lambda+\alpha_1, e_{p+2}-e_{p+1}\rangle/|\alpha|^2 = m-n-1 \geqslant 0.$$

The alternative is for

$$(6.13)\qquad 1-\frac{2\langle\mu, \alpha_1\rangle}{|\alpha|^2} + 2\#\{\beta\in\Delta_n^+ \mid \beta\perp\alpha_2, \beta+\alpha_1\in\Delta, \langle\Lambda, \beta+\alpha_1\rangle = 0\}$$

to be 0. We show this cannot happen. If it does happen, then $2\langle\mu, \alpha_1\rangle/|\alpha|^2 = 1$. Also the simple root adjacent to α_1 in the Dynkin diagram must be

$e_{p+2}-e_p$, not some other $e_{p+2}-e_i$, since otherwise $\beta = e_{p+2}-e_i$ contributes to the set in (6.13). Thus α_1, $e_{p+2}-e_p$, α_2 are consecutive roots in the Dynkin diagram. Referring to (6.10), we see that $\langle \Lambda, e_{p-1}-e_p \rangle = 0$ forces $2\langle \mu, \alpha_2 \rangle / |\alpha|^2 = 1$. Then (6.10) shows also that $\langle \Lambda, e_{p+2}-e_{p+1} \rangle = 0$ and hence $m = n$, contradiction.

Thus $a_0 = 0$ implies $\Lambda + \alpha_1$ is Δ_K^+ dominant. The same argument applies to L. The presence of the two minimal K-type parameters Λ and $\Lambda + \alpha_1$ shows that there are no parameters ν corresponding to unitary representations in G, and the same argument applies in L.

If $b_0 = 0$, we can prove similarly that $\Lambda - \alpha_2$ is Δ_K^+ dominant. Then the same considerations show that there are no unitary points in G or in L.

(b) Let $m = n$. In G all 8 members of the Weyl group of the restricted roots then fix σ, and the two reflections in the «non-real» restricted roots lead to the identity intertwining operator at $\nu = 0$. Then it follows that the -1 element of the Weyl group corresponds to the identity intertwining operator. Hence there will be unitary points near $\nu = 0$.

Let us see that the Langlands quotient is not well defined on the a axis and on the b axis. It is enough to see that the R group of [10] has order 2 at $\nu = 0$, since then the reflections in the axes will have to correspond to nontrivial intertwining operators. By [17] it is enough to deduce the existence of a second minimal K-type from the equality $a_0 = b_0 = 0$. We describe what happens without giving the details: One shows that $\mu = \mu_1(\alpha_1 + \alpha_2)$ with $|\mu_1| = \frac{1}{2}$. Then one shows that $\Lambda - 2\mu_1(\alpha_1 + \alpha_2)$ is a second minimal K-type parameter. The style of argument is similar to that in (a). Thus the Langlands quotient is not well defined on the a axis and on the b axis.

All the argument thus far applies equally well to L, and the same conclusions apply to L. To complete the proof, we must identify the unitary points off the axes. We have seen that there are unitary points in G and in L near $\nu = 0$, and unitarity must then extend to the first place where the induced representations are reducible, which is the line $a + b = 2$, by (6.3). We now apply Theorem 5.7. If $m_L = n_L = -(p'-1)$, the theorem applies directly. The other case is that $m_L = n_L = +(p'-1)$, and then a variant of the theorem is needed. The theorem says that there are no unitary points in G or in L outside the circle of radius 2 centered at 0. Referring again to the list of reducibility points in (6.3), we see that unitarity in G and in L must stop already at the line $a + b = 2$. This completes the proof of (b) and the lemma.

We are left with the situation that (6.11) holds and that a_0 and b_0 are > 0.

Within the closed fundamental rectangle, the unitary points are identified in Propositions 8.1 and 8.2 of [9]. They depend only on the parameters a_0, b_0, and $m-n$. Since these parameters are the same for G as for L, the unitary points within the closed fundamental rectangle are the same.

