Intertwining operators for semisimple groups

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This paper contains the detailed presentation of results previously announced in [20], [21], and [22]. Our objective is to analyze the intertwining integrals for semisimple Lie groups. To do so, it is necessary to investigate their meromorphic character, find their normalizing factors, and obtain the relations they satisfy. The main reason for making this study is that there is an intimate connection between these integrals and the irreducibility of the principal series, on the one hand, and the unitarity of the analytically continued representations (the complementary series), on the other hand. To describe these matters in greater detail, we require some notation.

Let G be a connected semisimple matrix group and $G = ANK$ an Iwasawa decomposition. We write M and M' for the centralizer and normalizer of A in K, and V for the subgroup contragradient to N. Then for each finite-dimensional irreducible representation of MAN, given by

$$(\sigma, \lambda)\colon man \longrightarrow \sigma(m)\lambda(a) ,$$

we obtain the representation $U(\sigma, \lambda)$ of G that it induces. The representations formed this way when λ is unitary are the principal series.

For each element w of M' and each (σ, λ) we can consider the intertwining integral $A(w, \sigma, \lambda)$ defined formally by

$$A(w, \sigma, \lambda)f(x) = \int_{V \cap w^{-1}Nw} f(vw^{-1}x)dv . \tag{0.1}$$

This integral depends in an essential way only on the residue of w modulo M, and thus in effect only on the element of the Weyl group that w represents. A basic property of the integral (0.1) is the intertwining relation

$$A(w, \sigma, \lambda)\, U(\sigma, \lambda) = U(w\sigma, w\lambda)A(w, \sigma, \lambda) . \tag{0.2}$$

Now the integral given by (0.1) actually converges only for certain characters λ, and only for some nonunitary ones at that. (The characters of A correspond to points in $\mathbf{C}^r$, where $r = \dim A$ is the real-rank of G; the unitary λ correspond to points with imaginary coordinates.) It can be shown that these

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integrals may be continued analytically in λ to give meromorphic functions defined in all of $\mathbf{C}^r$.

From our point of view the main problem is that of finding the normalizing factors $\gamma(w, \sigma, \lambda)$ for the integrals (0.1). Using these, one defines the normalized intertwining integrals

$$\mathcal{A}(w, \sigma, \lambda) = \gamma(w, \sigma, \lambda)^{-1} A(w, \sigma, \lambda) . \tag{0.3}$$

The importance of the normalization may be understood on several related levels: (a) The factors γ eliminate the inessential poles of A and maintain the essential ones. As such the normalizing factors (and, more precisely, their poles and zeros) provide decisive information concerning the irreducibility of principal series and the existence of complementary series, together with the "width" of the strip of complementary series. (b) With these factors the operators $\mathcal{A}$ become unitary for unitary λ. (c) The normalizing factors are necessary also in proving the cocycle relations that the intertwining operators satisfy, whose relations mirror those in the Weyl group.

The study of the intertwining operators and their normalizing factors can be reduced to a large extent to the case of real-rank one. In that case the operators one is led to deal with are natural extensions of the Euclidean singular integrals of Mihlin-Calderón-Zygmund, to the context of nilpotent groups. While in the Euclidean case the L^2 theory can be dealt with satisfactorily by the use of the Fourier transform, in our situation new techniques must be used, since invoking the group Fourier transform seems to lead to unmanageable problems. The L^2 theory of these singular integrals is carried out in Part I of this paper. The main result is that the singular integral leads to a bounded operator on L^2 if and only if a certain mean value vanishes.[1]

Part II deals with the application of the above singular integrals to the real-rank one case of semisimple groups. After some preliminaries we isolate in Proposition 27 a factor $c_\sigma(z)$ in terms of which the normalizing factor $\gamma_\sigma(z)$ is constructed. A closely related aspect is the connection of reducibility in the principal series with the vanishing of a certain mean value; such vanishing happens if and only if $c_\sigma(z)$ is regular at $z = 0$. (See Proposition 20 in § 8 and Proposition 27 (vii) in § 9.)

The main result of Part II is the explicit determination of the factor $c_\sigma(z)$. This is given by the identity $c_\sigma(z) = \text{constant} \times p_\sigma(z)^{-1}$, where $p_\sigma(it)dt$ is the measure for the nondiscrete part of the Plancherel formula for G. There are

[1] Once the L^2 theory is established, the L^p theory can be carried out in a way that has some similarity to the classical case. This was proved by Rivière [31], Coifman and de Guzman [6], and Korányi and Vági [23].

two steps in the proof of this. The first step relates $c_\sigma(z)$ with the asymptotic behavior of certain entry-functions of the representation $U(\sigma, \lambda)$. The second step relates this asymptotic behavior to the Plancherel measure. Since the functions whose asymptotic behavior we consider are in reality eigenfunctions of systems of Sturm-Liouville equations, the relationship of their asymptotic character to the Plancherel measure follows the broad lines of a known heuristic principle; see, e.g., Coddington and Levinson [6, pp. 255–257], Bargmann [1], and Harish-Chandra [12]. However, neither the existing theorems nor the recently announced asymptotic results in Harish-Chandra [14, § 13] are of the form where they can be applied here.* So we are compelled to derive the needed facts in §§ 10–11 in order to complete our identification of $c_\sigma(z)$ with $p_\sigma(z)^{-1}$. As a by-product we also obtain sharp error terms for the asymptotic behavior, valid in strips.

The above can then be used to describe completely the situation concerning irreducibility of principal series (§ 12) and to deal with the existence of complementary series (§§ 13–15) for the real-rank one case. A useful notion for the latter is the *critical abscissa*, which has the property that in the interior of the interval it defines there is a complementary series, but on the boundary the corresponding inner product ceases to be strictly positive definite. Again the function $c_\sigma(z)$ plays a crucial role in locating the critical abscissa. (See Theorem 6 and the definition that precedes it.)

In Part III the results for the real-rank one case are applied to the general situation of higher rank. The technique of reducing matters to the real-rank one case was begun by Kunze and Stein [26] and is facilitated by more recent results of Schiffmann [32]. Also used in a significant way are the normalizing factors of the real-rank one case determined in Part II. What we prove is the following:

(i) If $\sigma(w)\mathfrak{A}(w, \sigma, 1)$ is not a constant multiple of the identity, then $U(\sigma, \lambda)$ is reducible for all λ such that $w\lambda = \lambda$.

(ii) If $\sigma(w)\mathfrak{A}(w, \sigma, 1)$ is a constant multiple of the identity and if the formal symmetry condition obtains, then there is a complementary series. What are lacking in general (but we show in the real-rank one case) are converses to (i) and (ii). Finally,

(iii) The relations we prove for the $\mathfrak{A}$'s allow us in principle to reduce the conditions in (i) and (ii) to the explicitly known cases of real-rank one. Not all the consequences of this reduction have yet been worked out, and the completion of this program appears to be at present an ambitious effort. To indicate the promise of this kind of approach, we cite the following two results. First,

* See footnote at end of the paper.

in § 19 we show that for any *complex* G there is a complementary series whenever the formal symmetry condition obtains. (This is not true for the general case of real groups, as is already shown by the example $SL(2, \mathbf{R})$.) Second, in § 20 we show that whenever n is even, there are reducible representations of the principal series of $SL(n, \mathbf{R})$, contrary to the existing claims in the literature. (See Gelfand-Graev [10].)

It should be emphasized that in these questions the special case of representations of the principal series arising when σ is trivial on M is not indicative of the complexity of the general problem. For example, in that case there is always irreducibility in the principal series, and there is always a complementary series whenever the formal conditions of symmetry obtain. Moreover many of the results derived here were known earlier in the case when σ is trivial on M.[2]

We mention here several other recent papers that have a bearing on our work. Kunze [25] obtains in certain cases the complementary series for complex groups. The paper by Harish-Chandra [14] containing the asymptotics of generalized spherical functions has already been alluded to. In Schiffmann [33] an alternative and independent derivation is given for some of the material of §9, dealing with the analytic continuation of intertwining integrals; also Part III uses his earlier results in [32]. It is also our pleasure to acknowledge helpful suggestions from C. Fefferman and H. Garland, which have been incorporated into the text.

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[2] See Kostant [24] and Helgason [16], together with the previous literature cited there.

I. SINGULAR INTEGRALS

1. Notation, statements of theorems

The Calderón-Zygmund Theorem [5] gives conditions under which a principal-value convolution operator on $\mathbf{R}^n$ is a bounded operator on L^p for $1 < p < \infty$. In Theorem 1 we shall give a generalization of the L^2 part of this theorem to operators on nilpotent Lie groups. Theorem 2 goes in the converse direction, asserting that one of the conditions in Theorem 1 is necessary in order that some reasonable interpretation of the convolution operator (as a principal-value integral or otherwise) be bounded.

Let X be a connected, simply-connected nilpotent Lie group. A continuous one-parameter group $\{\delta_r, 0 < r < \infty\}$ of automorphisms of X will be called a one-parameter group of *dilations* if the differentials $(\delta_r)_*$ at the identity 1 of X satisfy $(\delta_r)_* = r^D$ for a diagonable transformation D with all eigenvalues positive. This condition implies that $(\delta_r)_*$ has all eigenvalues >1 if $r > 1$ and that $\lim_{r\to 0} \delta_r x = 1$ for all x in X.

If $\{\delta_r\}$ is a one-parameter group of dilations, a *norm function* on X is a C^∞ function $|x|$ from $X - \{1\}$ to the positive real numbers, having the three properties

(i) $|x^{-1}| = |x|$

(ii) $|\delta_r x| = r^q |x|$ for a fixed number $q > 0$

(iii) the measure $|x|^{-1}\, dx$ is invariant under dilations.

In the corollary to Proposition 2 we shall see that if a function $|x|$ satisfies all these properties but (iii), then some positive power of it is a norm function. The proof of the proposition and of Lemma 1 will show, for a norm function, that (iii) implies that the number q in (ii) is the trace of D. Condition (iii) has equivalent formulations that are given in Proposition 2.

Examples: (1) Let $X = \mathbf{R}^n$, $\delta_r =$ scalar multiplication by r, $||x|| =$ the usual Euclidean norm, and $|x| = ||x||^n$. Theorem 1 below for this special case was proved by Mihlin [27] in the case $t = 0$ and by Muckenhoupt [28] in the case $t \neq 0$. The L^p version of Mihlin's theorem is due to Calderón and Zygmund [5].

(2) Let $X = \mathbf{R}^{n+1} = \{(x_1, \cdots, x_n, t)\}$, fix a positive integer m, and let

$$\delta_r(x_1, \cdots, x_n, t) = (rx_1, \cdots, rx_n, r^m t) .$$

Let M be a positive constant, and put $|x| = \max(\|x\|^{2m}, Mt^2)$. Then $|x|$ is a norm function, and Theorem 1 is similar to the main result of Jones in [19]. (Actually Jones's result corresponds to the limiting case $M \to \infty$. But see example (3).)

(3) Let $X = \mathbf{R}^n$, let $\{d_j, 1 \leqq j \leqq n\}$ be positive scalars, and let

$$\delta_r(x_1, \cdots, x_n) = (r^{d_1}x_1, \cdots, r^{d_n}x_n) .$$

Put $|x| = \left(\min\{\rho \geqq 0 \mid \sum x_j^2/\rho^{2d_j} \leqq 1\}\right)^{\Sigma d_j}$. Then $|x|$ is a norm function, and Theorem 1 for this case when $t = 0$ is the L^2 part of Theorem 1 of Fabes and Rivière in [9]. Fabes and Rivière show that Jones's result [19] in example (2) follows from theirs.

(4) Let X be the group of 3-by-3 real matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ x_1 & 1 & 0 \\ x_3 & x_2 & 1 \end{pmatrix},$$

let $\delta_r(x_1, x_2, x_3) = (rx_1, rx_2, r^2x_3)$, and put

$$|(x_1, x_2, x_3)| = (x_1^2 + x_2^2)^2 + \left(x_3 - \frac{1}{2}x_1x_2\right)^2 .$$

Then $|x|$ is a norm function. It is substantially the same as the norm function that will arise in Part II in the special case that the semisimple group G is $SU(2, 1)$.

Return to general X. It will be shown in observation 2 of § 2 that any set $\{x \mid c \leqq |x| \leqq d\}$ with $0 < c < d$ is compact. Consequently the integrals appearing in the statement of Theorem 1 are well-defined.

THEOREM 1. *Let X be a connected simply-connected nilpotent Lie group, $\{\delta_r\}$ a one-parameter group of dilations, $|x|$ a norm function on X, H a separable Hilbert space, and $\Omega(x)$ a C^∞ function on $X - \{1\}$ to the space of bounded operators on H (with the norm topology). Suppose that $\Omega(\delta_r x) = \Omega(x)$ for all r and x. Let t be a fixed real number, and suppose that one or both of the following conditions hold:*

(α) $\int_{c \leqq |x| \leqq d} \Omega(x)dx = 0$ *for some (or equivalently all) c and d with $0 < c < d$.*

(β) $t \neq 0$.

Let $f \in L^2(X, H)$. If (α) holds, then the limit

$$Tf(x) = \lim_{\varepsilon \to 0, M \to \infty} \int_{\varepsilon \leqq |y| \leqq M} |y|^{-1+it}\, \Omega(y) f(yx)dy$$

exists in $L^2(X, H)$, and $f \to Tf$ is a bounded operator on $L^2(X, H)$. If (α) fail *but* (β) *holds, then the same conclusion is valid, provided that the limit is taken over appropriate sequences of* ε*'s and* M*'s tending to* 0 *and* ∞, *respectively.*

THEOREM 2. *Let* X, $\{\delta_r\}$, $|x|$, H, *and* $\Omega(x)$ *be as in Theorem* 1 *with* $\Omega(\delta_r x) = \Omega(x)$ *for all* r *and* x, *and suppose that* (α) *does not hold. Then there exists no* H*-valued distribution* $d\mu$ *defined on* $C_c^\infty(X, H)$ *satisfying the following two conditions*:

(γ) *Away from the identity,* $d\mu$ *is equal to the function* $\Omega(x)/|x|$.

(δ) *The mapping* $f \to \int d\mu(y) f(yx)$ *defined by* f *in* $C_c^\infty(X, H)$ *extends to a bounded operator mapping* $L^2(X, H)$ *into itself.*

2. Properties of norm functions

Throughout this section we assume that X is a connected, simply-connected nilpotent Lie group, that $\{\delta_r\}$ is a one-parameter group of dilations, and that $|x|$ is a function on $X - \{1\}$ having all the properties of a norm function except possibly (iii). We shall obtain some results about the measure $|x|^{-1}dx$ and then proceed to the proof of an important inequality (Lemma 6) used in the proof of Theorem 1.

LEMMA 1. *There is a real number* $h > 0$ *such that* $d(\delta_r x) = r^h dx$ *for all* $r > 0$.

Proof. Since δ_r is an automorphism of X, we have $d(\delta_r x) = \det(\delta_r)_* dx$. Since $\{\delta_r\}$ is a continuous group, $\det(\delta_r)_*$ is continuous in r and satisfies the functional equation $f(rs) = f(r)f(s)$. Hence $\det(\delta_r)_* = r^h$ for some h. Since $\det(\delta_r)_* \to 0$ as $r \to 0$, we have $h > 0$.

PROPOSITION 2. *The following conditions on* $|x|$ *are equivalent.*

(1) $|x|^{-1}dx$ *is invariant under dilations (and therefore* $|x|$ *is a norm function).*

(2) $\int_{1 \le |x| \le r} |x|^{-1} dx = C \log r$ *for a fixed* C *and all* $r \geqq 1$.

(3) $\int_{2^k \le |x| \le 2^{k+1}} |x|^{-1} dx$ *is independent of* k.

(4) *If* q *is the number in* (ii), *then* $d(\delta_r x) = r^q dx$.

Proof. If h is as in Lemma 1, we show all four statements are equivalent with the condition $h = q$. For (4) this is obvious. For (1) it follows from the identity $|\delta_r x|^{-1} d(\delta_r x) = r^{h-q} |x|^{-1} dx$. For (2) and (3), define

$$f(r) = \int_{1 \le |x| \le r} |x|^{-1} dx .$$

A simple change of variables shows that

$$f(rs) = f(r) + r^{h/q-1}f(s) . \tag{2.1}$$

Since f is clearly continuous, (2) holds if and only if $h = q$. By (2.1),

$$f(2^{k+1}) - f(2^k) = 2^{k(h/q-1)}f(2) ,$$

and thus (3) holds if and only if $h = q$.

COROLLARY. *Some positive power of* $|x|$ *is a norm function.*

Proof. In fact, $|x|^{h/q}$ is a norm function, by the equivalence of (1) and (4).

PROPOSITION 3. *Let* $|x|$ *be a norm function and let* C *be the constant in* (2) *of Proposition* 2. *If* H *is a separable Hilbert space and* $\Omega(x)$ *is a continuous function on* $X - \{1\}$ *whose values are bounded operators on* H *and which satisfies* $\Omega(\delta_r x) = \Omega(x)$, *then there exists a bounded operator* $\mathfrak{M}(\Omega)$ *such that*

$$\int_X \Omega(x) f(|x|)dx = C\mathfrak{M}(\Omega) \int_0^\infty f(r)dr \tag{2.2}$$

for every measurable complex-valued function f *on* $(0, \infty)$ *such that either side is defined.*

Remark. We can regard $\mathfrak{M}(\Omega)$ as the *mean value* of Ω over the "unit sphere" defined relative to $|x|$.

Proof. Letting

$$F(r) = \begin{cases} \int_{1 \leqq |x| \leqq r} \Omega(x) \, |x|^{-1} \, dx & \text{for } r \geqq 1 \\ - \int_{r \leqq |x| \leqq 1} \Omega(x) \, |x|^{-1} \, dx & \text{for } 0 < r \leqq 1 , \end{cases}$$

we obtain, just as in equation (2.1), $F(rs) = F(r) + F(s)$. Since F is continuous,

$$F(r) = C\mathfrak{M}(\Omega) \log r$$

for some bounded operator $\mathfrak{M}(\Omega)$. This proves (2.2) for functions f that are r^{-1} times characteristic functions of intervals. Forming linear combinations and passing to the limit, we obtain the proposition.

If $\mathfrak{X}$ is the Lie algebra of X, then the exponential mapping is a diffeomorphism of $\mathfrak{X}$ onto X. It will be notationally convenient to identify $\mathfrak{X}$ and X by means of the exponential mapping, so that N has four operations — addition, scalar multiplication, bracket multiplication, and group multiplication. Under this identification, Haar measure is Lebesgue measure and $\delta_r = (\delta_r)_*$. The identity will still be called 1. We have assumed that $(\delta_r)_* = r^D$ for a diagonable transformation D with positive eigenvalues. Fix a basis $\{X_i\}$ of eigenvectors for D and define positive numbers p_i by the equations

$$\delta_r(X_i) = r^{p_i} X_i \ .$$

We make some observations about the function $|x|$.

(1) $|x|$ extends to a continuous function from X to the non-negative real numbers under the definition $|1| = 0$.

(2) $\{x \mid |x| \leqq 1\}$ is a bounded set. Consequently the inverse image under $|x|$ of any compact set of non-negative reals is compact. [In fact, take an inner product on X, let B be the closed unit ball, and let S be the unit sphere. Since S is compact and is in $X - \{1\}$, $|x|$ has a positive minimum t on it. If $r = t^{-1/q}$, then $|x| \geqq 1$ on $\delta_r S$. It follows easily that $\{x \mid |x| \leqq 1\} \subseteq \delta_r B$ and hence that $\{x \mid |x| \leqq 1\}$ is bounded. The second statement follows from the homogeneity property (ii) for $|x|$.]

(3) There exists $c(\geqq 1)$ such that $|x + y| \leqq c(|x| + |y|)$ for all x and y. [In fact, by (ii) we can assume $|x| + |y| = 1$. By (2) and the continuity of $|x|$, the set of (x, y) such that $|x| + |y| = 1$ is compact. Then c can be taken as the maximum value of $|x + y|$ on this compact set.]

(4) If c is as in (3), then

$$|x_1 + \cdots + x_n| \leqq c^{n-1}(|x_1| + \cdots + |x_n|)$$

by induction on n from (3).

LEMMA 4. *If $||x||$ denotes a Euclidean norm on X, then there exists c' such that*

$$||yx| - 1| \leqq c' \, ||y|| \tag{2.3}$$

whenever $|y| \leqq 1$ and $|x| = 1$.

Proof. If $1/2 \leqq |y| \leqq 1$, the inequality is trivial. For $|y| \leqq 1/2$, the function $f\colon X \times X \to \mathbf{R}^+$ given by $f(y, x) = |yx|$ is smooth in a neighborhood of the compact set where $|y| \leqq 1/2$ and $|x| \leqq 1$, and therefore

$$|f(y, x) - f(1, x)| \leqq c' \, ||(y, x) - (1, x)|| = c' \, ||y|| \ .$$

LEMMA 5. *There exist constants $a > 0$ and $b > 0$ with the following properties: Whenever $\sum c_i X_i$ is such that $|c_i| \geqq a r^{p_i/q}$ for some i, then $|\sum c_i X_i| \geqq r$. Whenever $\sum |c_i r^{-p_i/q}| \leqq b$, then $|\sum c_i X_i| \leqq r$.*

Proof. $\{x \mid |x| = 1\}$ is compact by (2), and $P_i\colon \sum c_i X_i \to |c_i|$ is continuous. Thus we can choose a so that $|c_i| \leqq a$ for all i whenever $|\sum c_i X_i| = 1$. Also since $f\colon \sum c_i X_i \to \sum |c_i|$ is continuous, $\sum |c_i|$ assumes its minimum, which we define as b; b is not 0 because the identity is not in $\{x \mid |x| = 1\}$. Now if $|\sum c_i X_i| = r$, then $|\delta_{r^{-1/q}} \sum c_i X_i| = 1$, which implies that $|\sum c_i r^{-p_i/q} X_i| = 1$. Hence $|c_i| \leqq a r^{p_i/q}$ and $\sum |c_i r^{-p_i/q}| \geqq b$. The lemma follows.

As above, identify X with its Lie algebra by means of the exponential map, and let X_i be a basis of X such that $\delta_r(X_i) = r^{p_i}X_i$.

LEMMA 6. *Let* $||x||$ *be the Euclidean norm on* X *obtained by using the eigenvectors* X_i *of* $\{\delta_r\}$ *as an orthonormal basis. There exists* $d > 0$ *such that, for any* $M > 0$,

$$||x|| \leqq c(M)\,|x|^d$$

for all x *in* X *with* $|x| \leqq M$.

Proof. If $x = \sum c_i X_i$, then

$$||x|| = (\sum c_i^2)^{1/2} \leqq D^{1/2} \max |c_i| ,$$

where $D = \dim X$. Choose j so that $|c_j| \geqq |c_i|$ for all i. Then

$$|c_l| \geqq a(a^{-q/p_j}D^{-q/2p_j}\,||x||^{q/p_j})^{p_l/q}$$

for some l, namely $l = j$, and so

$$|x| \geqq a^{-q/p_j}D^{-q/2p_j}\,||x||^{q/p_j}$$

by Lemma 5. Hence

$$||x|| \leqq aD^{1/2}\,|x|^{p_j/q} ,$$

and the lemma follows with $d = \min(p_i)/q$.

LEMMA 7. *There exist constants* c_0 *and* d *with* $d > 0$ *such that*

$$\left|\frac{|yx|}{|x|} - 1\right| \leqq c_0\left(\frac{|y|}{|x|}\right)^d$$

whenever $|y| \leqq |x|$ *and* $x \neq 1$.

Remark. The same conclusion is valid if yx is replaced by xy.

Proof. By (ii) we may assume that $|x| = 1$. Clearly then $|y| \leqq 1$. The result then follows by combining (2.3) and Lemma 6.

3. The operators $T_{\varepsilon,M}$

Let X be a connected simply-connected nilpotent Lie group, $\{\delta_r\}$ a one-parameter group of dilations, and $|x|$ a norm function.

LEMMA 8. *As in Theorem* 1, *let* H *be a separable Hilbert space and let* $\Omega(x)$ *be a* C^∞ *function on* $X - \{1\}$ *whose values are bounded operators on* H *such that* $\Omega(\delta_r x) = \Omega(x)$ *for all* r *and* x. *Suppose that* Ω *has mean value* 0, *in the sense of Proposition* 3. *Fix* f *in* $C_c^\infty(X, H)$. *Then the limit as* $\varepsilon \to 0$ *and* $M \to \infty$ *of*

$$T_{\varepsilon,M}f(x) = \int_{\varepsilon \leqq |y| \leqq M} |y|^{-1}\,\Omega(y)f(yx)dy$$

exists both uniformly and in $L^2(X, H)$.

Proof. By Proposition 3 applied to $|y|^{-1}|\Omega(y)|$, $|y|^{-1}\Omega(y)$ is in $L^2 \cap L^\infty$ on the set $1 \leqq |y| < \infty$. Consequently $T_{1,M}f$ converges uniformly and in L^2 as $M \to \infty$. Since Ω has mean value 0,

$$\begin{aligned} T_{\varepsilon,1}f(x) &= \int_{\varepsilon \leqq |y| \leqq 1} |y|^{-1}\,\Omega(y)f(yx)dy \\ &= \int_{\varepsilon \leqq |y| \leqq 1} |y|^{-1}\,\Omega(y)[f(yx) - f(x)]dy\,. \end{aligned}$$

Let $S = \{(x, y) \mid |y| \leqq 1,\ f(x) \neq 0 \text{ or } f(yx) \neq 0\}$; S is bounded. Since multiplication is smooth, $yx = x + v_{x,y}$ with $\|v_{x,y}\| \leqq A\|y\|$ for (x, y) in S. Since f is smooth, we have

$$f(yx) = f(x) + (d_xf)(v_{x,y}) + R(x, y)\,,$$

where d_xf is a linear mapping from X to H, varying continuously with x, and where $\|y\|^{-1}|R(x, y)| \leqq B$ on S and $\lim_{y\to 0}\|y\|^{-1}R(x, y) = 0$. If $\delta \leqq \varepsilon$, then

$$\begin{aligned} &|T_{\delta,1}f(x) - T_{\varepsilon,1}f(x)| \\ &\leqq \int_{\delta \leqq |y| \leqq \varepsilon} |y|^{-1}\sup_x(|\Omega(y)|)\{A\sup_x\|d_xf\| + B\}\,\|y\|\,dy\,. \end{aligned}$$

The uniform convergence of $T_{\varepsilon,1}f(x)$ as $\varepsilon \to 0$ therefore follows from the finiteness of

$$\int_{0<|y|\leqq 1} |y|^{-1}\,\|y\|\,dy\,,$$

which is a consequence of Lemma 6 and Proposition 3. Since the functions $T_{\varepsilon,1}f$ have support contained in a common compact set, the convergence is also in L^2. This completes the proof.

The next two lemmas will not be used in proving Theorems 1 and 2 but will be used in Part II.

LEMMA 9. *With notation as in Lemma* 8, *the operator* $\lim_{\varepsilon\to 0, M\to\infty} T_{\varepsilon,M}$, *defined on functions in* $C_c^\infty(X, H)$, *cannot be a scalar multiple of the identity unless* Ω *is identically* 0.

Proof. Suppose the limiting operator is c times the identity. In view of the uniform convergence in Lemma 8, we then have

$$\lim_{\varepsilon\to 0}\int_{\varepsilon \leqq |y| < \infty} |y|^{-1}\,\Omega(y)f(y)dy = cf(0) \qquad (3.1)$$

for all f in $C_c^\infty(X, H)$. Choose $f(y) = g(|y|)\Omega(y)^*\xi$, where ξ is a member of H

such that $\Omega(y_0)\Omega(y_0)^*\xi \neq 0$ for some y_0 and where g is a smooth function $\geqq 0$ vanishing in a neighborhood of 0 and non-vanishing at y_0. Then the inner product of the left side of (3.1) with ξ is positive, and the inner product of the right side with ξ is 0, contradiction.

LEMMA 10. *With notation as in Lemma 8, define, for $s > 0$ and $f \in C_c^\infty(X, H)$,*

$$T_s f(x) = \int_X |y|^{-(1-s)}\, \Omega(y) f(yx) dy\ .$$

Then $\lim_{s\downarrow 0} T_s f = \lim_{\varepsilon\to 0, M\to\infty} T_{\varepsilon,M} f$ in L^2 for all f in C_c^∞, and the rate of convergence is controlled by $\| f \|_1$, by the support of f, and by a uniform Lipschitz constant for f.

Proof. First consider the difference

$$\int_{1\leqq|y|\leqq\infty} (|y|^{-1} - |y|^{-(1-s)})\Omega(y) f(yx) dy\ . \tag{3.2}$$

To see that (3.2) tends to 0 in L^2, apply Proposition 3. Then the restriction of $(|y|^{-1} - |y|^{-(1-s)})\Omega(y)$ to $1 \leqq |y| < \infty$ tends to 0 in L^2, by dominated convergence, and (3.2) tends to 0 in L^2 since f is in L^1. We are left with proving that

$$\lim_{\varepsilon\to 0} \int_{\varepsilon\leqq|y|\leqq 1} (|y|^{-1} - |y|^{-(1-s)})\Omega(y) f(y^{-1}x) dy \tag{3.3}$$

tends to 0 in L^2 as $s \downarrow 0$. To do so, it is enough to prove uniform convergence in x, and, for any fixed x, we can assume $f(x) = 0$ since Ω has mean value 0. An argument like that in the proof of Lemma 8 shows that the integral (3.3) with $0 < |y| \leqq 1$ is absolutely convergent and that the question reduces to showing that

$$\lim_{s\downarrow 0} \int_{0<|y|\leqq 1} \big|\, |y|^{-1} - |y|^{-(1-s)} \big|\, \| y \|\, dy = 0\ . \tag{3.4}$$

But (3.4) follows from Lemma 6 and Proposition 3.

4. Proof of boundedness theorem

We turn to the proof of Theorem 1. We are indebted to C. Fefferman for helping with the proof. Let X, $\{\delta_r\}$, $|x|$, and $\Omega(x)$ be as in the statement of the theorem. The proof will be based on Lemma 11 below. For commentary on the history of this lemma, see [20] and Cotlar [8].