The heart of Theorem 3.2 is its statement about the exterior of the fundamental rectangle. We handle these points largely by means of the results of §5, using one or another result depending on the values of m_L and n_L. Since the estimates in §5 are the same in L as in G (by Lemma 6.2), we are in effect proving Theorem 3.2. Here are the situations and applicable results:

1) $m_L > n_L \geqslant 0$. We use Corollary 5.5a to see that there are no unitary points for $a > \nu_1^+$, where ν_1^+ is a certain constant. In L, ν_1^+ coincides with a_0, which is the same number in L as in G. Thus there are no unitary points for $a > a_0$. In L also we have $b_0 \geqslant a_0$ for this case, and the same must be true in G. The points (a, b) with $a \leqslant a_0$ and $b > b_0$ lie outside the positive Weyl chamber. Thus we have proved that there are no unitary points outside the fundamental rectangle.

2) $m_L \geqslant 0 \geqslant n_L$ with $m_L \neq n_L$. We apply Corollary 5.5a to exclude $a > a_0$ and Corollary 5.5b to exclude $b > b_0$. Again we are identifying a_0 and b_0 with our estimates by examining the effect in L. The result is that there are no unitary points outside the fundamental rectangle.

3) $0 > m_L > n_L$. We apply Corollary 5.5b to exclude $b > b_0$ and Corollary 5.5c to exclude the region where $a > a_0$ and $b < b_0$. The remaining part of the exterior is the line with $a > a_0$ and $b = b_0$. We apply Theorem 5.7 to exclude the part of the line where $a > a_0 + 2$. The result is that there are no unitary points outside the fundamental rectangle except possibly on the line segment $a_0 < a \leqslant a_0 + 2, b = b_0$.

4) $0 \geqslant m_L = n_L$. (Here $a_0 = b_0$). We apply Theorem 5.6 to exclude the two regions $\{a > a_0, b < b_0\}$ and $\{a < a_0, b > b_0\}$. Then we apply Theorem 5.7 to exclude the region

$$\{a^2 + b^2 > (a_0 + 2)^2 + b_0^2\}.$$

The result is that there are no unitary points outside the fundamental rectangle except possibly on the set where $a \geqslant a_0$, $b \geqslant b_0$, either $a > a_0$ or $b > b_0$, and $a^2 + b^2 \leqslant (a_0 + 2)^2 + b_0^2$.

5) $m_L = n_L \geqslant 0$. This is completely analogous to (4) and is derived from obvious variants of Theorems 5.6 and 5.7.

To handle more of the undecided sets, we use the method of Duflo, as described in §10 of [9]. The K-types $\tau_{\Lambda + k\gamma}$, where $\gamma = e_{p+2} - e_{p+1}$ and

$k \geqslant 0$ is an integer, occur in the induced representation with multiplicity one. Consideration of the determinant of the intertwining operator on these K-types excludes all the remaining points listed above except those lying on some line

$$a = b + (m - n + 2k), \quad k \geqslant 1.$$

In (3) above, a_0 is $p' - |m_L| - 1$ and b_0 is $p' - |n_L| - 1$. Thus

$$a_0 - b_0 = m_L - n_L = m - n,$$

and the only candidate for a unitary point is $(a, b) = (a_0 + 2, b_0)$. In (4) we have $a_0 = b_0$, and the only candidate for a unitary point is $(a, b) = (a_0 + 2, a_0)$. In (5) the only candidate is the same point as in (4).

In short, the only candidates for unitary points outside the fundamental rectangle in L or in G are the points $(a, b) = (a_0 + 2, b_0)$ when $0 > m_L > n_L$ or when $m_L = n_L$. We mentioned in §3 that these points in L can be shown to be unitary. A little computation, which we omit, proves the following lemma.

LEMMA 6.6. *Let α_1 and α_2 be in the same simple component of Δ^L. Suppose that $\langle \Lambda, e_{p-1} - e_p \rangle = 0$ and that a_0 and b_0 are nonzero. Suppose further either that $0 > m_L > n_L$ or that $m_L = n_L$. Then*

$$\left\langle \lambda_0 + \frac{1}{2}(a_0 + 2)\alpha_1 + \frac{1}{2} b_0 \alpha_2, \beta \right\rangle \geqslant 0$$

for every β in Δ^+ that is not in Δ^L.

We can now apply Theorem 1.3a of Vogan [19] to conclude that these exceptional points correspond to unitary representations of G. This completes the proof of Theorem 3.2.

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