LEMMA 11. *Let $\varphi(n) \geqq 0$ be a function on the integers $-\infty < n < \infty$ with $\Phi = \sum \varphi(n)^{1/2} < \infty$. If $T_1, \cdots, T_N$ are linear operators on a Hilbert space with $\| T_i^* T_j \| \leqq \varphi(i-j)$ and $\| T_i T_j^* \| \leqq \varphi(i-j)$ for all i and j, then $\| T_1 + \cdots + T_N \| \leqq \Phi$, independently of N.*

Proof. Put $T = T_1 + \cdots + T_N$. We have $\| T_j \|^2 = \| T_j^* T_j \| \leqq \varphi(0) \leqq \Phi^2$. Also $\| T \|^2 = \| T^* T \|$. Since T^*T is self-adjoint, its norm is given by its spectral radius. Thus we are to estimate $\limsup \| (T^*T)^n \|^{1/n}$. We have

$$(T^*T)^n = \sum_{(i_1,\cdots,i_{2n})} T_{i_1}^* T_{i_2} T_{i_3}^* \cdots T_{i_{2n}} ,$$

where each index i_j extends from 1 to N. The assumptions of the lemma give us the two estimates

$$\| T_{i_1}^* T_{i_2} T_{i_3}^* \cdots T_{i_{2n}} \| \leqq \varphi(i_2 - i_1)\varphi(i_4 - i_3)\cdots\varphi(i_{2n} - i_{2n-1})$$
$$\| T_{i_1}^* T_{i_2} T_{i_3}^* \cdots T_{i_{2n}} \| \leqq \Phi\varphi(i_3 - i_2)\varphi(i_5 - i_4)\cdots\varphi(i_{2n-1} - i_{2n-2})\Phi .$$

The geometric mean of these inequalities is

$$\| T_{i_1}^* T_{i_2} T_{i_3}^* \cdots T_{i_{2n}} \| \leqq \Phi\varphi(i_2 - i_1)^{1/2}\varphi(i_3 - i_2)^{1/2}\cdots\varphi(i_{2n} - i_{2n-1})^{1/2} .$$

Sum on i_{2n}, then on i_{2n-1}, and so on through i_2, letting each sum extend from $-\infty$ to ∞. Then

$$\| (T^*T)^n \| \leqq \sum_{i_1=1}^{N} \Phi\cdot\Phi^{2n-1} = N\Phi^{2n}$$

and so

$$\| T \|^2 = \limsup_{n\to\infty} \| (T^*T)^n \|^{1/n} \leqq \Phi^2 .$$

Passing to the proof of Theorem 1, we consider first the case $t = 0$. At the end we shall indicate the changes in the argument needed for the case $t \neq 0$. The idea of the proof is to apply Lemma 11 to a sequence of operators T_k whose sum is to converge to T. Letting

$$\Omega_k(x) = \begin{cases} \Omega(x) & \text{for } 2^k \leqq |x| < 2^{k+1} \\ 0 & \text{otherwise,} \end{cases}$$

we shall take as our operators

$$T_k f(x) = \int_X |y|^{-1}\, \Omega_k(y) f(yx) dx . \tag{4.1}$$

First we observe that the T_k have uniformly bounded norms because $\| T_k f \|_2 \leqq \| \, |y|^{-1} \Omega_k(y) \|_1 \| f \|_2$ and

$$\| \, |y|^{-1} \Omega_k(y) \|_1 \leqq \sup_x (|\Omega(x)|) \int_{2^k \leqq |y| < 2^{k+1}} |y|^{-1} dy ,$$

with the right side independent of k by Proposition 2. In view of Lemma 11, all we need is an estimate for $\| T_j T_k^* \|$ and $\| T_j^* T_k \|$ when $|j - k|$ is large.

Next, straightforward changes of variables lead to the formulas

$$T_k^* f(x) = \int |y|^{-1}\, \Omega_k(y^{-1})^* f(yx) dy$$

and

$$T_j T_k^* f(x) = \int G_{jk}(y) f(yx) dy ,$$

where

$$G_{jk}(y) = \int |zy|^{-1} |z|^{-1} \Omega_j(zy) \Omega_k(z)^* dz .$$

By Minkowski's inequality, $\| T_j T_k^* \| \leqq \| G_{jk} \|_1$. Also $T_j^* T_k$ is of the same form as $T_j T_k^*$, except that the kernel $\Omega(y)$ is replaced by a new kernel $\Omega(y^{-1})^*$ satisfying the hypotheses of the theorem. Thus we need estimate only $\| G_{jk} \|_1$ for large $|j - k|$. Furthermore,

$$G_{jk}(y^{-1})^* = G_{kj}(y) ,$$

from which it follows that $\| G_{jk} \|_1 = \| G_{kj} \|_1$. Thus we need estimate $\| G_{jk} \|_1$ only for large positive $j - k$.

Now, since Ω_j has mean value 0 and $|\Omega_k(y)| \leqq \sup |\Omega|$,

$$\begin{aligned} |G_{jk}(x)| &\leqq \int_X \left| \frac{\Omega_j(yx)}{|yx|} - \frac{\Omega_j(x)}{|x|} \right| \left| \frac{\Omega_k(y)}{|y|} \right| dy \\ &\leqq (\sup |\Omega|)^2 \int_{2^k \leqq |y| \leqq 2^{k+1}} \left| \frac{1}{|yx|} - \frac{1}{|x|} \right| |y|^{-1} dy \\ &\quad + (\sup |\Omega|) |x|^{-1} \int_{2^k \leqq |y| \leqq 2^{k+1}} |\Omega_j(yx) - \Omega_j(x)| \, |y|^{-1} dy . \end{aligned}$$

Consequently, if E is the set $\{(x, y) \mid 2^k \leqq |y| \leqq 2^{k+1}, 2^j \leqq |yx| \leqq 2^{j+1}\}$, then

$$\begin{aligned} \| G_{jk} \|_1 &\leqq (\sup |\Omega|)^2 \int_E \left| \frac{1}{|yx|} - \frac{1}{|x|} \right| \frac{1}{|y|} dy dx \\ &\quad + \sup |\Omega| \int_E \frac{|\Omega_j(yx) - \Omega_j(x)|}{|x|} \frac{1}{|y|} dy dx \\ &= (\sup |\Omega|)^2 I_1 + (\sup |\Omega|) I_2 , \text{ say.} \end{aligned}$$

Before estimating I_1 and I_2, we shall specify a minimum size for $j - k$ in the estimates. By the same argument as in observation (3) of § 2, there is a constant c_1 such that $|xy| \leqq c_1(|x| + |y|)$ for all x and y. In our estimates we shall require that $2^{j-k} \geqq 8c_1$. Under this condition, let (x, y) be in the set E above. Then

$$|x| \leqq c_1(|y^{-1}| + |yx|) = c_1(|y| + |yx|) \leqq c_1(2^{k+1} + 2^{j+1}) \leqq 4c_1 2^j$$

and

$$|x| \geqq c_1^{-1} |yx| - |y| \geqq 2^j c_1^{-1} - 2^{k+1} = 2^j c_1^{-1} - 2^{j-2} 2^{-(j-k-3)} \geqq \frac{1}{2} c_1^{-1} 2^j .$$

So

$$\frac{1}{2}c_1^{-1}2^j \leqq |x| \leqq 4c_1 2^j . \tag{4.2}$$

Also

$$|y| \leqq 2^{k+1} = 2^{-2}2^{-(j-k-3)}2^j \leqq 2^{-2}c_1^{-1}2^j \leqq \frac{1}{2}|x| . \tag{4.3}$$

We consider I_1. If (x, y) is in E and $2^{j-k} \geqq 4c_1$, then (4.2), (4.3), and Lemma 7 show that

$$\begin{aligned}\left|\frac{1}{|yx|} - \frac{1}{|x|}\right| &= \frac{1}{|yx|}\left|1 - \frac{|yx|}{|x|}\right| \leqq c_0\left(\frac{|y|}{|x|}\right)^d \frac{1}{|yx|} \\ &\leqq c_0 2^{d(k+1)} c_1^d 2^{-d(j-1)} 2^{-j} 4c_1 2^j \, |x|^{-1} \\ &= \text{Const } 2^{-d(j-k)} \, |x|^{-1} .\end{aligned}$$

Hence

$$\begin{aligned} I_1 &\leqq \left(\int_{2^k \leqq |y| \leqq 2^{k+1}} |y|^{-1} dy\right)\left(\int_{\frac{1}{2}c_1^{-1}2^j \leqq |x| \leqq 4c_1 2^j} |x|^{-1} dx\right) \text{Const } 2^{-d(j-k)} \\ &\leqq \text{Const } 2^{-d(j-k)} \end{aligned} \tag{4.4}$$

by Proposition 2.

For I_2, we note that $\Omega_j(yx) = \Omega(yx)$ on E and hence

$$\begin{aligned} I_2 &\leqq \int_E \frac{|\Omega(yx) - \Omega(x)|}{|x|}\frac{1}{|y|}dydx + \int_E \frac{|\Omega(x) - \Omega_j(x)|}{|x|}\frac{1}{|y|}dydx \\ &= I_{2A} + I_{2B} . \end{aligned}$$

In I_{2A}, enlarge E to the set

$$\left\{2^k \leqq |y| \leqq 2^{k+1}, \frac{1}{2}c_1^{-1}2^j \leqq |x| \leqq 4c_1 2^j\right\} .$$

(This is possible by (4.2).) On this set we have $|y| \leqq |x|/2$ by (4.3). We claim that for $|y| \leqq |x|/2$,

$$|\Omega(yx) - \Omega(x)| \leqq \text{Const } (|y|/|x|)^d , \tag{4.5}$$

where d is the constant of Lemma 6.

To prove (4.5), we may assume by homogeneity that $|x| = 1$ and $|y| \leqq 1/2$. Then the same style of proof as for Lemma 4 shows that

$$|\Omega(yx) - \Omega(x)| \leqq \text{Const } \|y\| ,$$

and (4.5) follows from Lemma 6.

From (4.5) we obtain

$$\begin{aligned} |\Omega(yx) - \Omega(x)| \leqq \text{Const}\left(\frac{|y|}{|x|}\right)^d &\leqq \text{Const } 2^{d(k+1)-d(j-1)}c_1^d \\ &= \text{Const } 2^{-d(j-k)} \end{aligned}$$

on the enlarged set of integration. By Proposition 2

$$I_{2A} \leqq \text{Const } 2^{-d(j-k)} . \tag{4.6}$$

We consider

$$I_{2B} = \int_E \frac{|\Omega(x) - \Omega_j(x)|}{|x|} \frac{1}{|y|} dydx .$$

The integrand is 0 unless $|x| \leqq 2^j$ or $|x| \geqq 2^{j+1}$, and we examine the set of x for which this can happen. In any case, (x, y) in E implies $|y| \leqq |x|/2$ and $2^{-1} c_1^{-1} 2^j \leqq |x| \leqq 4c_1 2^j$ by (4.2) and (4.3). Suppose that $|x| \geqq 2^{j+1}$ and (x, y) is in E. Then $|yx|/|x| \leqq 1$ and Lemma 7 gives

$$|1 - 2^{j+1}|x|^{-1}| \leqq |1 - |yx|/|x|| \leqq c_0(|y|/|x|)^d \leqq c_0 2^{-d(j-k)} .$$

Hence

$$||x| - 2^{j+1}| \leqq c_0 2^{-d(j-k)} |x| \leqq 2c_0 c_1 2^{j+1} 2^{-d(j-k)} ,$$

and the contribution to I_{2B} from points x with $|x| \geqq 2^{j+1}$ is at most

$$\begin{aligned} &2 \sup |\Omega| \left(\int_{2^k \leq |y| \leq 2^{k+1}} |y|^{-1} dy\right)\left(\int_{2^{j+1} \leq |x| \leq 2^{j+1}(1+2c_0c_1 2^{-d(j-k)})} |x|^{-1} dx\right) \\ &= \text{Const} \log (1 + 2c_0 c_1 2^{-d(j-k)}) \end{aligned} \tag{4.7}$$

by Proposition 2. Similarly if $|x| \leqq 2^j$, then $|yx|/|x| \geqq 1$ and

$$|1 - 2^j |x|^{-1}| \leqq |1 - |yx|/|x|| \leqq c_0(|y|/|x|)^d \leqq c_0 2^{d(k+1)} |x|^{-d} .$$

Hence

$$||x| - 2^j| \leqq c_0 2^{d(k+1)} |x|^{1-d} \leqq c_0 2^d 2^j 2^{-d(j-k)} ,$$

and the contribution to I_{2B} from points x with $|x| \leqq 2^j$ is at most

$$\text{Const} \log (1 - c_0 2^d 2^{-d(j-k)})^{-1} . \tag{4.8}$$

(Here we impose a further lower bound on $j - k$ so that (4.8) is bounded.) Combining (4.7) and (4.8), we see that

$$I_{2B} \leqq \text{Const } 2^{-d(j-k)} . \tag{4.9}$$

Using the estimates (4.4), (4.6), and (4.9) and applying Lemma 11, we see that the partial sums

$$T_j + T_{j+1} + \cdots + T_k$$

have norms bounded independently of j and k. Consequently, in the notation of Lemma 8, the operators $T_{\varepsilon,M}$ have uniformly bounded norms. Since Lemma 8 gives the convergence of $T_{\varepsilon,M} f$ for a dense set of functions f, $T_{\varepsilon,M}$ converges strongly and the limiting operator is bounded. This completes the proof of Theorem 1 for the case that $t = 0$.

Turning to the case that $t \neq 0$, we begin by stating an immediate consequence of Proposition 3.

LEMMA 12. *If* Ω *is as in Theorem* 1 *and if* $t \neq 0$, *then the real number* $R = e^{2\pi/|t|}$ *has the property that*

$$\int_{R^k \leqq |y| \leqq R^{k+1}} \frac{\Omega(y)}{|y|^{1-it}} dy = 0$$

for every integer k.

We can now indicate the modifications in the proof of Theorem 1 for the case $t \neq 0$. Choose R as in Lemma 12, let

$$\Omega_k(x) = \begin{cases} \Omega(x) & \text{for } R^k \leqq |x| < R^{k+1} \\ 0 & \text{otherwise}, \end{cases}$$

and define T_k by (4.1). Lemma 12 and the same argument as with G_{jk} in the previous case show that it is enough to estimate

$$\iint \left| \frac{\Omega_j(yx)}{|yx|^{1-it}} - \frac{\Omega_j(x)}{|x|^{1-it}} \right| \left| \frac{\Omega_k(y)}{|y|^{1-it}} \right| dydx .$$

The subsequent arguments in the previous case make no further use of the mean value of Ω, and it is enough to estimate

$$I_1 = \int_E \left| \frac{1}{|yx|^{1-it}} - \frac{1}{|x|^{1-it}} \right| \frac{1}{|y|} dydx ,$$

where $E = \{(x, y) \mid R^k \leqq |y| \leqq R^{k+1}, R^j \leqq |yx| \leqq R^{j+1}\}$. In the estimate we can assume that $R^{j-k} \geqq 4Rc_1$. Then for (x, y) in E, computations similar to those for (4.2) and (4.3) lead to the inequalities

$$\frac{1}{2} c_1^{-1} R^j \leqq |x| \leqq 2Rc_1 R^j \tag{4.10}$$
$$|y| \leqq \frac{1}{2} |x| .$$

From the inequality or the argument of Lemma 7, we obtain

$$\left| 1 - \frac{|yx|^{1-it}}{|x|^{1-it}} \right| \leqq \text{Const} \left(\frac{|y|}{|x|} \right)^d \tag{4.11}$$

for $|y| \leqq |x|$. From (4.10) and (4.11) we obtain an inequality analogous to (4.4), namely

$$I_1 \leqq \text{Const}\, R^{-d(j-k)} .$$

Collecting results and applying Lemma 11, we see again that the partial sums

$$T_{jk} = T_j + T_{j+1} + \cdots + T_k$$

have norms bounded independently of j and k. The argument of Lemma 8 shows that $T_{jk}f$ converges as $j \to -\infty$, $k \to \infty$, provided f is in $C_c^\infty(X, H)$. Since the T_{jk} have uniformly bounded norms, T_{jk} converges strongly, and the limit is a bounded operator. This completes the proof of Theorem 1 for the case that $t \neq 0$.

5. Unboundedness theorem

We recall the statement of the theorem in question.

THEOREM 2. *Let* X, $\{\delta_r\}$, $|x|$, H, *and* $\Omega(x)$ *be as in Theorem* 1 *with* $\Omega(\delta_r x) = \Omega(x)$ *for all* r *and* x, *and suppose that* Ω *has mean value* $\mathfrak{M}(\Omega)$ (*see equation* (2.2)) *different from* 0. *Then there is no* H-*valued distribution* $d\mu$ *defined on* $C_c^\infty(X, H)$ *such that*

(γ) *Away from the identity,* $d\mu$ *is equal to the function* $\Omega(x)/|x|$.

(δ) *The mapping* L *given by* $Lf(x) = \int d\mu(y) f(yx)$ *for* f *in* $C_c^\infty(X, H)$ *extends to a bounded operator mapping* $L^2(X, H)$ *into itself.*

Before proving the theorem, we make two remarks. The first is that we can assume that $H = \mathbf{C}$ and that $\Omega(x) = 1$ identically. In fact, by Proposition 3 we can write Ω as the sum of a kernel of mean value 0 and a constant operator:

$$\Omega(x) = [\Omega(x) - \mathfrak{M}(\Omega)] + \mathfrak{M}(\Omega) .$$

By Theorem 1 we can ignore, for the purposes of Theorem 2, the contribution to L from the kernel of mean value 0. Thus assume that $\Omega(x) = \mathfrak{M}(\Omega)$ identically. If ξ is a vector in H with $(\mathfrak{M}(\Omega)\xi, \xi) \neq 0$, then the $\mathbf{C}$-valued distribution dT on $C_c^\infty(X)$ given by

$$\int f(x) dT(x) = \left(\int d\mu(x)[f(x)\xi], \xi\right)$$

has the properties that $lf(x) = \int f(yx) dT(y)$ is bounded on $L^2(X)$ with $\| l \| \leqq |\xi|^2 \| L \|$ and that, away from the identity, dT is equal to the function $(\mathfrak{M}(\Omega)\xi, \xi)/|x|$. Therefore we may assume that $H = \mathbf{C}$ and that $\Omega(x) = 1$ identically.

The second remark is that if $d\mu$ is any distribution defined on $C_c^\infty(X)$ and equal to $1/|x|$ away from the identity, then the operator L in (δ) does map $C_c^\infty(X)$ into $L^2(X)$. This remark follows from the fact that $|x|^{-2}$ is integrable at infinity.

The first lemma below is elementary, and its proof is omitted. The second lemma and the proof of the theorem use the notation $A \approx B$ to mean that A is bounded above and below by positive multiples of B independently of the parameters in question.

LEMMA 13. *If U is a bounded nonempty open set in a Euclidean space, then there is a sequence of real-valued C^∞ functions f_n with $0 \leqq f_n \leqq \chi_U$ and with $\lim f_n = \chi_U$ pointwise.*

LEMMA 14. *Let* $N = \max\{|xy| \mid |x| \leqq 1,\ |y| \leqq 1\} > 1$. *Then*

$$\frac{1}{|yzx|} \approx \frac{1}{|z|}$$

for all x, y, and z such that $|z| \geqq N^2|x| > 0$ *and* $|z| \geqq N^2|y| > 0$.

Proof. We have

$$(5.1) \qquad |xy| \leqq N\max\{|x|, |y|\}$$

for all x and y, by homogeneity. If $|b| < N^{-1}|a|$, then

$$|a| \leqq N\max\{|ab|, |b^{-1}|\} = \max\{N|ab|, N|b|\}$$

and so $|a| \leqq N|ab|$, or

$$(5.2) \qquad |ab| \geqq N^{-1}|a| .$$

Then (5.2) also holds when $|b| \leqq N^{-1}|a|$. By (5.1) and (5.2)

$$(5.3) \qquad N^{-1}|a| \leqq |ax| \leqq N|a|$$

whenever $|x| \leqq N^{-1}|a|$. Again by (5.2), $|y| \leqq N^{-1}|z|$ and $|x| \leqq N^{-2}|z|$ imply $|yz| \geqq N^{-1}|z| \geqq N|x|$ or $|x| \leqq N^{-1}|yz|$. That is, (5.3) applies with $a = yz$. So

$$N^{-1}|yz| \leqq |yzx| \leqq N|yz| \leqq N^2|z| .$$

But $N^{-2}|z| \leqq N^{-1}|yz|$ by (5.2) with $a = z^{-1}$, $b = y^{-1}$. The lemma follows.

For the proof of the theorem, suppose on the contrary that ($\Omega = 1$ and) L is bounded. Let U be a bounded non-empty open set in X, which is topologically a Euclidean space, and choose f_n as in Lemma 13. If x is not in the closure $\bar{U}$, then

$$(5.4) \qquad \int \chi_U(yx)\,|y|^{-1}\,dy = \lim \int f_n(yx)\,|y|^{-1}\,dy$$

by dominated convergence. By (γ),

$$(5.5) \qquad \begin{aligned} &\int_{x \notin \bar{U}} \left| \int_{y \in X} f_n(yx)\,|y|^{-1}\,dy \right|^2 dx = \int_{x \notin \bar{U}} |Lf_n(x)|^2\,dx \\ &\leqq \|Lf_n\|_2^2 \leqq \|L\|^2\,\|f_n\|_2^2 \leqq \|L\|^2\, m(U) , \end{aligned}$$

where $m(U)$ is the measure of U. Passing to the limit in (5.5) and using (5.4) and Fatou's Lemma, we obtain

$$\int_{x \notin \bar{U}} \left| \int_{y \in X} \chi_U(yx) \mid y \mid^{-1} dy \right|^2 dx \leqq \| L \|^2 m(U) . \tag{5.6}$$

Let B be a fixed bounded set. Applying Schwarz's inequality to the left side of (5.6), we obtain

$$\int_{x \in B - \bar{U}} \int_{y \in U} \mid yx^{-1} \mid dydx \leqq \| L \|^2 m(U)/m(B - \bar{U}) . \tag{5.7}$$

We shall produce sets $U = U_\varepsilon$ such that the left side of (5.7) is unbounded while the right side remains bounded, and this will prove the theorem.

Let N be as in Lemma 14, and define

$$B_\varepsilon = \{x \mid \mid x \mid < \varepsilon\} .$$

By Proposition 3, if m denotes Haar measure, then

$$m(B_\varepsilon) = C\varepsilon . \tag{5.8}$$

Fix $\varepsilon > 0$. As x varies over B_1, the sets $B_\varepsilon x$ cover B_1. By (the proof of) Lemma 2.1 of [34], there is a sequence $B_\varepsilon x_j$ of these sets so that the $B_\varepsilon x_j$ are disjoint in pairs and so that the union of the sets $B_{R\varepsilon} x_j$ covers B_1; here R is any number large enough so that $B_1 B_1 B_1 \subseteq B_R$, and thus R is independent of ε.

Clearly all the sets $B_\varepsilon x_j$ lie in B_N. Let U_ε be the union of the "annuli" $(B_{\varepsilon/N^2} - \bar{B}_{\varepsilon/N^4})x_j$. If n is the number of annuli contributing to U_ε, then (5.8) and the covering property of the $B_{R\varepsilon} x_j$ imply that

$$CR\varepsilon n = \sum m(B_{R\varepsilon} x_j) \geqq m(B_1) = C .$$

Thus

$$n \geqq R^{-1}\varepsilon^{-1} . \tag{5.9}$$

In (5.7) let $B = B_N$ and $U = U_\varepsilon$. Then (5.9) gives

$$m(B - \bar{U}) \geqq \sum_j m(B_{\varepsilon/N^4} x_j) \geqq (R^{-1}\varepsilon^{-1})(C\varepsilon N^{-4}) = R^{-1}CN^4 .$$

Thus the right side of (5.7) is bounded independently of ε.

The left side of (5.7) exceeds

$$\begin{aligned} &\sum_i \sum_{j \neq i} \int_{x \in B_{\varepsilon/N^4} x_j x_i} \int_{y \in (B_{\varepsilon/N^2} - B_{\varepsilon/N^4}) x_j} \mid yx^{-1} \mid^{-1} dydx \\ &= \sum_i \sum_{j \neq i} \int_{x \in B_{\varepsilon/N^4}} \int_{y \in (B_{\varepsilon/N^2} - B_{\varepsilon/N^4})} \mid y(x_j x_i^{-1})x^{-1} \mid^{-1} dydx . \end{aligned} \tag{5.10}$$

In the right side of (5.10), we have $\mid x^{-1} \mid \leqq \varepsilon/N^4$, $\mid y \mid \leqq \varepsilon/N^2$, and $\mid x_j x_i^{-1} \mid \geqq$ because $x_j \notin B_\varepsilon x_i$ and $x_j x_i^{-1} \notin B_\varepsilon$. Applying (5.8) and Lemma 14, we see that the right side of (5.10) is

$$\approx \sum_i \sum_{j \neq i} \varepsilon^2 \mid x_j x_i^{-1} \mid . \tag{5.11}$$

If y is such that $\mid y \mid \leqq R\varepsilon$ and $\mid yx_j x_i^{-1} \mid \geqq R\varepsilon N^2$, then $\mid y^{-1} \mid \leqq R\varepsilon$ and Lemma 1 gives $|x_jx_i^{-1}|^{-1} \approx |yx_jx_i^{-1}|^{-1}$. The set of such y is contained in $B_{R\varepsilon}$, and therefore (5.11) exceeds

$$\begin{aligned}
&\approx \sum_i \varepsilon \sum_{j\neq i} \int_{|y|\leqq R\varepsilon,\, |yx_jx_i^{-1}|>R\varepsilon N^2} |yx_jx_i^{-1}|^{-1}\, dy \\
&= \sum_i \varepsilon \sum_{j\neq i} \int_{(B_{R\varepsilon}x_j)x_i^{-1}-B_{R\varepsilon N^2}} |y|^{-1}\, dy \\
&= \sum_i \varepsilon \sum_{all\ j} \text{ (same)} \\
&\geqq \sum_i \varepsilon \int_{B_1x_i^{-1}-B_{R\varepsilon N^2}} |y|^{-1}\, dy\ .
\end{aligned} \tag{5.12}$$

If $|x_i| \leqq N^{-1}$, then $B_1x_i^{-1} \supseteq B_{1/N}$ and the i^{th} term of (5.12) is

$$\approx \varepsilon \log(1/\varepsilon)\ , \tag{5.13}$$

by Proposition 3. On the other hand, if $|x_i| \geqq N^{-1}$, then $B_{R\varepsilon}x_i$ is disjoint from B_{1/N^2} as soon as ε is sufficiently small, by (5.2). If n_0 is the number of x_i with $|x_i| \leqq N^{-1}$, then (5.8) and the covering property of the $B_{R\varepsilon}x_i$ imply that

$$CR\varepsilon n_0 = \sum_{|x_i|\leqq 1/N} m(B_{R\varepsilon}x_i) \geqq m(B_{1/N^2}) = CN^{-2}\ .$$

Thus $n_0 \approx \varepsilon^{-1}$ and it follows from (5.13) that (5.12) is $\approx \log(1/\varepsilon)$. In other words, the left side of (5.7) exceeds a multiple of $\log(1/\varepsilon)$. This contradiction proves the theorem.

II. Semisimple groups of real-rank one

6. Non-unitary principal series

Let G be a connected semisimple Lie group of matrices with Lie algebra $\mathfrak{g}$, let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ with respect to a Cartan involution θ, let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$, and let K and A be the analytic subgroups of G with Lie algebras $\mathfrak{k}$ and $\mathfrak{a}$. The dimension of A is called the *real-rank* of G. The content of Part II will be largely a study of a family of linear operators (the intertwining operators) naturally associated with some infinite-dimensional representations of G, under the assumption that G has real-rank one. As consequences of this study, we obtain

(i) explicit information about the principal series and complementary series of unitary representations of groups G of real-rank one and

(ii) machinery needed in Part III for the investigation of groups G of higher real-rank.

We shall use the following notation in both Parts II and III. ([15] and [26] are general notational references.) Under ad $\mathfrak{a}$, $\mathfrak{g}$ decomposes as

$$\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{n} \oplus \mathfrak{v}\ ,$$

where $\mathfrak{m}$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$, $\mathfrak{n} = \sum_{\alpha>0} \mathfrak{g}_\alpha$ is the sum of eigenspaces of the restricted roots that are positive relative to some ordering on the dual of $\mathfrak{a}$, and $\mathfrak{v}$ is $\theta\mathfrak{n}$. Let

ρ = half the sum of the positive restricted roots with multiplicities
$\mu(a) = e^{2\rho \log a}$ as a positive character on A
N, V = analytic subgroups corresponding to $\mathfrak{n}$ and $\mathfrak{v}$
M, M' = centralizer and normalizer of A in K
$w\lambda(a) = \lambda(w^{-1}aw)$ for w in M' and λ a character of A
$w\sigma(m) = \sigma(w^{-1}mw)$ for w in M' and σ a representation of M.

Then K is compact (since G is a matrix group), A is a vector group, N and V are simply-connected nilpotent, M and M' are compact, M'/M is a finite group (the Weyl group), and G has a global Iwasawa decomposition $G = ANK$:

$$x = \exp H(x) \cdot n \cdot \kappa(x) .$$

The set MAN is a closed subgroup of G whose finite-dimensional (continuous) irreducible unitary representations are all of the form

$$man \longrightarrow \lambda(a)\sigma(m) ,$$

where λ is a unitary character of A and σ is an irreducible unitary representation of M. The *principal series* of unitary representations of G is parametrized by (σ, λ) and is obtained by inducing these representations of MAN to G. Thus in the "induced picture" the continuous functions in the representation space are functions f on G with

$$f(manx) = \mu(a)^{1/2}\lambda(a)\sigma(m)f(x) , \tag{6.1}$$

and G operates by right translation; the values of f are in the finite-dimensional space E_σ on which σ operates.

These representations may be viewed as operating on a space of functions on K by restriction. To describe this "compact picture," we let H^σ be the closed subspace of $L^2(K, E_\sigma)$ of functions f such that for each m in M

$$f(mk) = \sigma(m)f(k) \tag{6.2}$$

for almost all k in K. The representation on H^σ is given by

$$U(\sigma, \lambda, x)f(k) = e^{\rho H(kx)}\lambda(\exp H(kx))f(\kappa(kx)) \tag{6.3}$$

for x in G. If H^σ is given the L^2 norm, then $U(\sigma, \lambda, x)$ is unitarily equivalent with the member of the principal series corresponding to the pair (σ, λ).

The definition (6.3) of a continuous representation in the Hilbert space H^σ makes sense also when λ is a not-necessarily-unitary character of A. In this case, $U(\sigma, \lambda, x)$ is a bounded operator with norm $\leq \sup_{k \in K} |\lambda(\exp H(kx))|$. We call these representations the *non-unitary principal series*. Let $C^\infty(\sigma)$ be the space of smooth functions in H^σ; then $C^\infty(\sigma)$ is dense and is exactly the space of C^∞ vectors for the representation $U(\sigma, \lambda, x)$. If k is in K, then $U(\sigma, \lambda, k)$ operates as right translation by k, independently of λ. Let D be an irreducible unitary representation of K and let H_D^σ be the subspace of functions in H^σ that transform under right translation by K according to D. The space H_D^σ is finite-dimensional and is contained in $C^\infty(\sigma)$, and $\sum_D H_D^\sigma$ is dense in H^σ. It is a simple matter to check also that $\sum_D H_D^\sigma$ is dense in $C^\infty(\sigma)$ in the usual topology on C^∞ functions. Later we shall be working with operators on $C^\infty(\sigma)$ or on $\sum H_D^\sigma$ that preserve each H_D^σ but are unbounded on H^σ. We shall allow ourselves some freedom with the notation of adjoint for such operators, using * for operators that are not the strict adjoints. Namely if A is such an operator, then A^* will denote either an operator on $C^\infty(\sigma)$ with the formal adjoint property or an operator on $\sum H_D^\sigma$ defined on the spaces H_D^σ one at a time; the exact meaning will always be clear from the context.

There is a third realization, the "noncompact picture," of the principal series. Before describing this realization, let us agree to normalize Haar measures so that

dk on K has total mass 1

dm on M has total mass 1

dv on V has $\int_V e^{2\rho H(v)} dv = 1$.

From the Gelfand-Naimark-Bruhat decomposition of G [4], every element g in G except for a set of lower dimension has a unique decomposition

$$g = m(g)a(g)nv(g), \quad m(g) \in M, \quad a(g) \in A, \quad n \in N, \quad v(g) \in V .$$

By means of this decomposition, we can extend representations of M and characters of A to almost-everywhere-defined functions on G. For example, let

$$\sigma(man\, v) = \sigma(m) \text{ and } \lambda(man\, v) = \lambda(a) .$$

The third realization of the principal series is obtained by restricting to V the functions in the induced picture. The Hilbert space is therefore $L^2(V, E_\sigma)$, and the representation is given by

$$U'(\sigma, \lambda, g)f(x) = \mu^{1/2}(xg)\lambda(xg)\sigma(xg)f(v(xg)) . \tag{6.4}$$

In this formula if g is fixed, then xg is in $MANV$ for almost every x in V, and thus (6.4) makes sense. We shall write $x\bar{g}$ for $v(xg)$ henceforth when x is in V and g is in G.

The representation $U'(\sigma, \lambda, g)$, for λ unitary, is unitarily equivalent with $U(\sigma, \lambda, g)$, and a unitary mapping $W(\sigma, \lambda)$ from the noncompact picture to the compact picture such that

$$U(\sigma, \lambda, g) W(\sigma, \lambda) = W(\sigma, \lambda) U'(\sigma, \lambda, g) \tag{6.5}$$

is given by

$$\begin{aligned} Wf(k) &= \mu^{1/2}(k)\lambda(k)\sigma(k)f(v(k)) \\ W^{-1}f(x) &= e^{\rho H(x)}\lambda(\exp H(x))f(\kappa(x)) \, . \end{aligned} \tag{6.6}$$

For the rest of Part II we shall assume that G has real-rank one. Then M'/M has order 2; we denote by w a representative of the nontrivial coset of M'/M. We know that w^2 is in M. Frequently we shall use the parametrization $\lambda(a) = e^{z\rho \log a}$ for the characters of A; here z is any complex number. It is known that the positive restricted roots are of one of the forms $\{\alpha\}$ or $\{\alpha, 2\alpha\}$. In either case, α is the smaller positive restricted root. Let $p = \dim \mathfrak{g}_\alpha$ and $q = \dim \mathfrak{g}_{2\alpha}$, so that $\rho = (1/2)(p + 2q)\alpha$.

Recall that $x\bar{g} = v(xg)$ for x in V. Since M'/M has order 2, there is at most one member x of V such that $x\bar{g}$ is not defined. In fact,

$$G = MAN \cup MANwMAN \, .$$

If $x_1 g$ is not in $MANV = MANwNw^{-1}$, then $x_1 gw$ is not in $MANwMAN$ and hence is in MAN. If also $x_2 g$ is not in $MANV$, then

$$x_1 x_2^{-1} = (x_1 gw)(x_2 gw)^{-1} \in MAN \cap V = \{1\} \, .$$

Hence $x\bar{g}$ is undefined for at most one x. Let V_g be V with this exceptional value of x deleted.

Let the character λ of A be not necessarily unitary. Then (6.4) defines $U'(\sigma, \lambda, g)$ as a mapping of $C_c(V_g, E_\sigma)$ onto $C_c(V_{g^{-1}}, E_\sigma)$, and (6.5) holds.

Observe that $V_w = V - \{1\}$. In particular, the kernel

$$\mu^{1/2}(xw)\sigma^{-1}(xw)$$

is a smooth function of x in V for $x \neq 1$. Similarly

$$\mu^{(1/2)(1-z)}(kw)\sigma^{-1}(kw)$$

is a smooth function of k in K for k not in M.

For later reference we recall from [12, p. 290] that $e^{(1+z)\rho H(x)}$ is an integrable function of x in V if $\mathrm{Re}(z) > 0$ and that $e^{-2\rho H(x)} \geqq 1$. The latter fact will be reproved in the course of Lemma 29.

We now collect a list of identities that will be used in Part II. The proofs take only one or two lines apiece and are omitted. In the identities, a_0 and m_0 are in A and M respectively, x and x' are members of V, and g is in G.

$$dx = \mu^{-1}(a_0)dx' \quad \text{when} \quad x = a_0 x' a_0^{-1} \tag{6.7}$$

$$\kappa(m_0 g) = m_0 \kappa(g) \tag{6.8}$$

$$a(\kappa(x)) = [\exp H(x)]^{-1} \tag{6.9}$$

$$m(\kappa(x)) = 1 \tag{6.10}$$

$$(x\bar{g})\overline{g^{-1}} = x \quad \text{for} \quad x \in V_g \tag{6.11}$$

$$m(\kappa(x)w) = m(xw) \tag{6.12}$$

$$a(m_0 a_0 g) = a_0 a(g) = a(g m_0 a_0) \tag{6.13}$$

$$a(m_0 x m_0^{-1} w) = a(xw) \tag{6.14}$$

$$a(a_0 x a_0^{-1} w) = a_0^2 a(xw) \tag{6.15}$$

$$a(xw) = a(x^{-1}w) \tag{6.16}$$

$$m(a_0 x a_0^{-1} w) = m(xw) \tag{6.17}$$

$$\mu(\kappa(x)w) = e^{-2\rho H(x)} \mu(xw) . \tag{6.18}$$

LEMMA 15. *On V the change of variables $y = t\bar{w}$ has $t = y\overline{w^{-1}}$ and $dy = \mu(tw)dt$.*

Proof. The inverse relation is $t = y\overline{w^{-1}}$, by (6.11). The equation for the change of measure follows directly from the fact that the operator $U'(\sigma, \lambda, w)$ with σ and λ both trivial is unitary.

7. Intertwining distributions

This section is devoted to showing that intertwining operators for the non-unitary principal series are of a special form. At present the results will be set in the noncompact picture, the passage to the compact picture being in § 9. The first lemma is substantially due to Bruhat [4] (except that Bruhat dealt with the induced picture and stated his final results only for unitary representations); we therefore omit the proof.

LEMMA 16. *Let $dT(y)$ be a* Hom $(E_\sigma, E_{\sigma'})$*-valued distribution on $C_c^\infty(V, E_\sigma)$, and let $A_0\colon C_c^\infty(V, E_\sigma) \to C^\infty(V, E_{\sigma'})$ be the operator defined by*

$$A_0 f(x) = \int dT(y) f(yx) . \tag{7.1}$$

Suppose for each g in G that A_0 satisfies

$$A_0 U'(\sigma, \lambda, g) f(x) = U'(\sigma', \lambda', g) A_0 f(x) \tag{7.2}$$

for all f in $C_c^\infty(V_{g^{-1}}, E_\sigma)$ and all x in V_g. If $dT(y)$ is supported not only at the identity, then σ' is equivalent with $w\sigma$ and λ' equals $w\lambda = \lambda^{-1}$. If $\sigma' = w\sigma$ (so that $E_{\sigma'} = E_\sigma$) and if $\lambda' = w\lambda$, then there is a constant c such that

$$dT(y) = c\mu^{1/2}(yw)\lambda^{-1}(yw)\sigma^{-1}(yw)dy$$

away from the identity.

LEMMA 17. *Let $dT(y)$ and A_0 be as in Lemma 16, and suppose (7.2) holds for all g in MA. Suppose that, for a in A, $\lambda'(a)\lambda(a)^{-1}$ is not of the form $e^{-n\alpha \log a}$ for an integer $n > 0$. If $dT(y)$ is not identically 0 and is supported at the identity, then $\lambda = \lambda'$ and σ is equivalent with σ'. If, in addition, $\sigma = \sigma'$, then A_0 is a scalar multiple of the identity.*

Proof. By a right-invariant vector field on V, we understand an operator of the form $Xf(x) = d/dt\{f((\exp tX)^{-1}x)\,|_{t=0}\}$. Since $dT(y)$ is supported at 1, it follows from known properties of distributions that A_0 is a right-invariant matrix-valued differential operator on V. If g is in MA, put $f_g(x) = f(g^{-1}xg)$ For X in $\mathfrak{v}$,

$$Xf_g(x) = \frac{d}{dt}f(g^{-1}(\exp tX)^{-1}gg^{-1}xg)\,|_{t=0} = (\mathrm{Ad}\,(g^{-1})X)f(g^{-1}xg)\ .$$

Thus

$$(A_0f_g)(x) = (\mathrm{Ad}\,(g^{-1})A_0)f(g^{-1}xg)\ .$$

Since g in MA implies

$$U'(\sigma, \lambda, g)f(x) = \mu^{1/2}(g)\lambda(g)\sigma(g)f(g^{-1}xg)\ ,$$

it follows from (7.2) that

$$\mu^{1/2}(g)\lambda(g)(\mathrm{Ad}\,(g^{-1})A_0)(\sigma(g)f)(g^{-1}xg)$$
$$= \mu^{1/2}(g)\lambda'(g)\sigma'(g)(A_0f)(g^{-1}xg)$$

or

$$\lambda(g)\,\mathrm{Ad}\,(g^{-1})A_0\sigma(g) = \lambda'(g)\sigma'(g)A_0\ . \tag{7.3}$$

Take $g = a$ in A. If a basis of $\mathfrak{v}$ is chosen consistently with the decomposition $\mathfrak{v} = \mathfrak{g}_{-\alpha} + \mathfrak{g}_{-2\alpha}$, then $\mathrm{Ad}\,(a)$ is diagonalized by the basis of monomials in the universal enveloping algebra of $\mathfrak{v}$, and the eigenvalues are $e^{-n\alpha\log a}$ with n an integer $\geqq 0$. Then (7.3) shows that $A_0 = 0$ or $\lambda'(a)\lambda(a)^{-1} = e^{-n\alpha\log a}$ for some $n \geqq 0$. In view of the assumption in the lemma, we may suppose that $n = 0$. The monomials belonging to the eigenvalue 1 are the constants and it follows that $A_0f = Cf$ for a constant matrix C. By (7.3), $A_0\sigma(m) = \sigma'(m)A_0$ for all m in M. Thus the rest follows by Schur's Lemma. This completes the proof.

In our later applications of these lemmas, we shall need the following elementary fact, which is due to A. H. Clifford. (Cf. [36, § 5.8].)

LEMMA 18. *Let $H \subseteq H'$ be compact groups with $[H'\colon H] = 2$, let h' be an* *element of H' not in H, and let R be a continuous irreducible unitary representation of H on V_R such that R and $h'R$ are equivalent.* (*Here* $h'R(h) = R(h'^{-1}hh')$.) *Then it is possible to define $R(h')$ as an operator on V_R in exactly two ways, differing by a minus sign, such that R extends to a unitary representation of H' on V_R.*

In the applications in Part II, we shall take $H = M$, $H' = M'$, $h' = w$, and $R = \sigma$. Then we can define $\sigma(w)$ if and only if σ is equivalent with $w\sigma$.

8. Irreducibility theorem, preliminary form

We are now in a position to apply Theorems 1 and 2 to the question of irreducibility of the principal series for semisimple groups of real-rank one. The main result of this section, Proposition 20, is only a preliminary form of the final irreducibility theorem, which is given in § 12 as Theorem 5. (See also § 16.)

LEMMA 19. *If V is regarded as a simply-connected nilpotent Lie group, then the mappings $x \to axa^{-1}$ for a in A provide a one-parameter group of dilations (in the sense of* § 1), *and $x \to \mu^{-1/2}(xw)$ is a norm function, invariant under $x \to mxm^{-1}$ for m in M. Moreover, the function $x \to \sigma(xw)$ is of class C^∞ away from $x = 1$ and has the homogeneity property*

$$\sigma\big((axa^{-1})w\big) = \sigma(xw)$$

for $x \neq 1$ and for all a in A.

Proof. The mappings $X \to \operatorname{Ad}(a)X$ are diagonable, and it follows that A does provide a group of dilations. Next, $x \to \sigma(xw)$ is of class C^∞ away from $x = 1$ because $V_w = V - \{1\}$, and it has the required homogeneity property by (6.17). Finally we show that $|x| = \mu^{-1/2}(xw)$ is a norm function, the M-invariance following from (6.14). The property $|x| = |x^{-1}|$ follows from (6.16). For the homogeneity of $|x|$ relative to dilations, we apply (6.15) to obtain

$$\mu^{-1/2}(axa^{-1}w) = \mu^{-1}(a)\mu^{-1/2}(xw)\ , \tag{8.1}$$

which is of the required form. For the invariance of $|x|^{-1}dx$ under dilations, we combine (8.1) and (6.7). If $x = ax'a^{-1}$, then (6.7) says that $dx = \mu^{-1}(a)dx'$. By (8.1), $|x| = \mu^{-1}(a)|x'|$. Hence $|x|^{-1}dx = |x'|^{-1}dx'$, and $|x|^{-1}dx$ is invariant under dilations. Since $V_w = V - \{1\}$, $|x|$ is nonvanishing for $x \neq 1$. Thus $|x|$ is a norm function.

PROPOSITION 20. *If λ is unitary, then the principal series representation $U'(\sigma, \lambda, g)$ is reducible if and only if*

(i) *σ is equivalent with $w\sigma$,*

(ii) *$\lambda = 1$, and*

(iii) *the mean value of* $\sigma(xw)^{-1}$ (*in the sense of Proposition* 3) *is* 0.

Remark. The necessity of (i) and (ii) was first proved by Bruhat [4].

Proof of necessity. If $U'(\sigma, \lambda, g)$ is reducible, then there exists a bounded linear operator A_0, not scalar, on $L^2(V, E_\sigma)$ such that

$$A_0 U'(\sigma, \lambda, g) = U'(\sigma, \lambda, g) A_0 \tag{8.2}$$

for all g in G. Taking g to be in V, we see that A_0 commutes with right translations, and it follows from the Schwartz Kernel Theorem that the restriction of A_0 to C_c^∞ is given by convolution with a distribution:

$$A_0 f(x) = \int dT(y) f(yx) \ . \tag{8.3}$$

Restrict (8.2) to C_c^∞. Then

$$A_0 U'(\sigma, \lambda, g) f(x) = U'(\sigma, \lambda, g) A_0 f(x) \tag{8.4}$$

at all simultaneous points of continuity of the two sides of (8.4). Thus (7.2) holds. Since A_0 is not scalar, Lemmas 16 and 17 imply that σ is equivalent with $w\sigma$ and that $\lambda = \lambda^{-1}$, i.e., $\lambda = 1$.

So suppose (i) and (ii) hold. By Lemma 18, it is possible to define $\sigma(w)$. Fix such a definition, and let $B_0 = \sigma(w)^{-1} A_0$. Then (8.4) becomes

$$B_0 U'(\sigma, 1, g) f(x) = \sigma(w)^{-1} U'(\sigma, 1, g) \sigma(w) B_0 f(x) \ .$$

But
$$\sigma(w)^{-1} U'(\sigma, 1, g) \sigma(w) = U'(w\sigma, 1, g),$$
and thus
$$B_0 U'(\sigma, 1, g) f(x) = U'(w\sigma, 1, g) B_0 f(x) \ .$$

By Lemma 16, away from the identity we have

$$dT(y) = c\mu^{1/2}(yw)\sigma(w)\sigma^{-1}(yw)dy$$

for a nonzero constant c. By Lemma 19, we are in a position to apply Theorem 2 to $dT(y)$. Theorem 2 says that A_0 does not extend to a bounded operator on $L^2(V, E_\sigma)$ unless $\sigma(w)\sigma^{-1}(yw)$ has mean value 0. Since $\sigma(w)$ is nonsingular, $\sigma(w)\sigma^{-1}(yw)$ has mean value 0 if and only if $\sigma^{-1}(yw)$ does. This proves the necessity of (iii).

Proof of sufficiency. Let σ be equivalent with $w\sigma$, so that $\sigma(w)$ exists by Lemma 18. Suppose $\sigma^{-1}(xw)$ has mean value 0; then $\sigma(w)\sigma^{-1}(xw)$ has mean value 0. By Lemma 19 and Theorem 1, we can use the kernel

$$\mu^{1/2}(yw)\sigma(w)\sigma^{-1}(yw)$$

to define a principal-value bounded "convolution" operator A_0 on $L^2(V, E_\sigma)$. A_0 is not scalar, by Lemma 9. A_0 will therefore exhibit the reducibility of $U'(\sigma, 1, g)$ if we show that the operator $B_0 = \sigma(w)^{-1}A_0$, whose kernel is

$\mu^{1/2}(yw)\sigma^{-1}(yw)$, satisfies

$$B_0 U'(\sigma, 1, g) = U'(w\sigma, 1, g)B_0 .$$

For $0 < s < 1$ and f in $C_c^\infty(V, E_\sigma)$, let T_s be the operator defined by

$$T_s f(x) = \int_V \mu^{(1/2)(1-s)}(yw)\sigma^{-1}(yw)f(yx)dy .$$

The kernel for T_s is locally integrable, by Proposition 3; thus there is no problem with convergence. Fix g in G. If f has compact support in V_g, then Lemma 23 and Theorem 2 of [26] show that

$$T_s U'(\sigma, \mu^{s/2}, g)f(x) = U'(w\sigma, \mu^{-s/2}, g)T_s f(x) \tag{8.5}$$

for x in V_g.

Let s decrease to 0 in (8.5). By Lemma 10, $\lim_{s\downarrow 0} T_s f = B_0 f$ in L^2. Choose a sequence $s_n \downarrow 0$ so that this convergence occurs almost everywhere. If the convergence occurs at a point x in V_g, then

$$\begin{aligned}\lim U'(w\sigma, \mu^{-s_n/2}, g)T_s f(x) &= \lim [\mu^{-s_n/2}(xg)] \lim [\mu^{1/2}(xg)w\sigma(xg)^{-1}T_{s_n}f(x\bar{g})] \\ &= U'(w\sigma, 1, g)B_0 f(x) .\end{aligned} \tag{8.6}$$

We claim that

$$\lim T_s U'(\sigma, \mu^{s/2}, g)f = B_0 U'(\sigma, 1, g)f \tag{8.7}$$

in L^2. To prove this, we write the difference of the left side of (8.5) and $B_0 U'(\sigma, 1, g)f$ as

$$(T_s - B_0)U'(\sigma, \mu^{s/2}, g)f + B_0[U'(\sigma, \mu^{s/2}, g) - U'(\sigma, 1, g)]f .$$

The second term tends to 0 in L^2 because B_0 is a bounded operator and f is compactly supported in V_g. The first term tends to 0 in L^2 by Lemma 10 because the functions $U'(\sigma, \mu^{s/2}, g)f$ have uniformly bounded L^1 norms and uniformly bounded uniform Lipschitz constants, f being supported in V_g. This proves (8.7). Combining (8.5), (8.6), and (8.7), we obtain

$$B_0 U'(\sigma, 1, g)f = U'(w\sigma, 1, g)B_0 f \tag{8.8}$$

for a dense set of functions f. Since all the operators in question are bounded, (8.8) holds for all f in L^2. Since g is arbitrary, this completes the proof of the proposition.

As a corollary of the proof, we have the following

COROLLARY. *If the principal series representation $U'(\sigma, 1, g)$ is reducible, then the intertwining operators are all of the form $aA_0 + bI$, where A_0 is the principal-value operator*

$$A_0 f(x) = \int \mu^{1/2}(yw)\sigma(w)\sigma^{-1}(yw)f(yx)dy\ .$$

We might point out that, once σ has been extended to M', the operator A_0 is independent of w. In addition, we shall see later that some real multiple of A_0 is self-adjoint and unitary.

9. Intertwining operators and their analytic continuation

Recall from § 6 that the representation $U(\sigma, \lambda, x)$ of the non-unitary principal series operates in the subspace H^σ of $L^2(K, E_\sigma)$ of functions f such that, for each m in M, $f(mk) = \sigma(m)f(k)$ for almost every k in K. The group action is

$$U(\sigma, \lambda, x)f(k) = e^{\rho H(kx)}\lambda(\exp H(kx))f(\kappa(kx))$$

with notation as in § 6. Recall that $C^\infty(\sigma)$ is the subspace of C^∞ functions in H^σ with the C^∞ topology. We say that the representation $U(\sigma, \lambda, x)$ is a member of the *complementary series* if there exists a positive definite continuous inner product $\langle\cdot, \cdot\rangle$ on $C^\infty(\sigma) \times C^\infty(\sigma)$ such that

$$\langle U(\sigma, \lambda, x)f,\ U(\sigma, \lambda, x)g\rangle = \langle f, g\rangle \tag{9.1}$$

for all x in G and all f and g in $C^\infty(\sigma)$. If there is a non-trivial positive semi-definite continuous inner product on $C^\infty(\sigma) \times C^\infty(\sigma)$ such that (9.1) holds, we shall say that $U(\sigma, \lambda, x)$ is in the *quasi-complementary series*.

Notice that our definition admits the (unitary) principal series as part of the complementary series. Notice also that our definition is closely related to, but not identical with, the condition that $U(\sigma, \lambda, x)$ be infinitesimally unitary in the sense of Harish-Chandra.

The most general character of A is of the form $\lambda = \mu^{z/2}$, where z is an arbitrary complex number. The character is unitary exactly when z is purely imaginary. Following Kunze and Stein [26], we introduce for $\mathrm{Re}\,(z) > 0$ an operator on $C^\infty(\sigma)$ that we shall denote $A(w, \sigma, \lambda)$ or $A(w, \sigma, z)$:

$$A(w, \sigma, \lambda)f(k_0) = \int_K \mu^{1/2}(kw)\lambda^{-1}(kw)\sigma^{-1}(kw)f(kk_0)dk\ . \tag{9.2}$$

It is proved in [26] that the kernel of $A(w, \sigma, z)$ is an integrable function of k for $\mathrm{Re}\,(z) > 0$ and hence that $A(w, \sigma, z)$ extends to a bounded operator on H^σ. Moreover, $A(w, \sigma, z)$ maps $C^\infty(\sigma)$ continuously into $C^\infty(w\sigma)$ and satisfies

$$A(w, \sigma, \lambda)\,U(\sigma, \lambda, x) = U(w\sigma, w\lambda, x)A(w, \sigma, \lambda) \tag{9.3}$$

for all x in G.

LEMMA 21. *Fix* σ, σ', $\lambda = \mu^{z/2}$, *and* λ', *and suppose* $\mathrm{Re}\,(z) > 0$. *Let*

L: $C^\infty(\sigma) \longrightarrow C^\infty(\sigma')$ be a non-zero continuous linear operator such that

$$LU(\sigma, \lambda, x) = U(\sigma', \lambda', x)L \tag{9.4}$$

for all x in G. Then there are just two possibilities:

(i) σ' is equivalent with σ and $\lambda' = \lambda$. In this case, if $\sigma' = \sigma$, then L is scalar.

(ii) σ' is equivalent with $w\sigma$ and $\lambda' = \lambda^{-1}$. In this case, if $\sigma' = w\sigma$, then L is a scalar multiple of $A(w, \sigma, \lambda)$.

Proof. Recall the mapping $W(\sigma, \lambda)$ of (6.6). Restrict it to a mapping of $C_c^\infty(V, E_\sigma)$ into $C^\infty(\sigma)$, and restrict $W(\sigma, \lambda)^{-1}$ to a mapping of $C^\infty(\sigma)$ into $C^\infty(V, E_\sigma)$. Define A_0: $C_c^\infty(V, E_\sigma) \rightarrow C^\infty(V, E_{\sigma'})$ by

$$A_0 = W(\sigma', \lambda')^{-1}L\,W(\sigma, \lambda) .$$

Then A_0 satisfies (7.2). To prove A_0 is of the form (7.1), we first prove that $f \longrightarrow A_0f(1)$ is continuous. In fact,

$$A_0f(1) = W(\sigma', \lambda')^{-1}L\,W(\sigma, \lambda)f(1) = L\,W(\sigma, \lambda)f(1);$$

since $W(\sigma, \lambda)$, L, and evaluation at 1 are continuous, $f \longrightarrow A_0f(1)$ is continuous. Thus

$$A_0f(1) = \int dT(y)f(y)$$

for a distribution $dT(y)$. Applying (7.2) with x in V, we obtain (7.1).

If $dT(y) = 0$, then $A_0 = 0$. This means that $L = 0$ on the image of $W(\sigma,\lambda)$. But the span of the right K-translates of this image is $C^\infty(\sigma)$, and L commutes with right translation, by (9.4) for x in K. Thus $L = 0$.

If $dT(y)$ is not 0 and is supported at the identity, Lemma 17 implies that σ' is equivalent with σ and $\lambda' = \lambda$. Also if $\sigma' = \sigma$, then A_0 is scalar. By the uniqueness proved in the preceding paragraph, L is scalar.

If $dT(y)$ is not supported at the identity, Lemma 16 implies that σ' is equivalent with $w\sigma$ and $\lambda' = \lambda^{-1}$. Let $\sigma' = w\sigma$. In view of (9.3), Lemma 16 implies that there is a constant c such that the distribution on V corresponding to $L - cA(w, \sigma, \lambda)$ is supported at the identity. By the result of the previous paragraph either $\lambda^{-1} = \lambda' = \lambda$ or $L - cA(w, \sigma, z) = 0$. The former alternative is ruled out since $\operatorname{Re} z > 0$, and thus $L = cA(w, \sigma, z)$. This proves the lemma.

LEMMA 22. *$U(\sigma, \lambda, x)$ is in the quasi-complementary series if and only if there exists a non-zero continuous positive semidefinite hermitian linear operator L: $C^\infty(\sigma) \longrightarrow C^\infty(\sigma)$ such that*

$$LU(\sigma, \lambda, x) = U(\sigma, \lambda, x^{-1})^*L \tag{9.5}$$

for all x in G. $U(\sigma, \lambda, x)$ is in the complementary series if and only if there exists a one-to-one such L.

Proof. Let P be the orthogonal projection of $L^2(K, E_\sigma)$ on the closure of $C^\infty(\sigma)$. Then P carries $C^\infty(K, E_\sigma)$ into $C^\infty(\sigma)$ and is continuous on $C^\infty(K, E_\sigma)$ since P commutes with right translation by K. Thus $\langle\cdot, \cdot\rangle$ extends continuously to $C^\infty(K, E_\sigma) \times C^\infty(K, E_\sigma)$ under the definition $\langle f, g\rangle = \langle Pf, Pg\rangle$. Let $j: E_\sigma \to E_\sigma^*$ be the conjugate linear mapping such that $(u, v)_{E_\sigma} = jv(u)$ for all u and v in E_σ, let $(Jg)(k) = j(g(k))$, and let B be the continuous bilinear form on $C^\infty(K, E_\sigma) \times C^\infty(K, E_\sigma^*)$ given by $B(f, g) = \langle f, Jg\rangle$. By the Schwartz Kernel Theorem

$$B(f, g) = \int dS(k, k')(f(k) \otimes g(k'))$$

for an $E_\sigma^* \otimes E_\sigma$-valued distribution dS on $K \times K$. Then (9.1) for x in K implies that $dS(kk_0, k'k_0) = dS(k, k')$. Hence $dS(k, k') = dT(k'k^{-1})dk'$ for an $E_\sigma^* \otimes E_\sigma$-valued distribution dT on $K \times K$. If L is defined by

$$Lf(k') = \int dT(k'k^{-1})f(k) ,$$

then $(Lf, g) = \langle f, g\rangle$ for all f and g in $C^\infty(K, E_\sigma)$. The lemma follows readily from this identity and (9.1).

For future reference we quote Lemmas 20 and 24 of [26].

LEMMA 23. $U(\sigma, \lambda, x)^* = U(\sigma, \bar{\lambda}^{-1}, x^{-1})$.

LEMMA 24. *Let $\lambda = \mu^{z/2}$ with* Re $(z) > 0$. *On $C^\infty(\sigma)$,*

$$A(w, \sigma, \lambda)^* = A(w^{-1}, w\sigma, \bar{\lambda}) .$$

PROPOSITION 25. *Fix σ and $\lambda = \mu^{z/2}$ with* Re $(z) > 0$. *Unless σ is equivalent with $w\sigma$ and z is real, $U(\sigma, \lambda, x)$ is not in the quasi-complementary series. If σ is equivalent with $w\sigma$, so that $\sigma(w)$ is defined, and if z is real, then*

$$\sigma(w)A(w, \sigma, \lambda) \tag{9.6}$$

is hermitian on $C^\infty(\sigma)$. Moreover, $U(\sigma, \lambda, x)$ is in the complementary series if and only if (9.6) *is positive or negative definite on $C^\infty(\sigma)$, and it is in the quasi-complementary series if and only if* (9.6) *is semidefinite.*

Remarks. In the statements of the proposition, $C^\infty(\sigma)$ can be replaced by the K-finite subspace $\sum H_D^\sigma$. In fact, if (9.6) is semidefinite or definite on $C^\infty(\sigma)$, then it has the same property on $\sum H_D^\sigma$. Conversely, if it is semidefinite on $\sum H_D^\sigma$, it is semidefinite on $C^\infty(\sigma)$ since $\sum H_D^\sigma$ is dense in $C^\infty(\sigma)$; if it is definite on $\sum H_D^\sigma$, it cannot vanish on any $f \not\equiv 0$ in $C^\infty(\sigma)$ because it would have to vanish on the projection of f in each H_D^σ.

Proof. By Lemma 22, it is equivalent to consider operators $L \neq 0$ satisfying (9.5). By Lemma 23, we are to look for operators $L \neq 0$ satisfying

$$LU(\sigma, \lambda, x) = U(\sigma, \bar{\lambda}^{-1}, x)L .$$

We apply Lemma 21. Alternative (i) is ruled out because $\lambda = \bar{\lambda}^{-1}$ requires that $\operatorname{Re}(z) = 0$. Thus (ii) holds, and σ is equivalent with $w\sigma$ and $\bar{\lambda}^{-1} = \lambda^{-1}$, i.e., z is real. Using (ii) and the same device as in the proof of Proposition 20, we find that L is a scalar multiple of (9.6). Thus the proof will be complete if we show that (9.6) is hermitian for z real. If $\operatorname{Re}(z) > 0$, then Lemma 24 shows that

$$\begin{aligned}[\sigma(w)A(w, \sigma, z)]^* &= A(w, \sigma, z)^*\sigma(w)^* = A(w^{-1}, w\sigma, \bar{z})\sigma(w)^{-1} \\ &= \sigma(w)^{-1}[\sigma(w)A(w^{-1}, w\sigma, \bar{z})\sigma(w)^{-1}] \\ &= \sigma(w^{-1})A(w^{-1}, \sigma, \bar{z}) .\end{aligned}$$

But the right side is independent of the representative in M' and thus equals $\sigma(w)A(w, \sigma, \bar{z})$. The proof is complete.

The proposition shows the importance of the intertwining operator $A(w, \sigma, z)$ for complementary series, and we have seen in effect that $A(w, \sigma, z)$ is the basic operator for studying irreducibility of the principal series. In order to proceed further, it is necessary to study the analyticity of $A(w, \sigma, z)$ in the parameter z.

As preparation for the next theorem, we discuss homogeneous functions on V. We regard V, the conjugations $x \to axa^{-1}$ for a in A, and $|x| = \mu^{-1/2}(xw)$ as an instance of the theory of Part I, as Lemma 19 permits. We say that a function $f(x)$ is *α-homogeneous of degree n* if

$$f(axa^{-1}) = e^{-n\alpha \log a}f(x) ,$$

where α is the smaller positive restricted root. There are two sources of α-homogeneous functions, for our purposes. Identify V with its Lie algebra $\mathfrak{v} = \mathfrak{g}_{-\alpha} + \mathfrak{g}_{-2\alpha}$ by the exponential map, and let $p = \dim \mathfrak{g}_{-\alpha}$ and $q = \dim \mathfrak{g}_{-2\alpha}$. Let $X_1, \cdots, X_p$ and $Y_1, \cdots, Y_q$ be bases of $\mathfrak{g}_{-\alpha}$ and $\mathfrak{g}_{-2\alpha}$, and let $x_1, \cdots, x_p$, $y_1, \cdots, y_q$ be coordinates in the resulting basis of $\mathfrak{v}$. Then the monomial

$$x_1^{a_1} \cdots x_p^{a_p} y_1^{b_1} \cdots y_q^{b_q}$$

is α-homogeneous of degree $(a_1 + \cdots + a_p) + 2(b_1 + \cdots + b_q)$. It follows that any polynomial on V is the sum of polynomials α-homogeneous of positive integral degrees. For another example, we have

$$\mu^{-1}(a) = e^{-2\rho \log a} = e^{-(p+2q)\alpha \log a};$$

here ρ is half the sum of the restricted roots, counted with multiplicities. By

(8.1), $\mu^{-1/2}(xw)$ is α-homogeneous of degree $p + 2q$.

THEOREM 3.* *Let f be in $C^\infty(\sigma)$. As a mapping into $C^\infty(\sigma)$, the function $z \to A(w, \sigma, z)f$, initially defined only for* Re $(z) > 0$, *has a meromorphic extension to the whole complex plane with singularities only at the nonnegative integral multiples of $-(p + 2q)^{-1}$. The singularities at these points are at most simple poles. Moreover, for each complex number z_0, there is a neighborhood $N(z_0)$ of z_0 such that the mapping $(z, f) \to (z - z_0)A(w, \sigma, z)f$ of $N(z_0) \times C^\infty(\sigma)$ into $C^\infty(\sigma)$ is continuous.*

Remarks. The point $z = 1$ corresponds to the character $\mu^{1/2}$ or the linear functional ρ. Therefore $z = (p + 2q)^{-1}$ corresponds to the linear functional $\alpha/2$, where α is the smaller positive restricted root. Not all multiples of $-\alpha/2$ give rise to poles, as the more detailed analysis in § 15 will show.

Proof. Write $A(z)$ for $A(w, \sigma, z)$. For most of the proof we shall examine the vector-valued mapping $(z, f) \to A(z)f(1)$. Let ψ be a fixed left M-invariant C^∞ function on K with values in $[0, 1]$ such that $\psi = 1$ in a neighborhood of M and $\psi = 0$ in a neighborhood of wM. We shall specify ψ shortly. Write $f = \psi f + (1 - \psi)f$. Then

$$(z, f) \longrightarrow A(z)(1 - \psi)f(1)$$

extends to be defined for all z, is continuous, and is entire as a function of z; the extension is

$$A(z)(1 - \psi)f(1) = \int_K \mu^{(1/2)(1-z)}(kw)[1 - \psi(k)]\sigma^{-1}(kw)f(k)dk \,, \tag{9.7}$$

and it has the required properties because the only singularities of the kernel occur for k in M.

We consider $(z, f) \to A(z)\psi f(1)$. From [12, p. 287] we have the change of variables formula

$$\int_K F(k)dk = \int_V\int_M F(m\kappa(y))e^{2\rho H(y)}dmdy \tag{9.8}$$

for our normalization of the Haar measure dy on V. (See § 6.) Taking into account the transformation laws for μ, ψ, σ, and f under left translation by M, we have

$$A(z)\psi f(1) = \int_V \mu^{(1/2)(1-z)}(\kappa(y)w)e^{2\rho H(y)}\psi(\kappa(y))\sigma^{-1}(\kappa(y)w)f(\kappa(y))dy$$

since $\int dm = 1$. By (6.18) and (6.13), the formula simplifies to

$$A(z)\psi f(1) = \int_V \mu^{(1/2)(1-z)}(yw)\sigma^{-1}(yw)\{\psi(\kappa(y))e^{(1+z)\rho H(y)}f(\kappa(y))\}dy \,.$$

* In connection with the statement of this theorem, compare Helgason [16], Schiffmann [32], [33], and our work [21].

We can now specify ψ. Let φ be a function in $C_c^\infty([0, \infty), [0, 1])$ equal to 1 in a neighborhood of 0. Put

$$\psi(k) = \begin{cases} \varphi(|y|) & \text{if } k = m\kappa(y) \text{ for some } m \in M,\ y \in V \\ 0 & \text{otherwise.} \end{cases}$$

Then ψ is of the correct form and

$$A(z)\psi f(1) = \int_V \mu^{(1/2)(1-z)}(yw)\varphi(|y|)\sigma^{-1}(yw)\{e^{(1+z)\rho H(y)}f(\kappa(y))\}dy . \tag{9.9}$$

As in the discussion before the theorem, identify V with the Lie algebra $\mathfrak{v}$. Let $X_1, \cdots, X_p, \ Y_1, \cdots, Y_q$ be a basis of $\mathfrak{v}$ compatible with the decomposition $\mathfrak{v} = \mathfrak{g}_{-\alpha} + \mathfrak{g}_{-2\alpha}$. By Taylor's Theorem write

$$e^{(1+z)\rho H(y)}f(\kappa(y)) = f(1) + f_1(z, y) + \cdots + f_n(z, y) + R_n(z, y)$$

with each term entire as a function of z, with f_j a homogeneous polynomial of degree j in y, and with $R_n(z, y)/\|y\|^{n+1}$ and $(d/dz)R_n(z, y)/\|y\|^{n+1}$ bounded for z in any compact set and y in the support of $\varphi(|y|)$. Rearrange the monomials appearing in the Taylor expansion so that

$$e^{(1+z)\rho H(y)}f(\kappa(y)) = f(1) + g_1(z, y) + \cdots + g_{2n}(z, y) + R_n(z, y)$$

with g_j entire as a function of z and α-homogeneous of degree j as a function of y. If $g_0(z, y) = f(1)$, then (9.9) gives

$$\begin{aligned} A(z)\psi f(1) = \textstyle\sum_{j=0}^{2n} &\int_V \mu^{(1/2)(1-z)}(yw)\varphi(|y|)\sigma^{-1}(yw)g_j(z, y)dy \\ &+ \int_V \mu^{(1/2)(1-z)}(yw)\varphi(|y|)\sigma^{-1}(yw)R_n(z, y)dy . \end{aligned} \tag{9.10}$$

If we fix a right half plane of z's, then n can be taken large enough so that the last term is analytic in the half plane. In fact, we are to prove that each z in the half plane has a neighborhood in which the z-derivative of the integrand is dominated by a single integrable function of y; then analyticity follows by dominated convergence. Lemma 6 shows that $\|y\| \leqq c\mu^{-d/2}(yw)$ on the support of $\varphi(|y|)$, where d is a positive constant. Thus on this support

$$\begin{aligned} \left|\mu^{(1/2)(1-z)}(yw)\sigma^{-1}(yw)\frac{d}{dz}R_n(z, y)\right| &\leqq \mu^{(1/2)(1-\operatorname{Re} z)}(yw) \text{ Const } \|y\|^{n+1} \\ &\leqq \text{Const } \mu^{(1/2)[(1-\operatorname{Re} z)-d(n+1)]}(yw) \end{aligned}$$

and similarly

$$\begin{aligned} &\left|\frac{d}{dz}[\mu^{(1/2)(1-z)}(yw)]\sigma^{-1}(yw)R_n(z, y)\right| \\ &\leqq \text{Const } [\log \mu^{-1/2}(yw)]\mu^{(1/2)[(1-\operatorname{Re} z)-d(n+1)]}(yw) . \end{aligned}$$

The assertion follows from Proposition 3.

With n fixed, consider one of the other terms

$$\int_V \mu^{(1/2)(1-z)}(yw)\varphi(|y|)\sigma^{-1}(yw)g_j(z, y)dy \tag{9.11}$$

in (9.10). We know that $\mu^{-1/2}(yw)$ is α-homogeneous of degree $p + 2q$. Thus $\mu^{j/(2(p+2q))}(yw)$ is α-homogeneous of degree $-j$, and

$$h_j(z, y) = \mu^{j/(2(p+2q))}(yw)\sigma^{-1}(yw)g_j(z, y)$$

is α-homogeneous of degree 0. Then there is a vector $\mathfrak{M}_z(h_j)$ such that

$$\int_V F(|y|)h_j(z, y)dy = C\mathfrak{M}_z(h_j)\int_0^\infty F(r)dr \tag{9.12}$$

for every measurable complex-valued function F on $(0, \infty)$ such that either side is defined; here C is the constant of Proposition 3. (This fact follows from Proposition 3 by defining $\Omega(x)$ to be a diagonal matrix whose entries are the entries of $h_j(z, x)$ and by restricting (2.2) to functions $f(r) = F(r)e$, where e is a vector.) Write (9.11) as

$$\int_V \mu^{(1/2)(1-z-j(p+2q)^{-1})}(yw)\varphi(|y|)\mathfrak{M}_z(h_j)dy$$
$$+ \int_V \mu^{(1/2)(1-z-j(p+2q)^{-1})}(yw)\varphi(|y|)[h_j - \mathfrak{M}_z(h_j)]dy .$$

The second term is 0 by (9.12), and the first term, by Proposition 3, is

$$C\left[\int_0^\infty r^{-(1-z-j(p+2q)^{-1})}\varphi(r)dr\right]\mathfrak{M}_z(h_j) . \tag{9.13}$$

Since $\varphi(r) \equiv 1$ near $r = 0$ and since φ has compact support, the term in brackets is the sum of an entire function and a multiple of $1/(z + j(p + 2q)^{-1})$. This gives the extension of (9.11).

Going back over the argument, we can check on the continuity of $(z, f) \to A(z)\psi f(1)$. It is easy to see that $(z, f) \to (z - z_0)A(z)\psi f(1)$ is continuous in $N(z_0) \times C^\infty(\sigma)$. Hence the same thing is true of

$$(z, f) \longrightarrow (z - z_0)A(z)f(1) . \tag{9.14}$$

To complete the proof, we must consider $A(z)f$. Let f_k be the right translate of f by k in K, and define

$$A(z)f(k) = A(z)f_k(1) .$$

This definition is consistent for Re $(z) > 0$ by (9.3). Also

$$(z, f, k) \longrightarrow (z - z_0)A(z)f(k) \tag{9.15}$$

is continuous, being the composition of $(f, k) \to f_k$ and of (9.14). Then near z_0, (9.16) $$(z - z_0)A(z)f = \sum_{n\geq 0} (z - z_0)^n f_n ,$$

where

(9.17) $$f_n(k) = \frac{1}{2\pi i}\oint \frac{(\xi - z_0)A(\xi)f(k)d\xi}{(\xi - z_0)^{n+1}} .$$

Formula (9.17) and the continuity of (9.15) prove that the convergence in (9.16) is rapid enough to take place in $C^\infty(\sigma)$. The theorem is proved.

As a corollary of the proof, we have the following.

COROLLARY 1. *Let $\mathfrak{M}(\sigma)$ be the mean value of $\sigma^{-1}(xw)$, in the sense of Proposition 3, and let C be the constant of Proposition 2. Then*

(9.18) $$\lim_{z\to 0} zA(w, \sigma, z)f = C\mathfrak{M}(\sigma)f$$

for all f in $C^\infty(\sigma)$.

Proof. We want the residue of $A(z)f(1)$ at $z = 0$. We trace through the proof of Theorem 3, dropping analytic terms. The residue at $z = 0$ of $A(z)f(1)$ is the same as that of (9.11) for $j = 0$, which is computed in the course of the proof and found to be $C\mathfrak{M}_0(h_0)$. But

$$h_0(0, y) = \sigma^{-1}(yw)g_0(0, y) = \sigma^{-1}(yw)f(1)$$

and

$$\mathfrak{M}_0(h_0) = \mathfrak{M}(\sigma^{-1}(yw))f(1) = \mathfrak{M}(\sigma)f(1) .$$

Since the operators on each side of (9.18) commute with right translation by K, the result follows.

COROLLARY 2. *As meromorphic functions of z in the complex plane,*

$$A(w, \sigma, z)\,U(\sigma, z, x) = U(w\sigma, -z, x)A(w, \sigma, z)$$

and

$$A(w, \sigma, \bar{z})^* = A(w^{-1}, w\sigma, z) .$$

Proof. Apply (9.3) and Lemma 24.

Before stating the third corollary, we comment on the result at the beginning of this section. Lemma 21, which was a result about uniqueness of intertwining operators, was proved under the assumption that $\operatorname{Re}(z) > 0$. Corollary 2 shows that the same proof applies at any point z where $A(w, \sigma, z)f$ has no pole for any f in $C^\infty(\sigma)$ and where $\lambda(a)^2 = \mu^z(a)$ is not of the form $e^{-n\alpha \log a}$ for n an integer >0, i.e., $z \neq -n(p + 2q)^{-1}$. That is, Lemma 21 is still valid at all points $z \neq -n(p + 2q)^{-1}$ for n an integer $\geqq 0$, and one can carry this result through the consequences of the lemma. At $z = 0$, the lemma is valid, provided $A(w, \sigma, z)f$ has no pole. If there is a pole, the lemma is valid, provided $\lim_{z \downarrow 0} zA(w, \sigma, z)$ is used in place of $A(w, \sigma, z)$ in (ii).

COROLLARY 3. *If σ is not equivalent with $w\sigma$, then the mean value $\mathfrak{M}(\sigma) = \mathfrak{M}(\sigma^{-1}(xw))$ satisfies $\mathfrak{M}(\sigma) = 0$.*

Proof. The constant operator $\mathfrak{M}(\sigma)$ satisfies

$$\mathfrak{M}(\sigma)\, U(\sigma, 0, x) = U(w\sigma, 0, x)\mathfrak{M}(\sigma)$$

by Corollaries 1 and 2. In view of the above remarks, Lemma 21 and its proof are applicable. The distribution $dT(y)$ (in the proof of the lemma) associated to $\mathfrak{M}(\sigma)$ is clearly supported at the identity, and the proof then shows either that σ is equivalent with $w\sigma$ or else that $\mathfrak{M}(\sigma) = 0$.

LEMMA 26. *If σ is equivalent with $w\sigma$, then the mean value*

$$\sigma(w)\mathfrak{M}(\sigma) = \mathfrak{M}(\sigma(w)\sigma^{-1}(xw))$$

is a scalar matrix independent of w.

Proof. It is independent of w because $\sigma(w)\sigma^{-1}(xw)$ is independent of w. By Schur's Lemma, it will be scalar as soon as we show that it commutes with $\sigma(m_0)$ for each m_0 in M. With $m(g)$ as the M-component in $MANV$, if $xw = manv$, then $(m_0 x m_0^{-1})w = m_0 manv w^{-1} m_0^{-1} w$ and so

$$m(m_0 x m_0^{-1} w) = m_0 m(xw) w^{-1} m_0^{-1} w \ .$$

Thus

$$\begin{aligned}\sigma(m_0)\sigma(w)\sigma^{-1}(xw)\sigma(m_0)^{-1} &= \sigma^{-1}(m_0 m(xw) w^{-1} m_0^{-1}) \\ &= \sigma(w)\sigma^{-1}(m_0 m(xw) w^{-1} m_0^{-1} w) \\ &= \sigma(w)\sigma^{-1}((m_0 x m_0^{-1})w) \ .\end{aligned}$$

Now we use Proposition 3. Substituting and making the change of variables $x' = m_0 x m_0^{-1}$, we can bring the m_0-dependence into the norm function $\mu^{-1/2}(xw)$. But this norm function is invariant under conjugation of x by members of M. Thus

$$\mathfrak{M}(\sigma(w)\sigma^{-1}(m_0 x m_0^{-1} w)) = \mathfrak{M}(\sigma(w)\sigma^{-1}(xw)) \ ,$$

and the lemma follows.

PROPOSITION 27. *There exists a meromorphic complex-valued function $c_\sigma(z)$ such that*

$$A(w^{-1}, w\sigma, -z)A(w, \sigma, z) = c_\sigma(z)I \ .$$

The function $c_\sigma(z)$ has the following properties:

(i) *$c_\sigma(z)$ is independent of w.*

(ii) *$c_\sigma(z)$ depends only on the equivalence class of σ, not on σ itself.*

(iii) $c_{w\sigma}(z) = c_\sigma(-z)$.

(vi) $c_\sigma(z) = \overline{c_\sigma(-\bar{z})}$.

(v) $c_\sigma(z) \geqq 0$ *for* z *purely imaginary.*

(vi) *At* $z = 0$, *either* $c_\sigma(z)$ *has exactly a double pole or* $c_\sigma(z)$ *is regular; at other points* $c_\sigma(z)$ *has at most simple poles, and the poles occur only for* z *real.*

(vii) $\lim_{z\to 0} z^2 c_\sigma(z)$ *is finite. It is* 0 *if and only if* $\mathfrak{M}(\sigma) = 0$.

Proof. By Corollary 2

$$A(w^{-1}, w\sigma, -z)A(w, \sigma, z)U(\sigma, z, x) = A(w^{-1}, w\sigma, -z)U(w\sigma, -z, x)A(w, \sigma, z)$$
$$= U(\sigma, z, x)A(w^{-1}, w\sigma, -z)A(w, \sigma, z)$$

as meromorphic functions. In particular, this equation holds for z purely imaginary and nonzero, where there are no poles. Lemma 21 and the remarks before Corollary 3 of Theorem 3 say that, for such z, $A(w^{-1}, w\sigma, -z)A(w, \sigma, z)$ is scalar. Thus

$$A(w^{-1}, w\sigma, -z)A(w, \sigma, z)f = c_\sigma(z)f \tag{9.19}$$

for these z and for all f in $C^\infty(\sigma)$. Fix $f \neq 0$ and take the L^2 inner product of both sides of (9.19) with f. The left side is meromorphic in z, and the right side is $c_\sigma(z)\,||f||_2^2$. Thus $c_\sigma(z)$ extends uniquely to a meromorphic function in the plane. If we take the inner product of both sides of (9.19) with g in $C^\infty(\sigma)$ and apply the same argument, we find by the uniqueness of the extension of $c_\sigma(z)$, that (9.19) holds for all z. Since f is arbitrary and the extension of $c_\sigma(z)$ is unique, (9.19) holds for all f and all z.

To prove (i) and (ii), we collect some identities; these hold initially for $\operatorname{Re}(z) > 0$ and then extend to all z by analytic continuation. We have

$$A(wm, \sigma, z) = \sigma(m)^{-1}A(w, \sigma, z)$$

for m in M, by direct computation. Replacing m by $w^{-1}mw$, we obtain

$$A(mw, \sigma, z) = w\sigma(m)^{-1}A(w, \sigma, z)\ .$$

Again by direct computation

$$A(w, E\sigma E^{-1}, z) = EA(w, \sigma, z)E^{-1}\ . \tag{9.20}$$

For (i), these identities give

$$A\big((wm)^{-1}, wm\sigma, -z\big)A(wm, \sigma, z)$$
$$= w^{-1}(wm\sigma)(m^{-1})^{-1}A(w^{-1}, wm\sigma, -z)\sigma(m)^{-1}A(w, \sigma, z)$$
$$= \sigma(m)A(w^{-1}, wm\sigma, -z)\sigma(m)^{-1}A(w, \sigma, z)\ .$$

Now $wm\sigma$ is equivalent with $w\sigma$ by $\sigma(m)^{-1}$, and so the above expression is

$$= \sigma(m)\sigma(m)^{-1}A(w^{-1}, w\sigma, -z)\sigma(m)\sigma(m)^{-1}A(w, \sigma, z)$$
$$= A(w^{-1}, w\sigma, -z)A(w, \sigma, z)\ .$$

This proves (i). Property (ii) is immediate from (9.20).

To prove (iv) and (v), we use the identity

$$A(w, \sigma, z)^* = A(w^{-1}, w\sigma, \bar{z})$$

from Corollary 2. We have

$$c_\sigma(z)I = A(w^{-1}, w\sigma, -z)A(w, \sigma, z) = A(w, \sigma, -\bar{z})^* A(w, \sigma, z) \; , \tag{9.21}$$

and (v) holds. Apply * to both sides (the operators being globally defined on $C^\infty(\sigma)$) to get

$$\overline{c_\sigma(z)} I = A(w, \sigma, z)^* A(w, \sigma, -\bar{z})$$

or

$$\overline{c_\sigma(-\bar{z})} I = A(w, \sigma, -\bar{z})^* A(w, \sigma, z) = c_\sigma(z) I \; .$$

This is (iv).

Except for the part about $z = 0$, (vi) is immediate from Theorem 3. At $z = 0$, $A(w, \sigma, z)f$ has at most a simple pole. Thus $c_\sigma(z)$ has at most a double pole. The pole cannot be simple, by (v). This proves (vi).

For (vii), Corollary 1 and the joint continuity in Theorem 3 give

$$\lim_{z\to 0} z^2 c_\sigma(z) I = -C^2 \mathfrak{M}\big(w\sigma(xw^{-1})^{-1}\big)\mathfrak{M}\big(\sigma(xw)^{-1}\big) \; .$$

By Corollary 3 the right side is 0 if σ is inequivalent with $w\sigma$. Thus we may suppose σ is equivalent with $w\sigma$. So $\sigma(w)$ exists. Then

$$\begin{aligned} \mathfrak{M}\big(w\sigma(xw^{-1})^{-1}\big)\mathfrak{M}\big(\sigma(xw)^{-1}\big) &= \sigma(w^{-1})\mathfrak{M}\big(\sigma(xw^{-1})^{-1}\big)\sigma(w)\mathfrak{M}\big(\sigma(xw)^{-1}\big) \\ &= \big[\sigma(w)\mathfrak{M}\big(\sigma(xw)^{-1}\big)\big]^2 \; . \end{aligned}$$

By Lemma 26, $\sigma(w)\mathfrak{M}\big(\sigma(xw)^{-1}\big)$ is scalar; hence its square is 0 if and only if it is 0, and it is 0 if and only if $\mathfrak{M}(\sigma)$ is 0. This proves (vii).

We are left with (iii). Let z be purely imaginary and not 0, and fix a K-finite subspace H_D^σ. Each operator A has no pole at z or $-z$, and each leaves H_D^σ stable, by Corollary 2. We have $A(w, \sigma, z)\colon H_D^\sigma \to H_D^{w\sigma}$ and $A(w^{-1}, w\sigma, -z)\colon H_D^{w\sigma} \to H_D^\sigma$ with their respective compositions equal to $c_\sigma(z)I$ and $c_{w\sigma}(-z)I$ on their domains. Since these spaces are finite-dimensional, the scalars are equal. The proof of the proposition is complete.

10. Asymptotic expansions

Our goal in this section and the next will be to connect $c_\sigma(z)$ with the Plancherel measure of G. After we have completed this identification, the main results about the real-rank one case will follow quickly.

The first step is to prove an asymptotic expansion for the matrix coefficients of the representation $U(\sigma, z, x)$ that is valid in a full strip about the imaginary axis. The motivation for proving such an expansion lies in Lemma 28 below and in the fact that the right side of the equality in the lemma is the kind of expression that arises as a coefficient in asymptotic expansions.

LEMMA 28. *If ψ and η are in $C^\infty(\sigma)$ and if* Re $(z) > 0$, *then*

$$(A(w, \sigma, z)\psi(w), \eta(1))_{E_\sigma} = \int_V e^{(1+z)\rho H(x)}(\psi(\kappa(x)), \eta(1))_{E_\sigma} dx .$$

Proof. We use (9.8), (6.13), (6.10), and (6.9). The left side in the statement equals

$$\begin{aligned}
&\int_K \mu^{(1/2)(1-z)}(kw)(\sigma^{-1}(kw)\psi(kw), \eta(1))_{E_\sigma} dk \\
&= \int_K \mu^{(1/2)(1-z)}(k)(\sigma^{-1}(k)\psi(k), \eta(1))_{E_\sigma} dk \\
&= \int_V\int_M \mu^{(1/2)(1-z)}(m\kappa(x))e^{2\rho H(x)}(\sigma^{-1}(m\kappa(x))\psi(m\kappa(x)), \eta(1))_{E_\sigma} dm\, dx \\
&= \int_V \mu^{(1/2)(1-z)}(\kappa(x))e^{2\rho H(x)}(\sigma^{-1}(\kappa(x))\psi(\kappa(x)), \eta(1))_{E_\sigma} dx \\
&= \int_V e^{(1+z)\rho H(x)}(\psi(\kappa(x)), \eta(1))_{E_\sigma} dx .
\end{aligned}$$

LEMMA 29. *The function $\mu^{-1}(xw)$ is a polynomial on V α-homogeneous of degree $p + 2q$. Moreover, there exist nonconstant polynomials $P_1(x), \cdots, P_s(x)$ all $\geqq 0$, α-homogeneous of degree $< p + 2q$, such that*

$$e^{-2\rho H(x)} = 1 + P_1(x) + \cdots + P_s(x) + \mu^{-1}(xw) .$$

Consequently

$$e^{2\rho H(x)} \leqq 1 \quad and \quad e^{2\rho H(x)} \leqq \mu(xw) \tag{10.1}$$

for all x. To each bounded set in V there correspond constants $d > 0$ and c_1 and c_2 such that

$$e^{-2\rho H(x)} - 1 \leqq c_1 \| x \| \leqq c_2 \mu^{-d/2}(xw) \tag{10.2}$$

for x in that bounded set.

Remark. Recall from Lemma 19 that $\mu^{-1}(xw)$ is the square of a norm function on V.

Proof. (Cf. [12], p. 280.) The lemma is valid for G if and only if it is valid for the simply-connected cover $\widetilde{G}$ of G, and it is valid for $\widetilde{G}$ if and only if it is valid for the largest quotient of $\widetilde{G}$ admitting a faithful matrix representation. This means that we can assume at the outset that G is contained in a simply-connected group G^C having Lie algebra $\mathfrak{g}^C$, the complexification of $\mathfrak{g}$. It is known that $\mathfrak{u} = \mathfrak{k} + i\mathfrak{p}$ is a compact form of $\mathfrak{g}^C$ and that $\mathfrak{a}^C$ extends to a

Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}^C$. Choose a compatible order on the roots. From page 248 of [18], one knows that half the sum of the positive roots is dominant integral. Choose an irreducible representation R of G^C on a space E with it as highest weight, and introduce an inner product so that $\mathfrak{u}$ acts by skew hermitian transformations. Lump together the weight spaces whose weights agree on $\mathfrak{a}$, and let the resulting orthogonal decomposition of E into "restricted weight spaces" be $E = \bigoplus_{i=0}^{s+1} E_i$. Say $R(H)X = \Lambda_i(H)X$ for H in $\mathfrak{a}$ and X in E_i. The highest restricted weight Λ_0 is ρ, and one can show that the lowest Λ_{s+1} is $-\rho$. Each E_i is stable under M.

Fix X in the highest restricted weight space E_0 with $||v|| = 1$. If $g = ank$, then $R(n^{-1})X = X$ and $R(k^{-1})$ is unitary; hence

$$|| R(g^{-1})X ||^2 = e^{2\rho \log (a^{-1})} || X ||^2 = e^{-2\rho H(g)} .$$

Let $P_j(g)$ be the norm squared of the E_j component of $R(g^{-1})X$. Then

$$e^{-2\rho H(g)} = P_0(g) + P_1(g) + \cdots + P_s(g) + P_{s+1}(g) .$$

If $g = x$ is in V, then $P_0(x) = 1$. If $g = x$ is in V and $x^{-1}w = manv$, then $R(w^{-1})X$ is in E_{s+1}, $R(v)$ leaves it fixed, and its image under $R(n)$ has the same E_{s+1} component. Thus, if $|| \cdot ||_{s+1}$ denotes the norm of the E_{s+1} component,

$$|| R(x^{-1})X ||_{s+1}^2 = || R(manv)R(w^{-1})X ||_{s+1}^2 = e^{-2\rho \log a} || R(w^{-1})X ||_{s+1}^2 = \mu^{-1}(x^{-1}w) .$$

By (6.16), $\mu^{-1}(x^{-1}w) = \mu^{-1}(xw)$, and thus $P_{s+1}(x) = \mu^{-1}(xw)$.

Each $P_j(x)$ is a polynomial since V acts by unipotent transformations. For the homogeneity, we have

$$P_j(axa^{-1}) = e^{2(\Lambda_j - \rho) \log a} P_j(x)$$

by an easy computation. The rest of the lemma follows from this homogeneity and from Lemma 6.

Fix an irreducible representation D of K such that $H_D^\sigma \neq 0$. In our notation we shall often suppress w, σ, and D, understanding that they are fixed. We write $D(k)$ for the operator on H_D^σ of right translation by k in K. Define linear transformations E and $\Gamma(z)$ mapping H_D^σ into $H_D^{w\sigma}$ by

(10.3) $$E\psi(k) = \psi(w^{-1}k)$$

(10.4) $$\Gamma(z) = A(w, \sigma, z) |_{H_D^\sigma} .$$

E is unitary; $\Gamma(z)$ is meromorphic in the whole plane, by Theorem 3. Define linear transformations P, $\Gamma_r(z)$, $\Gamma_l(z)$, and $\Phi(z, g)$ mapping H_D^σ into H_D^σ by

(10.5) $$(P\psi, \eta) = (\psi(1), \eta(1))_{E_\sigma}$$

(10.6) $$\Gamma_r(z) = PE^*\Gamma(z)$$

$$\Gamma_l(z) = D(w)^{-1}\Gamma_r(\bar{z})^* D(w) \tag{10.7}$$

$$\big(\Phi(z, g)\psi, \eta\big) = \big(U(\sigma, z, g)\psi, \eta\big) . \tag{10.8}$$

$\Gamma_r(z)$ and $\Gamma_l(z)$ are meromorphic in the whole plane, and $\Phi(z, g)$ is entire as a function of z. Also P is not 0. [In fact, if P were 0, we would have $D(k)\psi(1) = 0$ for all k in K and all ψ in H_D^σ, by (10.5), and this would mean that $H_D^\sigma = 0$.]

LEMMA 30. *If ψ and η are in H_D^σ and if* $\operatorname{Re}(z) > 0$, *then*

$$\big(\Gamma_r(z)\psi, \eta\big) = \int_V e^{(1+z)\rho H(x)}\big(\psi(\kappa(x)), \eta(1)\big)_{E_\sigma} dx \tag{10.9}$$

and

$$\big(\Gamma_l(z)\psi, \eta\big) = \int_V e^{(1+z)\rho H(x)}\big(\psi(x), \eta(\kappa(x)w)\big)_{E_\sigma} dx . \tag{10.10}$$

Proof. By (10.6),

$$\begin{aligned}\big(\Gamma_r(z)\psi, \eta\big) &= \big(PE^*\Gamma(z)\psi, \eta\big) = \big(E^*\Gamma(z)\psi(1), \eta(1)\big)_{E_\sigma} \\ &= \big(\Gamma(z)\psi(w), \eta(1)\big)_{E_\sigma} ,\end{aligned}$$

and (10.9) follows by Lemma 28. Also

$$\begin{aligned}\big(\Gamma_l(z)\psi, \eta\big) &= \big(\Gamma_r(\bar{z})^* D(w)\psi, D(w)\eta\big) \\ &= \overline{\big(\Gamma_r(\bar{z})D(w)\eta, D(w)\psi\big)} \\ &= \int_V e^{(1+z)\rho H(x)}\overline{\big(D(w)\eta(\kappa(x)), D(w)\psi(1)\big)}_{E_\sigma} dx .\end{aligned}$$

and (10.10) follows.

LEMMA 31. *There exists $\delta > 0$ such that, in the strip $|\operatorname{Re} z| \leq \delta$, $\Gamma(z)$ (and hence $\Gamma_r(z)$ and $\Gamma_l(z)$) has its only possible singularity a simple pole at $z = 0$ and $z^{-1}\Gamma(z)$ is bounded as $|\operatorname{Im} z|$ tends to infinity.*

Remark. z can be any positive number less than the number d of (10.2).

Proof. We inspect the proof of Theorem 3. The term (9.7) is harmless. In the Taylor series expansion of $e^{(1+z)\rho H(y)}f(\kappa(y))$, we choose $n = 0$, so that

$$e^{(1+z)\rho H(y)}f(\kappa(y)) = f(1) + R_0(z, y) .$$

Then $R_0(z, y)$ is bounded, and the estimate for the contribution to $A(z)\psi f(1)$ from the remainder term is $\leq$ Const $|1 - z|$ for z in any closed half plane contained in $\operatorname{Re}(z) > -d$. The contribution from the main term is (9.11) with $j = 0$, in which $\mathfrak{M}_z(h_0) = \mathfrak{M}(\sigma^{-1}(yw))$ is independent of z; except for the pole at $z = 0$, this term is bounded in any vertical strip.

PROPOSITION 32. *There are constants $\delta > 0$ and c such that for all z with*

$|\operatorname{Re}(z)| \leqq \delta$ *and for all a in the positive Weyl chamber of A*

$$\left|\frac{z}{(z-1)^2}\right| \left| \Phi(a, z) - e^{-(1-z)\rho \log a}\Gamma_r(z) - e^{-(1+z)\rho \log a}\Gamma_l(-z) \right| \leqq ce^{-(1+\delta)\rho \log a} .$$

Proof. Fix ψ and η in H_D^σ. First we show that

$$(10.11) \qquad \big(\Phi(a, z)\psi, \eta\big) = e^{-(1-z)\rho \log a} \int_V e^{(1+z)\rho H(x)} e^{(1-z)\rho H(axa^{-1})} \times \big(\psi(\kappa(x)), \eta(\kappa(axa^{-1}))\big)_{E_\sigma} dx .$$

In fact, by (9.8), (6.8), and (6.7),

$$\begin{aligned}
\big(\Phi(a, z)\psi, \eta\big) &= \int_K e^{(1+z)\rho H(ka)}\big(\psi(\kappa(ka)), \eta(k)\big)_{E_\sigma} dk \\
&= \int_V \int_M e^{(1+z)\rho[H(xa)-H(x)]} e^{2\rho H(x)} \big(\psi(\kappa(m\kappa(x)a)), \eta(m\kappa(x))\big)_{E_\sigma} dm\, dx \\
&= \int_V e^{(1+z)\rho H(xa)} e^{(1-z)\rho H(x)} \big(\psi(\kappa(\kappa(x)a)), \eta(\kappa(x))\big)_{E_\sigma} dx \\
(*) \qquad &= e^{(1+z)\rho \log a} \int_V e^{(1+z)\rho H(a^{-1}xa)} e^{(1-z)\rho H(x)} \big(\psi(\kappa(\kappa(x)a)), \eta(\kappa(x))\big)_{E_\sigma} dx \\
&= e^{-(1-z)\rho \log a} \int_V e^{(1+z)\rho H(y)} e^{(1-z)\rho H(aya^{-1})} \\
&\qquad \times \big(\psi(\kappa(\kappa(aya^{-1})a)), \eta(\kappa(aya^{-1}))\big)_{E_\sigma} dy
\end{aligned}$$

and (10.11) follows. Second we claim that

$$(10.12) \qquad \big(\Phi(a, -z)\psi, \eta\big) = e^{-(1-z)\rho \log a} \int_V e^{(1+z)\rho H(x)} e^{(1-z)\rho H(axa^{-1})} \times \big(\psi(\kappa(axa^{-1})w), \eta(\kappa(x)w)\big)_{E_\sigma} dx .$$

In fact, starting from $(*)$ with z replaced by $-z$, we make the change of variables $y = t\bar{w}$, for which Lemma 15 gives $t = \overline{yw^{-1}}$ and $dy = \mu(tw)dt$. The resulting expression then simplifies to the right side of (10.12).

The third part of the proof will be clearer if we introduce these notations: $x^a = axa^{-1}$ for a in A and x in V, $|x| = \mu^{-1/2}(xw)$ for x in V. Then $|x^a| = e^{-2\rho \log a}|x|$ by (8.1). The set $\{|x| \leqq 1\}$ is a bounded subset of V; by (10.2), we have

$$0 \leqq e^{-2\rho H(x)} - 1 \leqq c\,|x|^d$$

for $|x| \leqq 1$. Next fix a_0 in the positive Weyl chamber of A such that

$$ce^{-2d\rho \log a_0} < 1 .$$

We need work only with members a of A such that aa_0^{-1} is in the positive Weyl chamber. If $|x^{aa_0^{-1}}| \leqq 1$, then $|x^a| \leqq 1$ and so

$$(10.13) \qquad e^{-2\rho H(x^a)} - 1 \leqq c\,|x^a|^d = ce^{-2d\rho \log a_0}\,|x^{aa_0^{-1}}|^d < |x^{aa_0^{-1}}|^d .$$

Hence

$$0 \leqq e^{-2\rho H(x^a)} - 1 < 1 . \tag{10.14}$$

Now if $0 < \operatorname{Re}(z) \leqq 1$ and $0 \leqq E < 1$, we have

$$|(1 + E)^{-(1/2)(1-z)} - 1| \leqq \frac{1}{2} E |1 - z| . \tag{10.15}$$

If $|x^{aa_0^{-1}}| \leqq 1$, (10.14) shows that we can apply (10.15) with $E = e^{-2\rho H(x^a)} - 1$. By (10.13) and (8.1),

$$\begin{aligned} |e^{(1-z)\rho H(x^a)} - 1| &\leqq \frac{1}{2} |1 - z| \, |x^{aa_0^{-1}}|^d \\ &\leqq \left(\frac{1}{2} e^{2d\rho \log a_0}\right) |1 - z| \, e^{-2d\rho \log a} |x|^d . \end{aligned} \tag{10.16}$$

If $0 < \operatorname{Re}(z) < 1$, then Lemma 30 gives

$$\begin{aligned} |e^{(1-z)\rho \log a}\Phi(a, z) - \Gamma_r(z)| &= \Big| \int_V e^{(1+z)\rho H(x)} \big[e^{(1-z)\rho H(axa^{-1})} \\ &\quad \times \big(\psi(\kappa(x)), \eta(\kappa(axa^{-1}))\big)_{E_\sigma} - \big(\psi(\kappa(x)), \eta(1)\big)_{E_\sigma} \big] dx \\ &\leqq \|\psi\|_\infty \|\eta\|_\infty \int_V e^{(1+\operatorname{Re}(z))\rho H(x)} \, |e^{(1-z)\rho H(axa^{-1})} - 1| \, dx \\ &\quad + \int_V e^{(1+\operatorname{Re}(z))\rho H(x)} \, |\big(\psi(\kappa(x)), \eta(\kappa(axa^{-1})) - \eta(1)\big)_{E_\sigma}| \, dx \\ &= \|\psi\|_\infty \|\eta\|_\infty \mathrm{I} + \mathrm{II}, \text{ say} . \end{aligned}$$

Break each of the integrals I and II into the sum of an integral over $|x| \leqq 1$, an integral over $1 \leqq |x| \leqq e^{2\rho \log aa_0^{-1}}$, and an integral over $e^{2\rho \log aa_0^{-1}} \leqq |x|$. Call the six resulting integrals IA, IB, IC, IIA, IIB, IIC. We estimate each separately.

In IA and IB, we have $|x| \leqq e^{2\rho \log aa_0^{-1}}$ or $|x^{aa_0^{-1}}| \leqq 1$. Thus (10.16) applies to both IA and IB. Also $e^{(1+\operatorname{Re} z)\rho H(x)} \leqq 1$ by (10.1) and $e^{(1+\operatorname{Re} z)\rho H(x)} \leqq |x|^{-(1+\operatorname{Re} z)}$ by (10.1). By Proposition 3

$$\begin{aligned} \mathrm{IA} &\leqq \int_{|x| \leqq 1} \left(\frac{1}{2} e^{2d\rho \log a_0}\right) |1 - z| \, e^{-2d\rho \log a} |x|^d \, dx \\ &\leqq \text{Const} \, |1 - z| \, e^{-2d\rho \log a} \end{aligned}$$

with the constant independent of a, and, if $\operatorname{Re}(z) < d$,

$$\begin{aligned} \mathrm{IB} &\leqq \int_{1 \leqq |x| \leqq e^{2\rho \log aa_0^{-1}}} |x|^{-(1+\operatorname{Re}(z))} \left(\frac{1}{2} e^{2d\rho \log a_0}\right) |1 - z| \, e^{-2d\rho \log a} |x|^d \, dx \\ &\leqq \text{Const} \, |1 - z| \, \big(d - \operatorname{Re}(z)\big)^{-1} e^{-2d\rho \log a} \big[r^{d - \operatorname{Re}(z)} \big]_1^{e^{2\rho \log aa_0^{-1}}} \\ &\leqq \text{Const} \, |1 - z| \, \big(d - \operatorname{Re}(z)\big)^{-1} e^{-2d\rho \log a} e^{2(d - \operatorname{Re}(z))\rho \log a} \\ &= \text{Const} \, |1 - z| \, \big(d - \operatorname{Re}(z)\big)^{-1} e^{-2(\operatorname{Re} z)\rho \log a} . \end{aligned}$$

In IC and IIC all the factors in the integrand are bounded, and (10.1) gives $e^{(1+\mathrm{Re}(z))\rho H(x)} \leqq |x|^{-(1+\mathrm{Re}(z))}$. By Proposition 3

$$\begin{aligned} \mathrm{IC} + \mathrm{IIC} &\leqq \mathrm{Const} \int_{|x| \geqq e^{2\rho \log a a_0^{-1}}} |x|^{-(1+\mathrm{Re}(z))}\, dx \\ &\leqq \mathrm{Const}\, (\mathrm{Re}\, z)^{-1} e^{-2(\mathrm{Re}(z))\rho \log a a_0^{-1}} \\ &\leqq \mathrm{Const}\, (\mathrm{Re}\, z)^{-1} e^{-2(\mathrm{Re}(z))\rho \log a} . \end{aligned}$$

In IIA and IIB, $|x^a| \leqq 1$, and $\eta(\kappa(x))$ satisfies a Lipschitz condition on the unit ball. Also for IIA, $e^{(1+\mathrm{Re}(z))\rho H(x)} \leqq 1$ by (10.1). Thus for the constant d above and in (10.2), Lemma 6 gives

$$\begin{aligned} \mathrm{IIA} &\leqq \mathrm{Const} \int_{|x| \leqq 1} \| x^a \|\, dx \leqq \mathrm{Const} \int_{|x| \leqq 1} |x^a|^d\, dx \\ &= \mathrm{Const}\, e^{-2d\rho \log a} \int_{|x| \leqq 1} |x|^d\, dx = \mathrm{Const}\, e^{-2d\rho \log a} . \end{aligned}$$

Similarly for IIB, we have $e^{(1+\mathrm{Re}(z))\rho H(x)} \leqq |x|^{-(1+\mathrm{Re}(z))}$ by (10.1). By the same kind of argument as for IIA, we obtain, for $\mathrm{Re}\, z < d$,

$$\begin{aligned} \mathrm{IIB} &\leqq \mathrm{Const} \int_{1 \leqq |x| \leqq e^{2\rho \log a a_0^{-1}}} \| x^a \|\, |x|^{-(1+\mathrm{Re}(z))}\, dx \\ &\leqq \mathrm{Const}\, \big(d - \mathrm{Re}(z)\big)^{-1} e^{-2(\mathrm{Re}(z))\rho \log a} . \end{aligned}$$

If $0 < \mathrm{Re}(z) \leqq \delta < d$, then we can conclude from our estimates that

$$|\Phi(a, z) - e^{-(1-z)\rho \log a}\Gamma_r(z)| \leqq \mathrm{Const}\,\big[|1 - z| + \big(\mathrm{Re}(z)\big)^{-1}\big] e^{-(1+\mathrm{Re}(z))\rho \log a} . \tag{10.17}$$

Fix a and put

$$F(z) = \frac{z}{(z-1)^2}\big[\Phi(a, z) - e^{-(1-z)\rho \log a}\Gamma_r(z) - e^{-(1+z)\rho \log a}\Gamma_l(-z)\big] .$$

By Lemma 31, $F(z)$ is bounded and analytic in some strip $|\mathrm{Re}(z)| \leqq \delta < d$. Then (10.17) and Lemma 31 together give

$$|F(\delta + iy)| \leqq \mathrm{Const}\, e^{-(1+\delta)\rho \log a}$$

with the constant independent of a and y.

Starting from (10.12), we can give a similar argument for

$$|\Phi(a, z) - e^{-(1+z)\rho \log a}\Gamma_l(-z)|$$

when $\mathrm{Re}(z) < 0$ and arrive at the conclusion

$$|F(-\delta + iy)| \leqq \mathrm{Const}\, e^{-(1+\delta)\rho \log a} .$$

By the maximum modulus theorem

$$|F(z)| \leqq \mathrm{Const}\, e^{-(1+\delta)\rho \log a}$$

for $|\mathrm{Re}(z)| \leqq \delta$, with the constant independent of z and a. This completes the proof of the proposition.

Remark. In the proof δ can be any positive number less than d, and inspection of Lemma 6 shows that d is $(p + 2q)^{-1}$.

11. Role of the Plancherel measure

Our objective in this section is to give an explicit formula for $c_\sigma(z)$ in terms of the Plancherel measure for G. So far, what we have shown is that the matrix coefficients of $U(\sigma, z, a)$ admit an asymptotic expansion in the variable a such that the coefficients $\Gamma_r(z)$ and $\Gamma_l(-z)$ of the leading terms are related in a simple way to the intertwining operator $A(w, \sigma, z)$. As we shall see in Lemma 35, this simple relationship will allow us easily to relate $c_\sigma(z)$ to the Hilbert-Schmidt norm of $\Gamma_r(z)$ and $\Gamma_l(-z)$. Thus the remaining step is to use the asymptotic expansion to relate the norms of $\Gamma_r(z)$ and $\Gamma_l(-z)$ to the Plancherel measure. The core of this last identification lies in a property of the Plancherel formula of G. The Plancherel formula for G says that the mapping of $L^2(G)$ into a direct integral of Hilbert-Schmidt operators is an isometry; what we need is an explicit identification of the image of this isometry. We therefore begin with a brief discussion of the abstract Plancherel formula and then pass to the explicit identification of the image, which is a result implicit in the work of Naimark [29].

Let G_0 be a separable locally compact unimodular group. Fix a normalization of Haar measure. We shall suppose that G_0 is of type I. This property implies that G_0 has a Plancherel formula in the following sense: The space $\Pi = \{\pi\}$ of classes of irreducible unitary representations of G_0 (or of a representative from each class) admits a measure $d\nu(\pi)$ such that, for each f in $L^1(G_0) \cap L^2(G_0)$,

$$||f||_2^2 = \int_\Pi ||\pi(f)||_{\mathrm{HS}}^2 \, d\nu(\pi) ,$$

where $\pi(f)$ refers to $\pi(x)$ averaged by f and where HS refers to the Hilbert-Schmidt norm.

For the semisimple matrix group G of real-rank one, the Plancherel formula is known explicitly. (Knowledge about this formula is the only reason for restricting attention to matrix groups in Part II.) We shall describe the formula more explicitly in § 12, but for the moment we give only a few features of it. The representations contributing to the formula are all in the discrete series and the principal series, and the formula is

$$(11.1) \qquad ||f||_2^2 = \sum_{d \in \mathfrak{D}} ||\pi_d(f)||_{\mathrm{HS}}^2 \, p(d) + \sum_\sigma \int_0^\infty ||U(\sigma, it, f)||_{\mathrm{HS}}^2 \, p_\sigma(it) dt ,$$

where π_d is a discrete series representation, $\mathfrak{D}$ is a countable (possibly empty) set, $p(d)$ is bounded below by a positive constant, σ extends over all equivalence classes of irreducible unitary representations of M, and $p_\sigma(it)$ is a continuous function that extends to a meromorphic function $p_\sigma(z)$ in the whole plane without singularities on the imaginary axis and without zeros on the imaginary axis except possibly at $z = 0$. All the representations that appear in (11.1) are irreducible (with the understanding that $t > 0$).

LEMMA 33 (NAIMARK). *With σ fixed, let $t \to T_t$ be a measurable function from $(0, \infty)$ to the Hilbert-Schmidt operators on H^σ such that $\int_0^\infty \| T_t \|_{\mathrm{HS}}^2 p_\sigma(it)dt < \infty$. Then there exists a function f in $L^2(G)$ such that*

$$\int_G f(x)\overline{h(x)}dx = \int_0^\infty \mathrm{Tr}\,\big(T_t U(\sigma, it, h)^*\big)p_\sigma(it)dt \tag{11.2}$$

for all h in $C_c^\infty(G)$ and such that

$$\int_G | f(x) |^2\, dx = \int_0^\infty \| T_t \|_{\mathrm{HS}}^2\, p_\sigma(it)dt\ . \tag{11.3}$$

Formula (11.1) says that the mapping $f \to \{\pi_d(f),\ U(\sigma, it, f)\}$ defined on the dense subspace $L^1(G) \cap L^2(G)$ of $L^2(G)$ is an isometry into the obvious L^2 space, and the lemma says substantially that the continuous extension of this map to all of $L^2(G)$ is an isometry *onto*. Actually Naimark [29] stated and proved a result only for complex semisimple Lie groups. But the proof that he gives can be adjusted to yield the above lemma if one replaces the Hilbert-Schmidt kernels there by the actual Hilbert-Schmidt operators and makes other minor changes.

There is one detail that needs checking. For f in $L^1(G)$, it must be shown that the continuous function $\pi \to \pi(f)$ (for π occurring in (11.1)) vanishes in norm at infinity. It is enough to consider a dense set of such f, say an f in $C_c^\infty(G)$ that transforms under K on both the left and the right by a finite-dimensional representation of K. For this f, $\| \pi_d(f) \| \leqq \| \pi_d(f) \|_{\mathrm{HS}} \to 0$ by (11.1) and the fact that $p(d)$ is bounded below. Furthermore $U(\sigma, it, f) = 0$ unless σ occurs in the reduction of D under M; hence $U(\sigma, it, f) = 0$ for all t except for finitely many σ. For the exceptional σ's, only finitely many entries of $U(\sigma, it, f)$ are nonzero. Thus we are to show, for fixed σ, that each entry of $U(\sigma, it, f)$ tends to 0 as t tends to infinity. That is, we are to show that

$$\int_G \int_K e^{(1+it)\rho H(kx)}\big(\psi(\kappa(kx)),\ \eta(k)\big)_{E_\sigma} f(x)dk\,dx \tag{11.4}$$

has limit 0 as $t \to \infty$, if ψ and η are in H_D^σ. Since f has compact support, (11.4) in local coordinates transforms into a finite sum of terms

$$(11.5) \qquad \int_X e^{itF(x)} g(x)dx$$

where X is a bounded open subset of a euclidean space, $F(x)$ is a nonconstant real analytic function on the closure of X, and $g(x)$ is a bounded measurable function on X. Defining a measure on the line by

$$\tau(E) = \int_{F^{-1}(E)} g(x)dx ,$$

we see that (11.5) equals

$$(11.6) \qquad \int_{-\infty}^{\infty} e^{its} d\tau(s) .$$

If the Lebesgue measure of E is 0, then $F^{-1}(E)$ has measure 0, since F is nonconstant and real analytic. Thus $\tau(E) = 0$, and $d\tau$ is absolutely continuous. Since g is bounded, $d\tau$ is a finite measure, and the Riemann-Lebesgue Lemma shows that (11.6) tends to 0 as $t \to \infty$. Hence (11.4) tends to 0. This completes the verification of the detail needed for Lemma 33.

We shall now apply this lemma. As in § 10, let D be an irreducible unitary representation of K such that $H_D^\sigma \neq 0$. Then we have defined $\Phi(it, x)$ mapping H_D^σ into itself by

$$\big(\Phi(it, x)\psi, \eta\big) = \big(U(\sigma, it, x)\psi, \eta\big) .$$

Let $\beta(t)$ be a real-valued smooth function of compact support in the open interval $(0, \infty)$. Since $p_\sigma(it)$ vanishes only for $t = 0$, we obtain

$$(11.7) \qquad \int_0^\infty |\beta(t)|^2 p_\sigma(it)^{-1} dt < \infty .$$

Define

$$\Phi(\beta, x) = \int_0^\infty \beta(t)\Phi(it, x)^* dt .$$

LEMMA 34.

$$\int_G \| \Phi(\beta, x) \|_{\mathrm{HS}}^2 dx = (\dim H_D^\sigma)^2 \int_0^\infty |\beta(t)|^2 p_\sigma(it)^{-1} dt .$$

Proof. Let $\{\psi_i\}$ be an orthonormal basis of H_D^σ, express Φ in this basis, and let T_t^{ij} be the Hilbert-Schmidt operator on H^σ defined by

$$T_t^{ij}\eta = \beta(t)p_\sigma(it)^{-1}(\eta, \psi_j)\psi_i .$$

Since $p_\sigma(it)$ is real,

$$T_t^{ij*}\eta = \overline{\beta(t)}p_\sigma(it)^{-1}(\eta, \psi_i)\psi_j$$

and

$$T_t^{ij}T_t^{ij*}\eta = |\beta(t)|^2 p_\sigma(it)^{-2}(\eta, \psi_i)\psi_i .$$

Thus

$$\| T_t^{ij} \|_{\mathrm{HS}}^2 = |\beta(t)|^2 p_\sigma(it)^{-2} . \tag{11.8}$$

Let h be in $C_c^\infty(G)$. We have

$$\big(T_t^{ij} U(\sigma, it, h)\psi_k, \psi_l\big) = \beta(t)p_\sigma(t)^{-1}\big(U(\sigma, it, h)\psi_k, \psi_j\big)(\psi_i, \psi_l)$$

and so

$$\operatorname{Tr}\big(T_t^{ij} U(\sigma, it, h)\big) = \beta(t)p_\sigma(t)^{-1}\big(U(\sigma, it, h)\psi_i, \psi_j\big) . \tag{11.9}$$

By (11.7) and (11.8), T_t^{ij} satisfies the hypothesis of Lemma 33. Choosing $f = f^{ij}$ as in the lemma, we obtain

$$\begin{aligned}
\int_G f^{ij}(x)h(x^{-1})dx &= \int_0^\infty \operatorname{Tr}\big(T_t^{ij} U(\sigma, it, h)\big)p_\sigma(it)dt && \text{by (11.2)}\\
&= \int_0^\infty\int_G \beta(t)h(x)\Phi^{ij}(it, x)dx\,dt && \text{by (11.9)}\\
&= \int_0^\infty\int_G \beta(t)h(x^{-1})\Phi^{ij}(it, x^{-1})dx\,dt && \text{under } x \to x^{-1}\\
&= \int_0^\infty\int_G \beta(t)h(x^{-1})\Phi^{ij}(it, x)^*dx\,dt\\
&= \int_G \Phi^{ij}(\beta, x)h(x^{-1})dx && \text{after interchanging integrals.}
\end{aligned}$$

Since h is arbitrary, $f^{ij}(x) = \Phi^{ij}(\beta, x)$ almost everywhere, and (11.3) and (11.8) give

$$\int_G |\Phi^{ij}(\beta, x)|^2 dx = \int_0^\infty \| T_t \|_{\mathrm{HS}}^2 p_\sigma(it)dt = \int_0^\infty |\beta(t)|^2 p_\sigma(it)^{-1}dt .$$

Summing on i and j, we obtain the lemma.

We continue to assume that D is an irreducible representation of K such that $H_D^\sigma \neq 0$. Recall the definitions (10.5)–(10.7) of the operators P, $\Gamma_r(z)$, and $\Gamma_l(z)$ mapping H_D^σ into itself. We have seen that P is not 0.

LEMMA 35. *If t is real and if* HS *refers to Hilbert-Schmidt norms, then*

$$\| \Gamma_r(it) \|_{\mathrm{HS}}^2 = \| \Gamma_l(-it) \|_{\mathrm{HS}}^2 = \| P \|_{\mathrm{HS}}^2 c_\sigma(it) .$$

Proof. By (10.6),

$$\begin{aligned}
\| \Gamma_r(it) \|_{\mathrm{HS}}^2 &= \operatorname{Tr}\big(\Gamma_r(it)\Gamma_r(it)^*\big)\\
&= \operatorname{Tr}\big(PE^*\Gamma(it)\Gamma(it)^*EP^*\big)\\
&= c_\sigma(it) \operatorname{Tr}(PE^*EP^*) && \text{by (9.21)}\\
&= c_\sigma(it) \operatorname{Tr}(PP^*)
\end{aligned}$$

since E is unitary. The rest follows from (10.7).

THEOREM 4. *There exists a constant $c_G \neq 0$ depending only on the normalization of Haar measure in the Plancherel theorem of G such that*

$$c_\sigma(z) = c_G \| P \|_{\mathrm{HS}}^{-2} (\dim H_D^\sigma)^2 p_\sigma(z)^{-1}$$

for all complex z.

Proof. We shall compute $\int_G \| \Phi(\beta, x) \|_{\mathrm{HS}}^2 dx$ a second time, this time using the asymptotic expansion of Proposition 32. Write $G = KA^+K$, where A^+ is the positive Weyl chamber of A. Since $U(\sigma, z, k)$ preserves H_D^σ when k is in K, we have

$$\Phi(z, k_1 a k_2) = D(k_1)\Phi(z, a)D(k_2) .$$

Fix H_0 in the Lie algebra of A such that $\rho(H_0) = 1$, and introduce a real parameter $s \geqq 0$ by $a = \exp sH_0$. Then it is known [15] that Haar measure for G decomposes as

$$dx = \Delta(s) dk_1 ds dk_2 ,$$

where $\Delta(s)$ is a linear combination of exponentials in which the dominant exponential is e^{2s}. Define c_G by the condition that the coefficient of e^{2s} in $\Delta(s)$ is $(2\pi c_G)^{-1}$.

The form of the asymptotic expansion that we shall use is that

$$\Phi(it, a) = e^{-(1-it)\rho \log a}\Gamma_r(it) + e^{-(1+it)\rho \log a}\Gamma_l(-it) + R_0(it, a) ,$$

where $| R_0(it, a) | \leqq \text{Const}\, e^{-(1+\delta)\rho \log a}$ on any compact set of numbers t not containing $t = 0$. With our parametrization of a, this equation becomes

$$\Phi(it, \exp sH_0) = e^{-s}[e^{its}\Gamma_r(it) + e^{-its}\Gamma_l(-it)] + R(it, s) , \tag{11.10}$$

with $| R(it, s) | \leqq \text{Const}\, e^{-(1+\delta)s}$ on compact sets of t's excluding 0.

Put

$$\hat{\beta}(s) = \int_{-\infty}^{\infty} e^{ist}\beta(t)dt ,$$

so that $\| \hat{\beta} \|_2^2 = 2\pi \| \beta \|_2^2$. Fix $t_0 > 0$ and let $E_1, \cdots, E_4$ denote error terms to be computed later. Since $D(k)$ is unitary and $\int dk = 1$ and β is real, we have

(11.11)

$$\begin{aligned}
\int_G \| \Phi(\beta, x) \|_{\mathrm{HS}}^2 dx &= \int_0^\infty \| \Phi(\beta, \exp sH_0) \|_{\mathrm{HS}}^2 \Delta(s) ds \\
&= \int_0^\infty \Delta(s) e^{-2s} \left\| \int_{-\infty}^{\infty} [e^{its}\Gamma_r(it) + e^{-its}\Gamma_l(-it)]\beta(t)dt \right\|_{\mathrm{HS}}^2 ds + E_1 \\
&= (2\pi c_G)^{-1} \int_0^\infty \left\| \int_{-\infty}^{\infty} [e^{its}\Gamma_r(it) + e^{-its}\Gamma_l(-it)]\beta(t)dt \right\|_{\mathrm{HS}}^2 ds + E_1 + E_2
\end{aligned}$$

$$= (2\pi c_G)^{-1}\int_0^\infty \left\|\int_{-\infty}^\infty [e^{its}\Gamma_r(it_0) + e^{-its}\Gamma_l(-it_0)]\beta(t)dt\right\|_{\mathrm{HS}}^2 ds + E_1 + E_2 + E_3$$

$$= (2\pi c_G)^{-1}\int_0^\infty \Big\{|\,\hat\beta(s)\,|^2\,\|\,\Gamma_r(it_0)\,\|_{\mathrm{HS}}^2 + |\,\hat\beta(-s)\,|^2\,\|\,\Gamma_l(-it_0)\,\|_{\mathrm{HS}}^2$$

$$+ 2\,\mathrm{Re}\,\hat\beta(s)\overline{\hat\beta(-s)}\,\mathrm{Tr}\,[\Gamma_r(it_0)\Gamma_l(-it_0)^*]\Big\}ds + E_1 + E_2 + E_3$$

$$= (2\pi c_G)^{-1}\,\|\,P\,\|_{\mathrm{HS}}^2\, c_\sigma(it_0)\int_{-\infty}^\infty |\,\hat\beta(s)\,|^2\, ds + E_1 + E_2 + E_3 + E_4$$

$$= c_G^{-1}\,\|\,P\,\|_{\mathrm{HS}}^2\, c_\sigma(it_0)\int_{-\infty}^\infty |\,\beta(t)\,|^2\, dt + E_1 + E_2 + E_3 + E_4\,.$$

Let β run through a sequence of functions β_n each with L^2 norm equal to 1 and with β_n vanishing off the set $|\,t - t_0\,| \leqq 1/n$. On one hand, Lemma 34 gives

$$\lim_{n\to\infty}\int_G \|\,\Phi(\beta_n, x)\,\|_{\mathrm{HS}}^2\, dx = (\dim H_D^\sigma)^2 p_\sigma(it_0)^{-1}\,,$$

and on the other hand, (11.11) gives

$$\lim_{n\to\infty}\int_G \|\,\Phi(\beta_n, x)\,\|_{\mathrm{HS}}^2\, dx = c_G^{-1}\,\|\,P\,\|_{\mathrm{HS}}^2\, c_\sigma(it_0) + \lim\,(E_1 + E_2 + E_3 + E_4)\,.$$

We obtain the conclusion of the theorem for $z = it_0$ if we show that each error term tends to 0. For general z, the theorem then follows by analytic continuation, since both sides in the conclusion are meromorphic functions of z.

For E_1, we have

$$E_1 = \int_0^\infty \Delta(s)(\|\,f(s) + r(s)\,\|_{\mathrm{HS}}^2 - \|\,f(s)\,\|_{\mathrm{HS}}^2)ds,$$

where

$$f(s) = e^{-s}\int_{-\infty}^\infty [e^{its}\Gamma_r(it) + e^{-its}\Gamma_l(-it)]\beta_n(t)dt$$

$$r(s) = \int_{-\infty}^\infty R(it, s)\beta_n(t)dt\,.$$

Then

$$|\,E_1\,| \leqq \int_0^\infty 2\,|\,\mathrm{Re}\,\mathrm{Tr}\,[f(s)r(s)^*]\,|\,\Delta(s)ds + \int_0^\infty \|\,r(s)\,\|_{\mathrm{HS}}^2\,\Delta(s)ds$$

$$\leqq 2\int_0^\infty \|\,f(s)\,\|_{\mathrm{HS}}\,\|\,r(s)\,\|_{\mathrm{HS}}\,\Delta(s)ds + \int_0^\infty \|\,r(s)\,\|_{\mathrm{HS}}^2\,\Delta(s)ds$$

$$\leqq 2\Big(\int_0^\infty \|\,f(s)\,\|_{\mathrm{HS}}^2\,\Delta(s)ds\Big)^{1/2}\Big(\int_0^\infty \|\,r(s)\,\|_{\mathrm{HS}}^2\,\Delta(s)ds\Big)^{1/2} + \int_0^\infty \|\,r(s)\,\|_{\mathrm{HS}}^2\,\Delta(s)ds\,.$$

Now $\Delta(s) \leqq ce^{2s}$, and it follows readily from the Plancherel theorem for the line and the local boundedness of $\Gamma_r(it)$ and $\Gamma_l(-it)$ away from $t = 0$

that $\int_0^\infty \|f(s)\|_{\mathrm{HS}}^2 \Delta(s)ds$ is bounded as $n \to \infty$. Consequently $E_1 \to 0$ if $\int_0^\infty \|r(s)\|_{\mathrm{HS}}^2 \Delta(s)ds \to 0$. Using the known estimate for $R(it, s)$, we have

$$\begin{aligned}
&\int_0^\infty \left\|\int_{-\infty}^\infty \beta_n(t)R(it, s)dt\right\|_{\mathrm{HS}}^2 \Delta(s)ds \\
&\leqq \int_0^\infty \left(\int_{-\infty}^\infty |\beta_n(t)|\, \|R(it, s)\|_{\mathrm{HS}}\, dt\right)^2 \Delta(s)ds \\
&\leqq \mathrm{Const} \int_0^\infty \left(\int_{-\infty}^\infty |\beta_n(t)|\, e^{-(1+\delta)s}dt\right)^2 \Delta(s)ds \\
&\leqq \mathrm{Const}\, \|\beta_n\|_1^2 \int_0^\infty e^{-2\delta s}ds \\
&\leqq \mathrm{Const}\, \|\beta_n\|_1^2 \,.
\end{aligned}$$

By the Schwarz inequality

(11.12) $$\|\beta_n\|_1^2 \leqq 2n^{-1} \|\beta_n\|_2^2 = 2/n \longrightarrow 0 \,,$$

and thus $E_1 \to 0$.

Next, E_1 is a finite linear combination of terms

$$\begin{aligned}
&\int_0^\infty e^{-cs}\left\|\int_{-\infty}^\infty [e^{its}\Gamma_r(it) + e^{-its}\Gamma_l(-it)]\beta_n(t)dt\right\|_{\mathrm{HS}}^2 ds \\
&\leqq \int_0^\infty e^{-cs}c_0 \|\beta_n\|_1^2\, ds = cc_0 \|\beta_n\|_1^2 \,,
\end{aligned}$$

where c_0 is a bound for $\|e^{its}\Gamma_r(it) + e^{-its}\Gamma_l(-it)\|$ on the common support ot the functions β_n. Thus $E_2 \to 0$ by (11.12).

Next, $|E_3|$ is $(2\pi c_G)^{-1}$ times

$$\begin{aligned}
&\Big|\int_0^\infty \{\|(\Gamma_r(it)\beta_n(t))\hat{\ }(s) + (\Gamma_l(it)\beta_n(-t))\hat{\ }(s)\|_{\mathrm{HS}}^2 \\
&\qquad - \|(\Gamma_r(it_0)\beta_n(t))\hat{\ }(s) + (\Gamma_l(-it_0)\beta_n(-t))\hat{\ }(s)\|_{\mathrm{HS}}^2\}ds\Big| \\
&\leqq \int_0^\infty \big|\|\mathrm{I}\|^2 - \|\mathrm{II}\|^2\big|\, ds \\
&\leqq \int_{-\infty}^\infty \big|\|\mathrm{I}\|^2 - \|\mathrm{II}\|^2\big|\, ds \\
&\leqq \int_{-\infty}^\infty \big|\|\mathrm{I}\| - \|\mathrm{II}\|\big|\, \{\|\mathrm{I}\| + \|\mathrm{II}\|\}ds \\
&\leqq \int_{-\infty}^\infty \|\mathrm{I} - \mathrm{II}\|\, \{\|\mathrm{I}\| + \|\mathrm{II}\|\}ds \\
&\leqq \left(\int_{-\infty}^\infty \|\mathrm{I} - \mathrm{II}\|^2 ds\right)^{1/2}\left\{\left(\int_{-\infty}^\infty \|\mathrm{I}\|^2 ds\right)^{1/2} + \left(\int_{-\infty}^\infty \|\mathrm{II}\|^2 ds\right)^{1/2}\right\}.
\end{aligned}$$

By the Plancherel theorem for the line and by the fact that $\beta_n(t)$ and $\beta_n(-t)$ have disjoint supports, we have

$$\int_{-\infty}^\infty \|\mathrm{I}\|^2 ds \leqq 2\pi\{\sup_{|t-t_0|\leqq 1/n} \|\Gamma_r(it)\|_{\mathrm{HS}}^2 + \sup_{|t-t_0|\leqq 1/n} \|\Gamma_l(-it)\|_{\mathrm{HS}}^2\} \leqq \mathrm{Const}\,,$$

and similarly for $\int_{-\infty}^{\infty} \|\, \mathrm{II} \,\|^2 ds$. Also similarly

$$\int_{-\infty}^{\infty} \|\, \mathrm{I} - \mathrm{II} \,\|^2 ds \leqq 2\pi\{\sup_{|t-t_0|\leq 1/n} \|\, \Gamma_r(it) - \Gamma_r(it_0) \|_{\mathrm{HS}}^2 + \sup_{|t-t_0|\leq 1/n} \|\, \Gamma_l(-it) - \Gamma_l(-it_0) \|_{\mathrm{HS}}^2\},$$

which tends to 0 by the continuity of Γ_r and Γ_l. Hence $E_3 \to 0$.

Finally Lemma 35 shows that E_4 is given by

$$E_4 = (2\pi c_G)^{-1} \int_0^{\infty} 2 \operatorname{Re} \{\hat{\beta}_n(s)\overline{\hat{\beta}_n(-s)} \operatorname{Tr} [\Gamma_r(it_0)\Gamma_l(-it_0)^*]\} ds \,.$$

Thus it is enough to show that

$$\int_0^{\infty} \hat{\beta}_n(s)\overline{\hat{\beta}_n(-s)} ds \longrightarrow 0 \,.$$

If H denotes the Hilbert transform, then

$$\begin{aligned} 2\int_0^{\infty} \hat{\beta}_n(s)\overline{\hat{\beta}_n(-s)} ds &= \int_{-\infty}^{\infty} \hat{\beta}_n(s)\overline{\hat{\beta}_n(-s)} ds + \int_{-\infty}^{\infty} (\operatorname{sgn}(s))\hat{\beta}_n(s)\overline{\hat{\beta}_n(-s)} ds \\ &= 2\pi(\beta_n * \overline{\beta_n})(0) - 2\pi i H(\beta_n * \overline{\beta_n})(0) \,. \end{aligned}$$

Now $(\beta_n * \overline{\beta_n})(0) = 0$ as soon as $1/n < t_0/2$. Also as soon as this happens,

$$H(\beta_n * \overline{\beta_n})(0) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{(\beta_n * \overline{\beta_n})(x)}{x} dx \,.$$

The support of $\beta_n * \overline{\beta_n}$ remains in a fixed compact set on which $1/x$ is bounded, and $\|\, \beta_n * \overline{\beta_n} \,\|_1 \to 0$ by (11.12) since $\|\, \beta_n * \overline{\beta_n} \,\|_1 \leqq \|\, \beta_n \,\|_1^2$. Thus $E_4 \to 0$. The proof of the theorem is complete.

12. Irreducibility theorem

Let $p_\sigma(it)dt$ be the contribution to the Plancherel formula (11.2) by the principal series representation $U(\sigma, it, g)$, $0 < t < \infty$. Combining Propositions 20 and 27 (vii) with Theorem 4, we obtain the final form of the irreducibility theorem for the principal series of a semisimple matrix Lie group of real-rank one.

THEOREM 5. *The principal series representation $U(\sigma, it, g)$ is reducible if and only if*

(i) *σ is equivalent with $w\sigma$,*

(ii) *$t = 0$, and*

(iii) $p_\sigma(0) \neq 0$.

Proof. By Proposition 20, it is enough to show that if σ is equivalent with $w\sigma$, then $\mathfrak{M}(\sigma(xw)^{-1}) = 0$ if and only if $p_\sigma(0) \neq 0$. By Proposition 27 (vi–vii), $\mathfrak{M}(\sigma(xw)^{-1}) = 0$ if and only if $c_\sigma(z)$ is regular at $z = 0$. By Theorem 4,

$c_\sigma(z)$ is a constant multiple of $p_\sigma(z)^{-1}$. Thus $c_\sigma(z)$ is regular at 0 if and only if $p_\sigma(z)$ is nonvanishing at 0, and Theorem 5 follows.

The decisive fact for applying Theorem 5 is that the Plancherel measure is known so explicitly for the groups in question. For the proof of the formula see [30], [13], and [17]. We shall describe the result of these papers, but before doing so, we introduce some notation. Let $\mathfrak{h}^-$ be a maximal abelian subspace of $\mathfrak{m}$; since M is compact, $\mathfrak{h}^-$ is a Cartan subalgebra of $\mathfrak{m}$. Then $\mathfrak{h} = \mathfrak{h}^- \oplus \mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$. Form roots of $\mathfrak{g}^{\mathbf{C}}$ with respect to $\mathfrak{h}^{\mathbf{C}}$. We introduce an ordering in the usual way in the dual of $i\mathfrak{h}^- \oplus \mathfrak{a}$ so that positive elements on $i\mathfrak{h}^-$ are bigger than positive elements on $\mathfrak{a}$. Let Q be the set of positive roots in this ordering. Extend ρ, defined on $\mathfrak{a}$, to a functional ρ^+ on $\mathfrak{h}$ by setting $\rho^+(\mathfrak{h}) = 0$. Let ρ^- be half the sum of the members of Q that vanish on $\mathfrak{a}$. Let us assume temporarily that G is simple.

Except when G is locally $SL(2, \mathbf{R})$, M is connected. (The case of $SL(2, \mathbf{R})$ is well-known and will be excluded from the discussion for most of the rest of the section.) When M is connected, the representation σ is determined by its highest weight Λ^- on $\mathfrak{h}^-$. Extend Λ^- to all of $\mathfrak{h}$ by setting $\Lambda^-(\mathfrak{a}) = 0$. Put

$$\omega(z) = \prod\nolimits_{\alpha \in Q} \langle \alpha, z\rho^+ + \Lambda^- + \rho^- \rangle .$$

Then there are three cases:

Case 1: rank $G >$ rank K. Then

$$p_\sigma(z) = c\omega(z)$$

for some constant c independent of z.

Case 2: rank $G =$ rank K and $q = \dim \mathfrak{g}_{2\alpha}$ is 0. Then

$$p_\sigma(z) = c\omega(z) \tanh(\pi piz/2)$$

or

$$p_\sigma(z) = c\omega(z) \coth(\pi piz/2) .$$

The decision between tanh and coth is done roughly as follows. There is a distinguished element γ of $\exp \mathfrak{h}^-$ of order at most 2 (see [30, p. 121]). Select H in $\mathfrak{h}^-$ such that $\gamma = \exp H$. Then $\exp\{(\Lambda^- + \rho^-)(H)\} = \pm 1$. The coth is used when the sign is $+$, and the tanh is used when the sign is $-$.

Case 3: rank $G =$ rank K and $q = \dim \mathfrak{g}_{2\alpha}$ is not 0. Then

$$p_\sigma(z) = c\omega(z) \tanh\left(\pi(p + 2q)iz/4\right)$$

or

$$p_\sigma(z) = c\omega(z) \coth\left(\pi(p + 2q)iz/4\right)$$

with the decision between tanh and coth made as in Case 2.

In Case 1, G is covered by Spin $(2n+1, 1)$, and we shall see in § 16 that every representation of the principal series is irreducible.

In Cases 2 and 3 there is always irreducibility at $z=0$ in the tanh case. But in the coth case, $\omega(z)$ will vanish either to order 1 or order 3 at $z=0$, and irreducibility at $z=0$ will occur exactly when the vanishing is to order 3. Order 1 is generic, and thus most representations in the coth case are reducible when $z=0$. Some of these facts had been noted earlier by the authors in the examples of [20].

For $SL(2, \mathbf{R})$ there are two representations of M, one trivial and one not. The Plancherel measures in the two cases are multiples of $z \tanh(\pi i z/2)$ and $z \coth(\pi i z/2)$. Irreducibility occurs at $z=0$ in the first case, and reducibility occurs in the second case. This fact was of course already well known.

13. Normalizing factor for intertwining operators

We return to the question of complementary series. We shall now find it necessary to normalize the intertwining operators $A(w, \sigma, z)$ by a meromorphic scalar factor $\gamma_\sigma(z)$ with certain properties.

LEMMA 36. *If $c(z)$ is a meromorphic function in the plane such that*

(i) $c(z) = \overline{c(-\bar{z})}$ *for all* z

(ii) $c(z) \geqq 0$ *on the imaginary axis,*

then there exists a meromorphic function $\gamma(z)$ in the plane such that

$$c(z) = \gamma(z)\overline{\gamma(-\bar{z})} \; . \tag{13.1}$$

The function $\gamma(z)$ can be chosen so that all its zeros are in the closed right half plane and all its poles are in the closed left half plane. If also $c(z)$ satisfies

(iii) $c(z) = c(-z)$ *for all* z,

then γ can be chosen to be real for real z.

Proof. In the special case that $c(z)$ is entire and non-vanishing, take $\gamma(z)$ to be one of the two versions of $c(z)^{1/2}$. Then

$$\gamma(z)\overline{\gamma(-\bar{z})} = c(z)^{1/2}\overline{c(-\bar{z})^{1/2}} = \pm c(z)^{1/2}[\overline{c(-\bar{z})}]^{1/2} = \pm[c(z)^{1/2}]^2 = \pm c(z)$$

with the sign continuous for all z and positive for $z=0$. Thus (13.1) holds in this case. If also (iii) holds, then (i) shows that $c(z)$ is real on the real axis. By (ii), $c(z) > 0$ on the real axis. Hence $\gamma(z)$ is real on the real axis.

In the general case, the zeros and poles of $c(z)$ are symmetric about the imaginary axis by (i), and the ones on the axis occur with even multiplicity by (ii). Construct $\gamma_0(z)$ as the quotient of two Weierstrass canonical products, so that $\gamma_0(z)$ has the zeros of $c(z)$ that lie in the open right half plane, $\gamma_0(z)$ has the poles of $c(z)$ that lie in the open left half plane, $\gamma_0(z)$ has zeros and poles of half the orders of those for $c(z)$ on the imaginary axis, $\gamma_0(z)$ has no other zeros and poles, and $\gamma_0(z)$ is real on the real axis if $c(z)$ is real on the real axis. Then we can apply the special case to $c(z)\gamma_0(z)^{-1}\overline{\gamma_0(-\bar{z})^{-1}}$ and obtain a function $\gamma_1(z)$. If we put $\gamma(z) = \gamma_0(z)\gamma_1(z)$, then $\gamma(z)$ has the required properties.

PROPOSITION 37. *It is possible to choose simultaneously for all irreducible unitary representations σ of M functions $\gamma_\sigma(z)$ meromorphic in the whole plane such that*

(i) *$\gamma_\sigma(z)$ depends only on the equivalence class of σ*

(ii) *$\gamma_\sigma(z)$ has no poles in the open right half plane and no zeros in the open left half plane.*

(iii) $\gamma_\sigma(z)\overline{\gamma_\sigma(-\bar{z})} = c_\sigma(z)$

(iv) $\gamma_{w\sigma}(z) = \overline{\gamma_\sigma(\bar{z})}$.

Proof. Since w^2 is in M, we can partition the set of equivalence classes of representations of M into a number of one-element sets and a number of two-element sets, with each set closed under the operation $[\sigma] \to w[\sigma]$. To each one-element set $\{\sigma\}$, we apply the full version of Lemma 36 to $c_\sigma(z)$. In Proposition 27, part (ii) shows that $c_\sigma(z)$ depends only on the class of σ, part (iii) shows that condition (iii) in the lemma is satisfied, (iv) shows (i), and (v) shows (ii). The lemma produces $\gamma_\sigma(z)$ with the four required properties.

If $\{\sigma, w\sigma\}$ is a two-element set, we apply just the first half of Lemma 36 to $c_\sigma(z)$. Again Proposition 27 (ii) shows $c_\sigma(z)$ depends only on the class of σ, (iv) shows (i) holds in the lemma, and (v) shows (ii). The lemma produces $\gamma_\sigma(z)$ satisfying properties (i), (ii), and (iii) of the present proposition. Define

$$\gamma_{w\sigma}(z) = \overline{\gamma_\sigma(\bar{z})} .$$

Then (iv) follows for σ, and (i) and (ii) follow for $w\sigma$. For (iii) with $w\sigma$, we have

$$\gamma_{w\sigma}(z)\overline{\gamma_{w\sigma}(-\bar{z})} = \overline{\gamma_\sigma(\bar{z})}\gamma_\sigma(-z) = c_\sigma(-z) = c_{w\sigma}(z) ,$$

the last step following from Proposition 27 (iii). Finally for (iv) with $w\sigma$, we have

$$\gamma_{w(w\sigma)}(z) = \gamma_{w^2\sigma}(z) = \gamma_\sigma(z) = \overline{\gamma_{w\sigma}(\bar{z})} ,$$

the middle equality following from (i) for σ, since w^2 is in M. The proof is complete.

Fix a choice of functions $\gamma_\sigma(z)$ as in Proposition 37, and define normalized intertwining operators by

$$\mathcal{A}(w, \sigma, z) = \gamma_\sigma(z)^{-1}A(w, \sigma, z) .$$

$\mathcal{A}(w, \sigma, z)$ maps $C^\infty(\sigma)$ into $C^\infty(w\sigma)$, and the z dependence is meromorphic.

PROPOSITION 38. *The operators $\mathcal{A}(w, \sigma, z)$ have the following properties*

(i) $\mathcal{A}(w^{-1}, w\sigma, -z)\mathcal{A}(w, \sigma, z) = I$

(ii) $\mathcal{A}(w, \sigma, z)^* = \mathcal{A}(w^{-1}, w\sigma, \bar{z})$

(iii) $\mathcal{A}(w, \sigma, z)$ *is unitary for imaginary* z

(iv) *If σ is equivalent with $w\sigma$, so that $\sigma(w)$ is defined, then*

$$[\sigma(w)\mathcal{A}(w, \sigma, z)]^* = \sigma(w)\mathcal{A}(w, \sigma, z)$$

for z real.

Proof. (i) follows from the definition of $c_\sigma(z)$ and from the equality

$$\gamma_{w\sigma}(-z)\gamma_\sigma(z) = \overline{\gamma_\sigma(-\bar{z})}\gamma_\sigma(z) = c_\sigma(z) ,$$

which follows from parts (iii) and (iv) of Proposition 37. Next, (ii) follows from Corollary 2 of Theorem 3 and from Proposition 37 (iv), and (iii) is a consequence of (i) and (ii). Finally (iv) follows from Proposition 25 and from Proposition 37 (iv).

14. Existence of complementary series

If σ is an irreducible representation of M and $\lambda = \mu^{z/2}$ is a character of A, we have made the definition (in § 9) that $U(\sigma, z, x)$ is in the complementary series if there is a positive definite continuous inner product on $C^\infty(\sigma) \times C^\infty(\sigma)$ with respect to which $U(\sigma, z, x)$ is unitary. If there is a nontrivial positive semidefinite continuous inner product of this kind, then $U(\sigma, z, x)$ is in the quasi-complementary series. In considering which σ and z lead to such representations, there is substantially no loss of generality in restricting attention to $\operatorname{Re}(z) > 0$, because, except on an isolated set, $\mathcal{A}(w, \sigma, z)$ exhibits $U(\sigma, z, x)$ and $U(w\sigma, -z, x)$ as equivalent representations on $C^\infty(\sigma)$ and because, as we shall see, the z for which $U(\sigma, z, x)$ is in the quasi-complementary series form a closed set.

We assume then that $\operatorname{Re}(z) > 0$. By Proposition 25, $U(\sigma, z, x)$ is not in the quasi-complementary series unless σ is equivalent with $w\sigma$ and z is real. There is a further necessary condition. As will be implicit in the proof of Theorem 6, there is no quasi-complementary series near $z = 0$ unless the Plancherel measure satisfies $p_\sigma(0) = 0$ (see §§ 11–12 for a description of p_σ).[3] We shall have the condition $p_\sigma(0) = 0$ as an assumption in our existence theorem.

Recall that $p = \dim \mathfrak{g}_\alpha$ and $q = \dim \mathfrak{g}_{2\alpha}$. By Theorem 3, the only possible poles of the intertwining operator $A(w, \sigma, z)$ are simple poles at non-negative

[3] In the cases of Lorentz groups and their two-fold covers and in the case of the hermitian Lorentz groups $SU(n, 1)$, there is no quasi-complementary series at all for $\operatorname{Re}(z) > 0$ unless $p_\sigma(0) = 0$.

integral multiples of $-(p+2q)^{-1}$. (However, not all of these multiples do give rise to poles; this point will be discussed further in §15.) Also from the explicit formulas for the Plancherel measure $p_\sigma(z)$ in § 12, we see that $p_\sigma(z)$ has at most simple poles and poles can occur only at integral multiples of $(p+2q)^{-1}$.

LEMMA 39. *The only zeros of $p_\sigma(z)$ other than at $z=0$ are simple and can occur only at integral multiples of $(p+2q)^{-1}$.*

Proof. By Theorem 4, zeros of $p_\sigma(z)$ come from poles of $c_\sigma(z)$, which in turn come only from poles of $A(w, \sigma, z)$ or $A(w^{-1}, w\sigma, -z)$. The lemma follows from Theorem 3.

Suppose σ is equivalent with $w\sigma$. Then $\sigma(w)$ exists, and the operator $\sigma(w)\mathfrak{A}(w,\sigma,z)$ maps H^σ into itself (see § 13 for definition of $\mathfrak{A}$). Since $A(w,\sigma,z)$ commutes with right translation by K, $\sigma(w)\mathfrak{A}(w,\sigma,z)$ maps each H_D^σ into itself. Now H_D^σ is finite-dimensional since $H^\sigma \subseteq L^2(K)\otimes E_\sigma$. Hence there is a system of orthogonal finite-dimensional invariant subspaces for $\sigma(w)\mathfrak{A}(w,\sigma,z)$ whose sum is dense in $C^\infty(\sigma)$. This fact will be of crucial importance in what follows. The next lemma is elementary.

LEMMA 40. *Let $F(x)$ be a continuous function from a space X to the vector space of $n \times n$ hermitian matrices such that $F(x_0)$ is positive definite for some x_0 and such that* det $F(x)$ *is non-vanishing for x in a dense connected subset Y. Then $F(x)$ is positive definite for x in Y, and $F(x)$ is positive semi-definite for all x in X.*

If the Plancherel measure for σ satisfies $p_\sigma(0)=0$, we define the *critical abscissa* z_c for σ to be the least positive integral multiple of $(p+2q)^{-1}$ such that either of these conditions holds:*

(i) $p_\sigma(z_c)=\infty$

(ii) $A(w, \sigma, z)$ does have a pole at $z=-z_c$, and $p_\sigma(z_c)\neq 0$.

THEOREM 6. *Let σ be an irreducible unitary representation of M such that the Plancherel measure satisfies $p_\sigma(0)=0$, and let z_c be the critical abscissa. Then $U(\sigma, \mu^{z/2}, x)$ is in the complementary series for $0<z<z_c$ and is in the quasi-complementary series but not the complementary series for $z=z_c$.*

Remark. It can happen that the unitary representation that arises at z_c is the trivial representation of G.

Proof. By Corollary 3 of Theorem 3, by Proposition 27 (vii), and by

* An examination of the argument below gives the following equivalent characterization of the critical abscissa z_c: $A(w,\sigma,z)$ has its only singularities at integral multiples of $(p+2q)^{-1}$, and z_c is the least positive integral multiple $z=n(p+2q)^{-1}$ such that either $A(w,\sigma,z)$ or $A(w,\sigma,-z)$ has a pole at z.

Theorem 4, σ is equivalent with $w\sigma$. Form the normalizing factors $\gamma_\sigma(z)$ and the normalized operators $\mathcal{A}(w, \sigma, z)$ of § 13. Since σ is equivalent with $w\sigma$, $\sigma(w)$ exists. From the identities

$$U(w\sigma, -z, x)A(w, \sigma, z) = A(w, \sigma, z)\,U(\sigma, z, x)$$

and

$$\sigma(w)^{-1}U(\sigma, -z, x)\sigma(w) = U(w\sigma, -z, x)$$

we obtain

$$U(\sigma, -z, x)[\sigma(w)\mathcal{A}(w, \sigma, z)] = [\sigma(w)\mathcal{A}(w, \sigma, z)]\,U(\sigma, z, x)\ . \tag{14.1}$$

Now $\sigma(w)\mathcal{A}(w, \sigma, 0)$ is unitary by Proposition 38 (iii). Since $p_\sigma(0) = 0$, Theorem 5 shows that $U(\sigma, 0, x)$ is irreducible. By (14.1), $\sigma(w)\mathcal{A}(w, \sigma, 0)$ is scalar. By Proposition 38 (iv), the scalar is real. Hence $\sigma(w)\mathcal{A}(w, \sigma, 0) = \pm I$. Since the definition of $\sigma(w)$ involves an ambiguous sign, we may suppose that

$$\sigma(w)\mathcal{A}(w, \sigma, 0) = I\ . \tag{14.2}$$

Proposition 38(iv) shows that $\sigma(w)\mathcal{A}(w, \sigma, z)$ is hermitian for $0 \leqq z \leqq z_c$. In view of Proposition 25 and the orthogonality of the H_D^σ, it is enough in proving the theorem to show that each $\sigma(w)\mathcal{A}(w, \sigma, z)|_{H_D^\sigma}$ is finite and positive definite for $0 \leqq z < z_c$ and to show that $\sigma(w)A(w, \sigma, z)$ is positive semidefinite at $z = z_c$ for all D and not definite for some D.

For $0 < z < z_c$, we know that $p_\sigma(z)$ is finite and that $p_\sigma(z) = 0$ if $A(w, \sigma, -z)$ has a pole. (Lemma 39 and the remarks before it account for $z \neq n(p + 2q)^{-1}$, and the definition of z_c accounts for the remaining values of z.) Theorem 4 and the definition of $\gamma_\sigma(z)$ show, for $0 < z < z_c$, that $\gamma_\sigma(z) \neq 0$ and that $\gamma_\sigma(z) = \infty$ if $A(w, \sigma, -z)$ has a pole. That is, $\sigma(w)\mathcal{A}(w, \sigma, z) = \gamma_\sigma(z)^{-1}\sigma(w)A(w, \sigma, z)$ is regular for $|z| < z_c$, hence continuous.

Also $\sigma(w)\mathcal{A}(w, \sigma, z)$ is nonsingular on H_D^σ for $|z| < z_c$ because

$$\begin{aligned}[\sigma(w)\mathcal{A}(w, \sigma, -z)][\sigma(w)\mathcal{A}(w, \sigma, z)] &= [\sigma(w^{-1})\mathcal{A}(w^{-1}, \sigma, -z)\sigma(w)]\mathcal{A}(w, \sigma, z)\\ &= \mathcal{A}(w^{-1}, w\sigma, -z)\mathcal{A}(w, \sigma, z) = I\ ,\end{aligned} \tag{14.3}$$

by Proposition 38(i).

On the other hand, $\sigma(w)\mathcal{A}(w, \sigma, 0)$ is positive definite on each H_D^σ by (14.2). Applying Lemma 40, we find that $\sigma(w)\mathcal{A}(w, \sigma, z)|_{H_D^\sigma}$ is positive definite for $0 \leqq z < z_c$. For $z_c - \varepsilon \leqq z \leqq z_c$, $\sigma(w)A(w, \sigma, z)|_{H_D^\sigma}$ is continuous, and it is nonsingular and definite for $z_c - \varepsilon \leqq z < z_c$. By the lemma, it is semidefinite at z_c.

Finally we are to show that $\sigma(w)A(w, \sigma, z_c)|_{H_D^\sigma}$ fails to be definite for some D. There are two possibilities. If $p_\sigma(z_c) = \infty$, then $c_\sigma(z_c) = 0$ by Theorem 4 and $c_\sigma(-z_c) = 0$ by Proposition 27(iv). Now

$$A(w, \sigma, z_c)A(w^{-1}, w\sigma, -z_c) = c_\sigma(-z_c)I = 0\ . \tag{14.4}$$

Also $A(w^{-1}, w\sigma, -z_c)$ is not identically 0, by the proof of Theorem 3. Then $A(w^{-1}, w\sigma, -z_c)h \neq 0$ for some h, and we may suppose that h is in some H_D^σ since $\sum H_D^\sigma$ is dense in $C^\infty(\sigma)$. If $f = A(w^{-1}, w\sigma, -z_c)h$, then f is in H_D^σ, and (14.4) shows that $A(w, \sigma, z_c)f = 0$. The second possibility is that $A(w, \sigma, -z_c)$ has a pole and $p_\sigma(z_c) \neq 0$, and the proof is completely analogous.

15. The critical abscissa

There are two questions suggested by Theorem 6. At exactly which multiples of $(p + 2q)^{-1}$ does $A(w, \sigma, z)$ have a pole? To what extent does the complementary series extend to the right of the critical abscissa?

We shall give some partial results about the poles of $A(w, \sigma, z)$ for general G and then examine the groups $SO(n, 1)$ and $SU(n, 1)$ in more detail. We begin by defining the element γ of §12 that plays an important role in the Plancherel formula. With α as the smaller positive restricted root, let β be the larger positive restricted root (α or 2α). Let Y be in $\mathfrak{g}_\beta$. Then

$$\mathbf{R}(\theta Y) + \mathfrak{a} + \mathbf{R}Y$$

is a subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}(2, \mathbf{R})$. Choose H in $\mathfrak{a}$ such that $\beta(H) = 2$, and let γ be the element of $G^{\mathbf{C}}$ defined by

$$\gamma = \exp i\pi H . \tag{15.1}$$

Then γ is in G because it is the image in G of the 2×2 matrix $(-1, 0; 0, -1)$ under the homomorphism of $SL(2, \mathbf{R})$ into G corresponding to the Lie algebra inclusion $\mathfrak{sl}(2, \mathbf{R}) \subseteq \mathfrak{g}$.

LEMMA 41. *If* $\mathfrak{g}_{2\alpha} = 0$, *then* γ *is in the center of* G. *If* $\mathfrak{g}_{2\alpha} \neq 0$, *then* γ *is in the center of* M, $\mathrm{Ad}(\gamma)$ *operates as* -1 *on* $\mathfrak{g}_\alpha$ *and* $\mathfrak{g}_{-\alpha}$, *and* $\mathrm{Ad}(\gamma)$ *operates as* 1 *on* $\mathfrak{g}_{2\alpha}$ *and* $\mathfrak{g}_{-2\alpha}$.

Proof. It is clear that

$$\mathrm{Ad}(\gamma)X = e^{\pi i k \beta(H)}X = e^{2\pi i k}X \tag{15.2}$$

for X in $\mathfrak{g}_{k\beta}$. If $\mathfrak{g}_{2\alpha} = 0$, then $\beta = \alpha$ and $\mathfrak{g} = \mathfrak{g}_{-\alpha} + \mathfrak{g}_0 + \mathfrak{g}_\alpha$. Thus (15.2) says $\mathrm{Ad}(\gamma)$ is the identity, and γ is central in G. If $\mathfrak{g}_{2\alpha} \neq 0$, then $\beta = 2\alpha$ and $\mathrm{Ad}(\gamma)$ is -1 on $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ and is 1 on $\mathfrak{g}_{2\alpha}$, $\mathfrak{g}_0$, and $\mathfrak{g}_{-2\alpha}$. Since γ is in K and $\mathrm{Ad}(\gamma) = 1$ on $\mathfrak{g}_0$, γ is in M and commutes with $\mathfrak{m}$. Since M is connected whenever $\mathfrak{g}_{2\alpha} \neq 0$, γ is in the center of M.

LEMMA 42. *If* σ *has the property that*

$$\sigma\big(\exp(-X + Y)w\big) = \sigma\big(\exp(X + Y)w\big)$$

for all X *in* $\mathfrak{g}_{-\alpha}$ *and* Y *in* $\mathfrak{g}_{-2\alpha}$, *then* $A(w, \sigma, z)$ *has no poles at odd multiples of* $(p + 2q)^{-1}$.

Proof. There will be such poles if, when j is odd, the function

$$h_j(z, y) = \mu^{(1/2)j/(p+2q)}(yw)\sigma^{-1}(yw)g_j(z, y) \tag{15.3}$$

in the proof of Theorem 3 has mean value 0. Here g_j is a polynomial α-homogeneous of degree j, thus odd in the $\mathfrak{g}_\alpha$ variables. The hypothesis is that $\sigma^{-1}(yw)$ is even in the $\mathfrak{g}_\alpha$ variables, and it follows readily from Proposition 3 that $h_j(z, y)$ has mean value 0.

PROPOSITION 43. *If* $\mathfrak{g}_{2\alpha} \neq 0$, *then* $A(w, \sigma, z)$ *has no poles at odd multiples of* $(p + 2q)^{-1}$.

Proof. Let γ be as in (15.1) and let $v = \exp(X + Y)$ as in Lemma 42. Notice that $\gamma^2 = 1$. Then by (15.1),

$$w^{-1}\gamma^{-1}w = w^{-1}\exp(-i\pi H)w = \exp i\pi H = \gamma = \gamma^{-1}\ .$$

Since, by Lemma 41, γ is in the center of M, we conclude

$$m(\gamma v\gamma^{-1}w) = \gamma m(vw)w^{-1}\gamma^{-1}w = \gamma m(vw)\gamma^{-1} = m(vw)\ .$$

The proposition therefore follows from Lemma 42.

We shall now consider the poles of $A(w, \sigma, z)$ for special groups. Further information about special cases will be given in §16. For the Lorentz groups $SO_e(n, 1)$, V is isomorphic with $\mathbf{R}^{n-1}$ and conjugation by a member of A is just a dilation in the ordinary sense. The norm function is a scalar multiple of $\|x\|^{n-1}$, and M is the rotation group $SO(n - 1)$. For the standard representation σ_0 of M, $\sigma_0(xw)$ is of the form $\|x\|^{-2}P(x)$ for a matrix of homogeneous polynomials of degree 2. Since σ_0 separates points of M and is real, it follows that $\sigma(xw)$ is an even function of x for every σ. By Lemma 42, $A(w, \sigma, z)$ can have poles at only the even multiples of $-1/p$. (Note $q = 0$ here.) If $p_\sigma(0) = 0$, then $\sigma^{-1}(yw)$ has mean value different from 0. Thus in (15.3), we can choose as a polynomial

$$g_j(z, y) = \|y\|^j$$

if j is even and obtain from it a pole of the operator $A(w, \sigma, z)$ at $z = -j/p$. We can summarize as follows.

PROPOSITION 44. *For the Lorentz groups* $SO_e(n, 1)$, *if* $p_\sigma(0) = 0$, *then* $A(w, \sigma, z)$ *has poles at every nonpositive integral multiple of* $2/p$ *and only there.*

In the case of the hermitian Lorentz groups $SU(n, 1)$, the Lie algebra of V is isomorphic with the Lie algebra of matrices

$$\begin{pmatrix} 0 & 0 & 0 \\ X & 0 & 0 \\ \frac{1}{2}iY & X^* & 0 \end{pmatrix},$$

where X is a column vector in $\mathbf{C}^{n-1}$ and Y is in $\mathbf{R}$. The norm function is a multiple of

$$(\|X\|^4 + |Y|^2)^{n/2} .$$

By Proposition 43, $A(w, \sigma, z)$ has poles only at even multiples of $(p + 2)^{-1}$. (Note $q=1$ here.) On the other hand, we can argue with powers of $\|X\|^4+|Y|^2$, just as in the Lorentz case, to show that if there is a pole at $-z$, then there is another pole at $-z - 4(p + 2)^{-1}$. And, of course if $p_\sigma(0) = 0$, then there is a pole at 0. In addition, it is possible to relate the behavior of $A(w, \sigma, z)$ at $z = -2(p + 2)^{-1}$ with the behavior of two operators $A(w, \sigma_1, z)$ and $A(w, \sigma_2, z)$ at $z = 0$; in this way the existence or non-existence of a pole of $A(w, \sigma, z)$ at $z = -2(p + 2)^{-1}$ again can be obtained from the Plancherel measure by inspection. This relationship is described as follows: M is the group of matrices

$$\begin{pmatrix} e^{i\theta} & & \\ & \omega & \\ & & e^{i\theta} \end{pmatrix}$$

with ω in the unitary group $U(n - 1)$ and with $e^{2i\theta} \det \omega = 1$. Let σ_0 be the one-dimensional representation $\sigma_0(e^{i\theta}, \omega, e^{i\theta}) = e^{i\theta}$, and define $\sigma_1 = \sigma \otimes \sigma_0$ and $\sigma_2 = \sigma \otimes \bar{\sigma}_0$. Then $A(w, \sigma, z)$ has no pole at $z = -2(p + 2)^{-1}$ if and only if $p_{\sigma_1}(0) \neq 0$ and $p_{\sigma_2}(0) \neq 0$. We omit the simple proof. This completes our discussion of the poles of $A(w, \sigma, z)$.

For the second question about the critical abscissa, one may guess that there is no complementary series to the right of z_c. As evidence for such a conjecture, we have the following proposition.

PROPOSITION 45. *Let $G = SO(n, 1)$ or $SU(n, 1)$. If $p_\sigma(0) \neq 0$, then $U(\sigma, \mu^{z/2}, x)$ is not in the quasi-complementary series for any z with* Re $z > 0$. *If $p_\sigma(0) = 0$, then $U(\sigma, \mu^{z/2}, x)$ is not in the quasi-complementary series for any $z > z_c$.*

The proof begins with two lemmas, the first of which is due to Kostant [24]. (See also [21], which shows how Lemma 46 is implied by our present Theorem 6 and Lemma 42.)

LEMMA 46. *Let $G = SO(n, 1)$ or $SU(n, 1)$. If σ is the trivial representation, then $U(\sigma, \mu^{z/2}, x)$ is in the complementary series for $0 < z < 1$.*

For the purposes of the second lemma, recall that a function F in $L^1(K, \operatorname{Hom} E_\sigma)$ is positive definite if

$$\iint (F(x^{-1}y)\varphi(x), \varphi(y))_{E_\sigma} dxdy \geqq 0 \tag{15.4}$$

for all φ in $C(K, E_\sigma)$. Let $\{e_k\}$ be an orthonormal basis of E_σ. In case F is continuous, (15.4) is equivalent with the condition that, for any finite set $\{x_i\}$ in K, the matrix

$$C_{ik,jl} = \big(F(x_i^{-1}x_j)e_k,\, e_l\big)$$

is positive semidefinite.

LEMMA 47. *Let $1 < p < \infty$ and let $p' = p/(p-1)$. If F in $L^p(K, \operatorname{Hom} E_\sigma)$ and f in $L^{p'}(K, \mathbf{C})$ are positive definite, then fF is positive definite.*

Proof. For ψ in $C(K, \mathbf{C})$, define

$$F_\psi(x) = \iint \psi(s)\overline{\psi(t)}F(s^{-1}xt)\,ds\,dt\ , \tag{15.5}$$

and define f_θ similarly for θ in $C(K, \mathbf{C})$. Then f_θ and F_ψ are continuous positive definite. If $\{x_i\}$ is given, it follows that the matrices $\big(F_\psi(x_i^{-1}x_j)e_k,\, e_l\big)$ and $f_\theta(x_i^{-1}x_j)$ are positive semidefinite hermitian. Then we can find a unitary matrix U_{ij} such that

$$f_\theta(x_i^{-1}x_j) = \sum\nolimits_m U_{im}\bar{U}_{jm}D_{mm}$$

for suitable numbers $D_{mm} \geqq 0$. If constants c_{ik} are given, the result is that

$$\begin{aligned}&\sum c_{ik}\bar{c}_{jl}\big(f_\theta F_\psi(x_i^{-1}x_j)e_k,\, e_l\big)\\ &\quad= \sum\nolimits_m D_{mm}\{\sum c_{ik}U_{im}\overline{c_{jl}U_{jm}}\big(F_\psi(x_i^{-1}x_j)e_k,\, e_l\big)\} \geqq 0\ .\end{aligned}$$

Therefore $f_\theta F_\psi$ is continuous and positive definite. Passing to the limit in ψ and then θ, we see that fF is positive definite.

Proof of Proposition 45. For $\operatorname{Re}(z) > 0$, the operator $\sigma(w)A(w, \sigma, z)$ extends to a bounded integral operator on $L^2(K, E_\sigma)$ with values in H^σ. Since the operator is hermitian, it is 0 on the orthogonal complement of H^σ in $L^2(K, E_\sigma)$. Therefore $\sigma(w)A(w, \sigma, z)$ is semidefinite on $C^\infty(\sigma)$ if and only if it is semidefinite on $L^2(K, E_\sigma)$. In view of Proposition 25, this means that $U(\sigma, z, x)$ is in the quasi-complementary series if and only if

$$\mu^{(1/2)(1-z)}(kw)\sigma(w)\sigma^{-1}(kw) \tag{15.6}$$

is a positive definite function.

Let $U(\sigma, z_1, x)$ be in the quasi-complementary series and suppose $\max\{0, z_1 - 1\} < z_2 < z_1$. Write

$$\begin{aligned}&\mu^{(1/2)(1-z_2)}(kw)\sigma(w)\sigma^{-1}(kw)\\ &\quad= \{\mu^{(1/2)(1-(1-z_1+z_2))}(kw)\}\{\mu^{(1/2)(1-z_1)}(kw)\sigma(w)\sigma^{-1}(kw)\}\ .\end{aligned}$$

The first factor is positive definite by (15.6) and Lemma 46, and the second factor is positive definite by (15.6). By Lemma 47, the product is positive definite. Therefore the set of $z > 0$ such that $U(\sigma, z, x)$ is in the quasi-complementary series forms an interval with 0 as left endpoint.

If there is any quasi-complementary series for $z > 0$, then it follows that

$\sigma(w)\mathfrak{A}(w, \sigma, 0)$, which is known to be unitary, is also semidefinite. Then it must be scalar, and $p_\sigma(0)$ must be 0.

If $p_\sigma(0) = 0$ and $U(\sigma, z, x)$ is in the quasi-complementary series for some $z > z_c$, then it follows from the above discussion and from the nonsingularity of $\mathfrak{A}(w, \sigma, z)$ at points $z \neq n(p + 2q)^{-1}$ that $U(\sigma, z_c + \varepsilon, x)$ is actually in the complementary series for some $\varepsilon > 0$.

Let

$$F(k) = \mu^{(1/2)(1-z_c-\varepsilon)}(kw)\sigma(w)\sigma^{-1}(kw)$$
$$f(k) = \mu^{(1/2)(1-(1-\varepsilon))}(kw) ,$$

and define F_ψ and f_θ as in (15.5). The function f_1 is constant and not 0; call the constant γ. Fix an irreducible representation D of K such that $H_D^\sigma \neq 0$, and let φ be a nonzero member of H_D^σ. If χ is the product of the degree of D and the character of D, then $\chi*\varphi = \varphi$. Since $U(\sigma, z_c + \varepsilon, x)$ is in the complementary series,

$$\begin{aligned} &\iint (f_1(x)F_{\bar\chi}(x)\varphi(xy), \varphi(y))_{E_\sigma} dx\,dy \\ (15.7) \qquad &= \gamma \iint (F(xy^{-1})(\chi*\varphi)(x), (\chi*\varphi)(y))_{E_\sigma} dx\,dy \\ &= \gamma\big(\sigma(w)A(w, \sigma, z_c + \varepsilon)\varphi, \varphi\big) > 0 . \end{aligned}$$

Since $U(\sigma, z_c, x)$ is in the quasi-complementary series,

$$(15.8) \qquad \iint (f_\theta(x)F_\psi(x)\varphi(xy), \varphi(y))_{E_\sigma} dx\,dy \geqq 0 .$$

If $\theta*1 = 0$ and $\psi*\bar\chi = 0$, then $f_{\theta+1} = f_\theta + f_1$ and $F_{\psi+\bar\chi} = F_\psi + F_{\bar\chi}$. Adding (15.7) and (15.8) and changing notation, we conclude that

$$\begin{aligned} &\iint (f_\theta(x)F_\psi(x)\varphi(xy), \varphi(y))_{E_\sigma} dx\,dy \\ &\geqq \gamma\big(\sigma(w)A(w, \sigma, z_c + \varepsilon)\varphi, \varphi\big) > 0 \end{aligned}$$

whenever the projection of θ in the space of constant functions is 1 and the projection of ψ in the space of $\bar D$ is $\bar\chi$. Passing to the limit as θ and ψ peak at the identity, we see that

$$\big(\sigma(w)A(w, \sigma, z_c)\varphi, \varphi\big) > 0 .$$

That is, $U(\sigma, z_c, x)$ is in the complementary series, in contradiction with Theorem 6. This completes the proof.

16. Examples of intertwining operators

In some of the groups we have been considering, the intertwining operators are known in more classical settings. For instance, in $SL(2, \mathbf{R})$, the operator in $L^2(V)$ that exhibits the reducibility of the one member of the principal series that is reducibleis the Hilbert transform on the line. In this section, we shall give the form of the kernels of the intertwining operators for the other classical groups[4] of real-rank one, and we shall examine special properties of these groups.

1) *Unimodular group* $SL(2, \mathbf{C})$. This is the group of 2-by-2 complex matrices of determinant one. We can choose

$$M = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \quad A = \begin{pmatrix} e^{t} & 0 \\ 0 & e^{-t} \end{pmatrix}, \quad N = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix},$$

with θ and t real and with x and y complex. We shall use the letter y ambiguously to mean either the complex number y or the matrix $(1, 0; y, 1)$. Let $w = (0, 1; -1, 0)$. Then an easy calculation gives

$$a(yw) = \begin{pmatrix} |y|^{-1} & 0 \\ 0 & |y| \end{pmatrix}, \quad m(yw) = \begin{pmatrix} [y]^{-1} & 0 \\ 0 & [y] \end{pmatrix},$$

where $[y] = y/|y|$. Thus $\mu^{-1/2}(yw) = |y|^2$. Let

$$\sigma_n\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} = e^{in\theta}.$$

Then $\sigma_n(yw) = [y]^{-n}$. Thus the kernel of $A(w, \sigma_n, z)$ is

$$|y|^{-2(1-z)}[y]^n.$$

For this group, $p = 2$ and $q = 0$. By (15.3), $A(w, \sigma_n, z)$ has a pole at $-j/2$ if and only if there is some polynomial $P_j(y, \bar{y})$ homogeneous of degree j such that

$$|y|^{-j}[y]^n P_j(y, \bar{y}) \tag{16.1}$$

has mean value different from 0.

PROPOSITION 48. *In* $SL(2, \mathbf{C})$, $A(w, \sigma_n, z)$ *has no poles for* $|z| < |n|/2$. *The normalized operator* $\mathcal{A}(w, \sigma_n, z)$ *has no poles and is nonsingular for* $|z| < (|n| + 2)/2$.

Proof. The first statement follows from the fact that (16.1) has mean value 0 if $j < |n|$, by an integration in polar coordinates. Next, except for a constant factor, the Plancherel measure is

$$p_{\sigma_n}(z) = \frac{1}{4}n^2 - z^2.$$

Therefore $\gamma_{\sigma_n}(z)$ has no zeros, and its first pole is at $z = -|n|/2$. This pole

[4] There is only one non-classical simple Lie group of real-rank one, and it is a real form of F_4. It has rank G = rank K and $\mathfrak{g}_{2\alpha} \neq 0$.

cancels the first pole of $A(w, \sigma_n, z)$, and thus $\mathcal{A}(w, \sigma_n, z)$ has no poles until the second pole of $A(w, \sigma_n, z)$, which does not occur before $-(|n| + 2)/2$, by (16.1). Thus $\mathcal{A}(w, \sigma_n, z)$ is regular for $|z| < (|n| + 2)/2$. Since

$$\mathcal{A}(w^{-1}, \sigma_{-n}, -z)\mathcal{A}(w, \sigma_n, z) = I ,$$

$\mathcal{A}(w, \sigma_n, z)$ is nonsingular for $|z| < (|n| + 2)/2$.

2) *Isometry groups of quadratic forms.* Let **K** be the reals, the complex numbers, or the quaternions **H**. Let G be the identity component of the group of all automorphisms of $\mathbf{K}^{n+1}$ preserving the hermitian quadratic form $|x_1|^2 + \cdots + |x_n|^2 - |x_{n+1}|^2$. In case $\mathbf{K} = \mathbf{C}$, we consider only automorphisms of determinant 1 (the condition determinant 1 being automatic in the other two cases). We suppose that $n \geqq 3$ if $\mathbf{K} = \mathbf{R}$ and $n \geqq 2$ if $\mathbf{K} = \mathbf{C}$ or **H**. A Cartan decomposition of the Lie algebra of G is $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ with

$$\mathfrak{k} = \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix}, \quad \mathfrak{p} = \begin{pmatrix} 0 & Y \\ \bar{Y}^t & 0 \end{pmatrix},$$

where X_1 is an n-by-n skew hermitian matrix with entries in **K**, X_2 is a skew member of **K**, $X_2 + \mathrm{Tr}(X_1) = 0$ if $\mathbf{K} = \mathbf{C}$, and Y is a column vector in $\mathbf{K}^n$. In each case the Cartan involution θ is negative conjugate transpose. A maximal abelian subspace of $\mathfrak{p}$ is obtained by letting Y be an arbitrary real number in the first component and be 0 in the other components. Let us write matrices symbolically in 9 blocks, corresponding to a partition of the $n + 1$ coordinates of $\mathbf{K}^{n+1}$ into blocks of 1, $n - 1$, and 1. Instead of considering $\mathfrak{g}$, we consider $g\mathfrak{g}g^{-1}$, where

$$g = \begin{pmatrix} 2^{-1/2} & 0 & 2^{-1/2} \\ 0 & I & 0 \\ -2^{-1/2} & 0 & 2^{-1/2} \end{pmatrix}.$$

Then A and M become

$$A = \begin{pmatrix} e^t & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & e^{-t} \end{pmatrix}, \quad M = \begin{pmatrix} u & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & u \end{pmatrix},$$

with t real and M described as follows: If $\mathbf{K} = \mathbf{R}$, $u = 1$ and ω is in $SO(n - 1)$; if $\mathbf{K} = \mathbf{C}$, $u = e^{i\theta}$ and ω is unitary and $e^{2i\theta} \det \omega = 1$; if $\mathbf{K} = \mathbf{H}$, u is a unit quaternion and ω is in the quaternionic unitary group $Sp(n - 1)$. With an appropriate choice of ordering on the restricted roots, we obtain

$$\mathfrak{g}_\alpha = \begin{pmatrix} 0 & \bar{X}^t & 0 \\ 0 & 0 & X \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathfrak{g}_{-\alpha} = \begin{pmatrix} 0 & 0 & 0 \\ X & 0 & 0 \\ 0 & \bar{X}^t & 0 \end{pmatrix} \quad \text{with } X \text{ in } \mathbf{K}^{n-1} ,$$

$$\mathfrak{g}_{2\alpha} = \begin{pmatrix} 0 & 0 & \frac{1}{2}Y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathfrak{g}_{-2\alpha} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{2}Y & 0 & 0 \end{pmatrix} \quad \text{with } Y \text{ in } \mathbf{K},\ \bar{Y} = -Y .$$

If $\mathbf{K} = \mathbf{R}$, notice that $\mathfrak{g}_{2\alpha} = \mathfrak{g}_{-2\alpha} = 0$. For w, we can choose

$$w = \begin{pmatrix} 0 & 0 & 1 \\ 0 & b & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & i \\ 0 & I & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & I & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

in the cases $\mathbf{K} = \mathbf{R}$, $\mathbf{C}$, and $\mathbf{H}$; here b is an orthogonal matrix with $\det b = -1$. If X and Y also denote the matrices in $\mathfrak{g}_{-\alpha}$ and $\mathfrak{g}_{-2\alpha}$ listed above, then $a(\exp(X + Y)w)$ has

$$e^{-t} = \frac{1}{2}(\|X\|^4 + |Y|^2)^{1/2} .$$

For $m(\exp(X + Y)w)$, we have

$$u = 1 \qquad \text{and} \quad \omega^{-1} = \Big(I - \frac{2b^{-1}XX^tb}{\|X\|^2}\Big)b^{-1} \qquad \text{if} \quad \mathbf{K} = \mathbf{R}$$

$$u = \frac{i(\|X\|^2 + Y)}{(\|X\|^4 + |Y|^2)^{1/2}} \quad \text{and} \quad \omega^{-1} = \Big(I - \frac{2(\|X\|^2 + Y)X\bar{X}^t}{\|X\|^4 + |Y|^2}\Big) \quad \text{if} \quad \mathbf{K} = \mathbf{C}$$

$$u = \frac{\|X\|^2 + Y}{(\|X\|^4 + |Y|^2)^{1/2}} \quad \text{and} \quad \omega^{-1} = \Big(I - \frac{2X(\|X\|^2 + Y)\bar{X}^t}{\|X\|^4 + |Y|^2}\Big) \quad \text{if} \quad \mathbf{K} = \mathbf{H} .$$

2a) *Lorentz groups $SO_e(n, 1)$ with n odd and $\geqq 3$.* This is the special case of the above analysis in which $\mathbf{K} = \mathbf{R}$ and n is odd. Here $M \cong SO(n - 1)$ and the action of the group element w is by conjugation of M by b. The action is therefore of the orthogonal group operating on the rotation group. An irreducible representation σ of M is specified by its highest weight, which here can be regarded as a sequence of $\frac{1}{2}(n - 1)$ integers with

$$j_1 \geqq j_2 \geqq \cdots \geqq |j_{(1/2)(n-1)}| .$$

Taking b to have diagonal entries $1, \cdots, 1, -1$, we see that $w\sigma$ has the highest weight $\{j_1, j_2, \cdots, -j_{(1/2)(n-1)}\}$. Thus σ and $w\sigma$ are equivalent if and only if $j_{(1/2)(n-1)} = 0$. From [17], we have, up to a constant factor,

$$p_\sigma(z) = \Big\{\prod_r\Big[\Big(\frac{1}{2}(n-1) - r + j_r\Big)^2 - \Big(\frac{1}{2}(n-1)z\Big)^2\Big]\Big\}$$
$$\times \Big\{\prod_{r<s}\Big[\Big(\frac{1}{2}(n-1) - r + j_r\Big)^2 - \Big(\frac{1}{2}(n-1) - s + j_s\Big)^2\Big]\Big\} .$$

Then $p_\sigma(0) = 0$ if and only if $j_{(1/2)(n-1)} = 0$. Combining these results with Theorems 5 and 6 and with Propositions 44 and 45, we obtain

PROPOSITION 49. *If n is odd and $\geqq 3$, then every representation of the principal series of $SO_e(n, 1)$ is irreducible. If σ has highest weight $\{j_1, \cdots, j_{(1/2)(n-1)}\}$ and if l is the greatest index such that $j_l \neq 0$, then the interval of complementary series for $U(\sigma, z, x)$ in the right half-plane is exactly* $0 < z < 1 - 2l/(n-1)$.

2b) *Lorentz groups $SO_e(n, 1)$ with n even and $\geqq 4$.* Again $\mathbf{K} = \mathbf{R}$. We can choose $b = -I$ to determine the element w. Then w commutes with each element of $M \cong SO(n-1)$, and σ is always equivalent with $w\sigma$. The highest weight of σ can be regarded as a sequence of integers with

$$j_1 \geqq j_2 \geqq \cdots \geqq j_{(1/2)(n-2)} \geqq 0 .$$

In the Plancherel measure the distinguished element γ (cf. § 12) is the identity, by Lemma 41. It follows that the choice of tanh or coth depends only on n, not on σ. It is then easy to verify that the correct choice is tanh. Combining Case 2 of § 12 with the results of [17], we see that the z-dependent part of $p_\sigma(z)$ is

$$z\Big\{\prod_r\Big[\Big(\frac{1}{2}(n-1) - r + j_r\Big)^2 - \Big(\frac{1}{2}(n-1)z\Big)^2\Big]\Big\}\tanh(\pi(n-1)iz/2) .$$

We therefore obtain

PROPOSITION 50. *If n is even and $\geqq 4$, then every representation of the principal series of $SO_e(n, 1)$ is irreducible.*[5] *If σ has highest weight $\{j_1, \cdots, j_{(1/2)(n-2)}\}$ and if l is the greatest index such that $j_l \neq 0$, then the interval of complementary series for $U(\sigma, z, x)$ in the right half-plane is exactly* $0 < z < 1 - 2l/(n-1)$.

2c) *$SU(n, 1)$ with $n \geqq 2$.* This is the case that $\mathbf{K} = \mathbf{C}$. In M we have $u = e^{i\theta}$, ω unitary, and $e^{2i\theta} \det \omega = 1$. The irreducible representations of M are of the form

$$\sigma(e^{i\theta}, \omega) = e^{im\theta}\sigma_0(\omega) ,$$

where m is an integer and σ_0 is an irreducible representation of $U(n-1)$; this decomposition of σ is not unique, because of the condition $e^{2i\theta} \det \omega = 1$. The element w commutes with every member of M. The distinguished element γ (cf. § 12) in the Plancherel formula has $e^{i\theta} = -1$ and $\omega = I$. From this fact one verifies that the trivial representation of M goes with coth if n is even and tanh if n is odd. It follows immediately that $\sigma = (m, \sigma_0)$ goes with coth if and only if $m \equiv n \bmod 2$. As a result of Theorems 5 and 6 and Proposition 43,

[5] This result for $n = 4$ is claimed by Takahashi in [35], but the proof he gives is incorrect.

we obtain

PROPOSITION 51. *In* $SU(n, 1)$ *with* $n \geqq 2$, *let* $\sigma = (m, \sigma_0)$. *If* $m \not\equiv n \bmod 2$, *then the principal series representation* $U(\sigma, 0, x)$ *is irreducible and* $U(\sigma, z, x)$ *is in the complementary series at least for* $0 < z < 1/n$.

Remark. It can be shown for "generic" σ with $m \not\equiv n \bmod 2$ that the critical abscissa is exactly $1/n$. Proposition 45 shows that for such σ there is no complementary series to the right of $1/n$. The resulting interval $0 < z < 1/n$ of complementary series should be compared with the interval $0 < z < 1$ arising in Lemma 46 for the case that σ is trivial on M.

The proposition deals with the tanh case. In the coth case one can use the Plancherel measure to decide on irreducibility. But it is instructive also to use the mean-value criterion of Proposition 20: If $\sigma = (m, \sigma_0)$, we know that the kernel of the intertwining operator at $z = 0$ is a constant times

$$(\|X\|^4 + |Y|^2)^{-n/2}\left(\frac{\|X\|^2 + Y}{(\|X\|^4 + |Y|^2)^{1/2}}\right)^{-m}\sigma_0\left(I - \frac{2(\|X\|^2 + Y)X\bar{X}^t}{\|X\|^4 + |Y|^2}\right).$$

Since the mean value of this is a scalar matrix (Lemma 26), the mean value is 0 exactly when the mean value of the trace is 0. By Proposition 3, we are to consider

$$\int_{c \leqq \|X\|^4 + |Y|^2 \leqq d} (|X|^2 + Y)^{-m} \operatorname{Tr} \sigma_0\left(I - \frac{2(\|X\|^2 + Y)X\bar{X}^t}{\|X\|^4 + |Y|^2}\right) dX dY .$$

The matrix $\sigma_0(-)$ in the integrand is conjugate by a suitable $\sigma_0(\omega)$ for $m = (e^{i\theta}, \omega)$ in M to a matrix $\sigma_0(-)$ with $\|X\|^{-1}X$ the column vector $(1, 0, \cdots, 0)$. Making this change in the integral and performing the obvious substitutions, we can reduce the integral to a positive radial factor times

$$\int_0^\pi e^{-im\theta} \operatorname{Tr} \sigma_0 \begin{pmatrix} e^{2i\theta} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \sin^{n-2}\theta \, d\theta . \tag{16.2}$$

The reducibility condition is that (16.2) is 0. By Proposition 51, (16.2) is not 0 if $m \not\equiv n \bmod 2$. However, if $m \equiv n \bmod 2$, then the integrand is a trigonometric polynomial of period π, and the integral, for fixed σ_0, is 0 if $|m|$ is sufficiently large.

2d) $Sp(n, 1)$ *with* $n > 2$. This is the case that $\mathbf{K} = \mathbf{H}$. M is the product of the group $SU(2) = \{u\}$ of unit quaternions and the group $Sp(n - 1) = \{\omega\}$ of quaternionic unitary matrices. A representation σ of M is uniquely of the form

$$\sigma(u, \omega) = R^{(m)}(u) \otimes \sigma_0(\omega) ,$$

where $R^{(m)}$ is the irreducible representation of $SU(2)$ of degree $m+1$. The element w commutes with every member of M.

The distinguished element γ (cf. § 12) has $u = -1$ and $\omega = I$, from which one verifies that the trivial representation of M goes with tanh for every n. It follows immediately that $\sigma = (m, \sigma_0)$ goes with coth if and only if $m \equiv 1 \bmod 2$, and we obtain

PROPOSITION 52. *In* $Sp(n,1)$ *with* $n \geqq 2$, *let* $\sigma = (m, \sigma_0)$. *If* $m \equiv 0 \bmod 2$, *then the principal series representation* $U(\sigma, 0, x)$ *is irreducible and* $U(\sigma, z, x)$ *is in the complementary series at least for* $0 < z < 1/(2n+1)$.

With $\sigma = (m, \sigma_0)$, the kernel of the intertwining operator at $z = 0$ is a constant times

$$(||X||^4 + |Y|^2)^{-(1/2)(2n+1)} R^{(m)}\Big(\frac{||X||^2 - Y}{(||X||^4 + |Y|^2)^{1/2}}\Big) \otimes \sigma_0\Big(I - \frac{2X(||X||^2 + Y)\bar{X}^t}{||X||^4 + |Y|^2}\Big) .$$

As with $SU(n, 1)$, the mean value of the trace of this matrix is a positive radial factor times a spherical integral, namely

$$\int_0^{\pi/2} \operatorname{Tr} R^{(m)}\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta}\end{pmatrix} \operatorname{Tr} \sigma_0 \begin{pmatrix} e^{-2i\theta} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \cos^{2n-3}\theta \sin^2\theta \, d\theta \ , \tag{16.3}$$

where $e^{-2i\theta}$ is the quaternion $\cos 2\theta - \boldsymbol{i} \sin 2\theta$. The reducibility condition is that (16.3) is 0. If m is odd, (16.3) is never 0, but if m even, (16.3) is 0 for fixed σ_0 as soon m is sufficiently large.

3) *Spin groups*, Spin$(n, 1)$, $n \geqq 3$. These are the only remaining classical simple groups of real-rank one, except for quotients of groups we have already considered. These are the simply-connected double covering groups of $SO_e(n, 1)$. Since M must be connected and contain the center of G, it follows that $M =$ Spin$(n-1)$.

If n is odd, the highest weight of a representation of M is, just as with $SO_e(n, 1)$, a sequence

$$j_1 \geqq j_2 \geqq \cdots \geqq |j_{(1/2)(n-1)}| ,$$

except that the j_l's are half-integers for the faithful representations of M. The Weyl group action sends $j_{(1/2)(n-1)}$ into $-j_{(1/2)(n-1)}$. Since $j_{(1/2)(n-1)}$ is a half integer and not 0, we have

PROPOSITION 53. *If* n *is odd and* $\geqq 3$, *then every representation of the principal series of* Spin$(n, 1)$ *is irreducible. If* σ *is a faithful representation* *of M, then no* $U(\sigma, z, x)$ *is in the complementary series for* $\operatorname{Re} z > 0$.

If n is even, the highest weight of a faithful representation of M is a sequence

$$j_1 \geqq j_2 \geqq \cdots \geqq j_{(1/2)(n-2)} \geqq 0$$

of half-integers and σ is always equivalent with $w\sigma$. Choose w to be one of the two lifts of the element w chosen for $SO_e(n, 1)$. Recall that the Clifford algebra of dimension 2^{n-1} has generators $\{e_0 = 1, e_1, \cdots, e_{n-1}\}$ and relations

$$\left(\sum_{j=1}^{n-1} x_j e_j\right)^2 = -\left(\sum x_j^2\right) e_0$$

and that the group $\operatorname{Spin}(n-1)$ is a certain subgroup of the group of invertible elements in this algebra. We omit the proof of the following lemma.

LEMMA 54. *For* $\operatorname{Spin}(n, 1)$ *with n even, one of the choices of w has the property that*

$$m((\exp X)w)^{-1} = -\|X\|^{-1}\left(\sum_{j=1}^{n-1} x_j e_j\right)(e_1 e_2 \cdots e_{n-1}) \tag{16.4}$$

for all $X = \{x_j\}$ *in* $\mathfrak{g}_{-\alpha} \cong \mathbf{R}^{n-1}$.

COROLLARY 1. *For* $\operatorname{Spin}(n, 1)$ *with n even, let* σ_0 *be the left regular representation of* $M = \operatorname{Spin}(n-1)$ *on the Clifford algebra. Then the entries of* $\sigma_0^{-1}((\exp X)w)$*, as functions on* $\mathbf{R}^{n-1}$*, are homogeneous polynomials of degree* 1*, divided by* $\|X\|$.

Proof. A basis of the Clifford algebra is the set of 2^{n-1} elements

$$\{1, e_1, \cdots, e_{n-1}, e_1e_2, e_1e_3, \cdots, e_2e_3, \cdots, e_1 \cdots e_{n-1}\}$$

and the linear transformation of left multiplication by (16.4), when expressed as a matrix in this basis, has some entries equal to $x_j\|X\|^{-1}$ and the rest equal to 0. This proves the corollary.

From the proof of Corollary 1, we see that the kernel of the intertwining operator $A(w, \sigma, 0)$ has some $x_j/\|X\|^n$'s in its nonzero entries. That is, $A(w, \sigma, 0)$ is a matrix operator composed of classical Riesz transforms in $\mathbf{R}^{n-1}$.

COROLLARY 2. *For* $\operatorname{Spin}(n, 1)$ *with n even, let* σ *be a faithful irreducible representation of M. Then there exists an odd integer d such that the entries of* $\sigma^{-1}((\exp X)w)$ *are* $\|X\|^{-d}$ *times homogeneous polynomials of degree d. Consequently, the mean value* $\mathfrak{M}(\sigma)$ *is* 0 *for all such* σ.

Proof. σ is the Cartan composition of the spin representation σ_0 and some representation of $SO_e(n, 1)$. For the latter, the entries are normalized homogeneous polynomials of even degree. The first statement follows. The second statement is an immediate consequence of the first.

PROPOSITION 55. *In* Spin(n, 1) *if n is even and* $\geqq 4$ *and if σ is a faithful representation of M, then the principal series representation* $U(\sigma, 0, x)$ *is reducible and no* $U(\sigma, z, x)$ *is in the complementary series for* $\operatorname{Re} z > 0$.

For $n = 4$, this result is due to Takahashi [35].

III. SEMISIMPLE GROUPS OF HIGHER REAL-RANK

17. Intertwining integrals

As in Part II, let G be a connected semisimple Lie group of matrices with Lie algebra $\mathfrak{g}$, and let the standard subgroups and subalgebras of G and $\mathfrak{g}$ be defined as in § 6. We no longer assume that G has real-rank one.

The principal series and the non-unitary principal series in the compact picture are defined as in equations (6.2) and (6.3); complementary series will be defined in § 19. The main results of Part III are as follows:

(i) We study intertwining integrals for general G, show how to normalize them, and prove the relations among them.

(ii) We obtain complementary series whenever a formal symmetry condition obtains and a certain operator (19.1) is a scalar multiple of the identity. For a complex semisimple group, (19.1) will automatically be scalar, and the formal symmetry condition therefore implies the existence of complementary series. For general real groups, this is not so, as we have seen already in the real-rank one case ($SL(2, \mathbf{R})$, for example).

(iii) We obtain reducible representations of the principal series whenever a symmetry condition obtains and the operator (19.1) is not scalar.

To each restricted root α there corresponds a member H_α of $\mathfrak{a}$ such that $\alpha(H) = B(H, H_\alpha)$ for all H in $\mathfrak{a}$, where B is the Killing form. The Weyl group M'/M contains the reflection of $\mathfrak{a}$ that is -1 on H_α and is $+1$ on the orthogonal complement; we denote this reflection by p_α. It is well-known that every element of the Weyl group is the product of *simple reflections* (i.e., members p_α such that α is a simple restricted root); the least number of simple reflections required in such a decomposition of a member p of the Weyl group is called the *length* of p.

The main technique in Part III is to use this decomposition of a member of the Weyl group into the product of simple reflections in order to reduce questions about G to questions about groups of real-rank one. Significant progress in this direction was made by Schiffmann [32, 33], whose results we shall summarize presently.

First we discuss the real-rank one groups that will arise from G. Let α be any simple root, and let G_α be the analytic subgroup whose Lie algebra is the smallest subalgebra of $\mathfrak{g}$ containing $\mathfrak{g}_{-2\alpha}$, $\mathfrak{g}_{-\alpha}$, $\mathfrak{g}_\alpha$, and $\mathfrak{g}_{2\alpha}$. Then G_α is simple, and θ is a Cartan involution. Thus $K_\alpha = K \cap G_\alpha$. We can take $\mathfrak{a}_\alpha = \mathbf{R}H_\alpha$, $\mathfrak{n}_\alpha = \mathfrak{g}_\alpha + \mathfrak{g}_{2\alpha}$, $\mathfrak{v}_\alpha = \mathfrak{g}_{-\alpha} + \mathfrak{g}_{-2\alpha}$, and $M_\alpha = M \cap G_\alpha$. In particular, G_α has real-rank one. If w_α is a representative in M'_α of the nontrivial element of the Weyl group M'_α/M_α, then w_α is in M' and $w_\alpha M = p_\alpha$. That is, the simple reflection p_α for G has a representative in G_α. Finally $\rho_\alpha = \rho|_{\mathfrak{a}_\alpha}$ and $\mu_\alpha = \mu|_{\mathfrak{a}_\alpha}$ by [26, p. 399].

With α simple we have defined intertwining operators $A_\alpha(w_\alpha, \sigma, \lambda)$ for G_α if σ is an irreducible representation of M_α, and this definition extends immediately to the case that the representation of M_α is a finite direct sum of irreducible representations. To connect our work with Schiffmann's, we need the following lemma.

LEMMA 56. *Let σ be a representation of M, and let* $\mathrm{Re}(z) > 0$. *If f is in $C^\infty(\sigma)$ and k_0 is in K, then*

$$A_\alpha(w_\alpha, \sigma|_{M_\alpha}, \mu_\alpha^{z/2})f(k_0) = \int_{V_\alpha} e^{(1+z)\rho H(v)} f(\kappa(v)w_\alpha^{-1}k_0)dv ,$$

provided dv is normalized so that $\int_{V_\alpha} e^{2\rho H(v)}dv = 1$.

Proof. The left side by definition equals

$$\int_{K_\alpha} \mu_\alpha^{(1/2)(1-z)}(kw_\alpha)\sigma^{-1}(kw_\alpha)f(kk_0)dk = \int_{K_\alpha} \mu^{(1/2)(1-z)}(k)\sigma^{-1}(k)f(kw_\alpha^{-1}k_0)dk$$

$$= \int_{V_\alpha}\int_{M_\alpha} \mu^{(1/2)(1-z)}(m\kappa(v))e^{2\rho H(v)}\sigma^{-1}(m\kappa(v))f(m\kappa(v)w_\alpha^{-1}k_0)dm\,dv \quad \text{by (9.8)}$$

$$= \int_{V_\alpha} \mu^{(1/2)(1-z)}(\kappa(v))e^{2\rho H(v)}\sigma^{-1}(\kappa(v))f(\kappa(v)w_\alpha^{-1}k_0)dv$$

$$= \int_{V_\alpha} e^{(1+z)\rho H(v)} f(\kappa(v)w_\alpha^{-1}k_0)dv$$

by (6.9) and (6.10).

We come to Schiffmann's results, which are announced in [32] and proved in his thesis [33]. For w in M', define

$$V_w = V \cap w^{-1}Nw .$$

Clearly V_w depends only on the coset wM, and it is easy to see that $V_{w_\alpha} = V_\alpha$ if α is a simple restricted root. Normalize the Haar measure on each V_w so that

$$\int_{V_w} e^{2\rho H(v)}dv = 1 ,$$

and consider the formal integral

$$A(w, \sigma, \lambda)f(k) = \int_{V_w} e^{\rho H(v)}\lambda(\exp H(v))f(\kappa(v)w^{-1}k)dv , \tag{17.1}$$

where σ is an irreducible representation of M, λ is a character of A, and f is in $C^\infty(\sigma)$. Schiffmann proved the following: Let $wM = p$, and let p decompose into a product $p_{\alpha_1} \cdots p_{\alpha_n}$ of simple reflections in as short a fashion as possible. If w decomposes as $w = m_0 w_{\alpha_1} \cdots w_{\alpha_n}$ with m_0 in M and w_{α_j} in M'_{α_j} and $w_{\alpha_j}M = p_{\alpha_j}$ and if $\langle \mathrm{Re}\,\Lambda, \alpha\rangle > 0$ for every positive restricted root α such that $w\alpha < 0$, then the integral (17.1) for $A(w, \alpha, \exp \Lambda)$ converges absolutely, each integral

$$A\big(w_{\alpha_i}, w_{\alpha_{i+1}} \cdots w_{\alpha_n}(\sigma, \exp \Lambda)\big) \tag{17.2}$$

converges absolutely, and $A(w, \sigma, \exp \Lambda)$ is the product

$$A(w, \sigma, \exp\Lambda) = \sigma(m_0)^{-1} A\big(w_{\alpha_1}, w_{\alpha_2} \cdots w_{\alpha_n}(\sigma, \exp \Lambda)\big) \cdots A(w_{\alpha_n}, \sigma, \exp \Lambda)\ . \tag{17.3}$$

Moreover, $A(w, \sigma, \exp \Lambda)$ is in $C^\infty(w\sigma)$, and

$$A(w, \sigma, \exp \Lambda)\, U(\sigma, \exp \Lambda, x) = U(w\sigma, w \exp \Lambda, x) A(w, \sigma, \exp \Lambda) \tag{17.4}$$

for all x in G. Notice that if m is in M, then

$$A(w, \sigma, \exp \Lambda) = \sigma(m)^{-1}\ . \tag{17.5}$$

On smooth functions or K-finite functions,

$$A(w, \sigma, \lambda)^* = A(w^{-1}, w\sigma, w\bar{\lambda}^{-1})\ . \tag{17.6}$$

From (17.3) Schiffmann obtains the further identity

$$A(w_1 w_2, \sigma, \exp \Lambda) = A(w_1, w_2\sigma, w_2 \exp \Lambda) A(w_2, \sigma, \exp \Lambda) \tag{17.7}$$

under the assumptions that the length of $w_1 w_2 M$ is the sum of the lengths of $w_1 M$ and $w_2 M$ and that $\langle \mathrm{Re}\, \Lambda, \alpha\rangle > 0$ for every positive restricted root α such that $w_1 w_2 \alpha < 0$.

From the decomposition (17.3) and from the identification given in Lemma 56 with the real-rank one case, it follows that $A(w, \sigma, \lambda)$ extends to a global meromorphic function of λ. Then (17.4), (17.6), and (17.7) hold for the extensions as well. The main point of § 18 will be to describe a normalization of the extended operators $A(w, \sigma, \lambda)$ so that (17.7) holds without any restriction on the lengths of w_1 and w_2.

Let us emphasize the connection between the general intertwining operators just defined and the operators in the real-rank one case. Let p_α be a simple reflection, let w_α be an element of M'_α such that $w_\alpha M = p_\alpha$, and let f be in $C^\infty(\sigma)$. If f_k denotes the restriction to K_α of the right translate of f by k, then the key formula is

$$A(w_\alpha, \sigma, \lambda) f(k) = A_\alpha(w_\alpha, \sigma|_{M_\alpha}, \lambda|_{A_\alpha}) f_k(1)\ . \tag{17.8}$$

This formula follows by analytic continuation from equation (17.1) and Lemma 56.

18. Normalization of intertwining operators

Let $Z_B(C)$ denote the centralizer of C in B. We begin by stating a lemma whose proof can be given in the spirit of [2] by first showing that $M = M_0 \oplus \sum \mathbf{Z}_2$.[6]

LEMMA 57. *If α is a simple restricted root, then*

$$M = M_\alpha Z_M(M_\alpha) .$$

COROLLARY. *Let σ be an irreducible representation of M, and let α be a simple restricted root. Then the restriction $\sigma|_{M_\alpha}$ is primary* (i.e., is equivalent with the sum of the same irreducible representation a number of times).

Proof. Let B be an operator on E_σ in the center of the ring of operators on E_σ that commute with all $\sigma(m_\alpha)$. Then B commutes with $\sigma(m_\alpha)$ since B is in the ring, and B commutes with $\sigma(z)$ for $z \in Z_M(M_\alpha)$ since B is in the center of the ring. By Lemma 57, B commutes with all $\sigma(m)$, and by Schur's Lemma, B is scalar. Thus $\sigma|_{M_\alpha}$ is primary.

The corollary enables us to normalize the intertwining operators $A(w, \sigma, \lambda)$. First let w_α be a member of M'_α such that $w_\alpha M$ is the simple reflection p_α. By the corollary, if σ is an irreducible representation of M, then $\sigma|_{M_\alpha}$ is equivalent with a multiple of a single irreducible representation $\bar{\sigma}$ of M_α. With the normalizing factors of Part II fixed once and for all, define

$$\mathcal{A}(w_\alpha, \sigma, \lambda) = \gamma_{\bar{\sigma}}(\log \lambda|_{A_\alpha})^{-1} A(w_\alpha, \sigma, \lambda) . \tag{18.1}$$

This definition is unambiguous by Proposition 37(i). For a general element w in M, decompose $A(w, \sigma, \lambda)$ as in (17.3), obtain the γ-factor for each component operator, and use the product of these γ-factors as a normalizing factor for $A(w, \sigma, \lambda)$. Call the normalized operators $\mathcal{A}(w, \sigma, \lambda)$. We must show that the normalizing factor does not depend upon the decomposition of w, and we shall do so in the next lemma. Then we obtain from (17.4) the following identity of meromorphic functions:

$$\mathcal{A}(w, \sigma, \lambda) U(\sigma, \lambda, x) = U(w\sigma, w\lambda, x) \mathcal{A}(w, \sigma, \lambda) \tag{18.2}$$

for all x in G. Also from (17.1) we obtain

$$\mathcal{A}(w, E\sigma E^{-1}, \lambda) = E\mathcal{A}(w, \sigma, \lambda)E^{-1} \tag{18.3}$$

for any unitary operator E on E_σ.

LEMMA 58. *The normalizing factor that defines $\mathcal{A}(w, \sigma, \lambda)$ is independent of the particular decomposition $w = m_0 w_{\alpha_1} \cdots w_{\alpha_n}$ used in* (17.3).

[6] We are indebted to Howard Garland for bringing such a proof of the lemma to our attention.

Proof. For each restricted root β such that $(1/2)\beta$ is not a restricted root, define G_β to be the analytic subgroup corresponding to the smallest Lie algebra containing $\mathfrak{g}_{-2\beta}$, $\mathfrak{g}_{-\beta}$, $\mathfrak{g}_\beta$, and $\mathfrak{g}_{2\beta}$, and let M_β be the M for G_β. Since $\mathrm{Ad}(w)\mathfrak{g}_\beta = \mathfrak{g}_{w\beta}$, we have the identity $M_{w\beta} = wM_\beta w^{-1}$.

In the decomposition $w = m_0 w_{\alpha_1} \cdots w_{\alpha_n}$, one knows that the set of restricted roots $\{\beta_j\}$ with

$$\beta_j = w_{\alpha_n} w_{\alpha_{n-1}} \cdots w_{\alpha_{j+1}}(\alpha_j)$$

is intrinsic to w and does not depend on the decomposition. (The β_j are the positive restricted roots such that $(1/2)\beta_j$ is not a restricted root and $w\beta_j < 0$. See [3, p. 158] and compare with [33].) Now

$$\begin{aligned} w_{\alpha_{j+1}} \cdots w_{\alpha_n}\sigma|_{M_{\alpha_j}} &= w_{\alpha_{j+1}} \cdots w_{\alpha_n}\sigma|_{M_{w_{\alpha_{j+1}}\cdots w_{\alpha_n}\beta_j}} \\ &= w_{\alpha_{j+1}} \cdots w_{\alpha_n}\sigma|_{w_{\alpha_{j+1}}\cdots w_{\alpha_n} M_{\beta_j} w_{\alpha_n}^{-1}\cdots w_{\alpha_{j+1}}^{-1}} , \end{aligned}$$

and the right side is canonically identified with $\sigma|_{M_{\beta_j}}$. Similarly $w_{\alpha_{j+1}} \cdots w_{\alpha_n} \lambda|_{A_{\alpha_j}}$ is canonically identified with $\lambda|_{A_{\beta_j}}$. This means that the set of γ-factors needed for (17.3) is canonical, and so the product of the γ-factors is canonical.

By an argument similar to the one in Lemma 58, we see from (17.7) that whenever the length of w_1w_2M is the sum of the lengths of w_1M and w_2M,

$$\mathcal{A}(w_1w_2, \sigma, \lambda) = \mathcal{A}(w_1, w_2\sigma, w_2\lambda)\mathcal{A}(w_2, \sigma, \lambda) . \tag{18.4}$$

The next lemma is implicit in § 15.1 of [11], and its proof is consequently omitted.

LEMMA 59. *Let H' be an abstract group, let H be a normal subgroup, and suppose H'/H is given by generators t_α with relations $r_\beta(t_\alpha)$. For each α let s_α be a representative in H' of the coset t_α in H'/H, and let h_β be the element of H given by $h_\beta = r_\beta(s_\alpha)$. Define a group F by generators and relations as follows: The generators are an element $\bar{h}$ for each h in H and an element $\bar{s}_\alpha$ for each representative s_α, and the relations are*

$$\bar{1}, \quad (\overline{h_1h_2})^{-1}\bar{h}_1\bar{h}_2, \quad \bar{s}_\alpha\bar{h}\bar{s}_\alpha^{-1}(\overline{s_\alpha h s_\alpha^{-1}})^{-1}, \quad \bar{h}_\beta^{-1}r_\beta(\bar{s}_\alpha) . \tag{18.5}$$

Then the homomorphism φ of F into H' given by $\varphi(\bar{h}) = h$ and $\varphi(\bar{s}_\alpha) = s_\alpha$ is an isomorphism onto.

THEOREM 7. *For any w_1 and w_2 in M',*

$$\mathcal{A}(w_1w_2, \sigma, \lambda) = \mathcal{A}(w_1, w_2\sigma, w_2\lambda)\mathcal{A}(w_2, \sigma, \lambda)$$

as an identity of meromorphic functions.[7]

Proof. It is known [37, p. 160] that the Weyl group M'/M can be given

[7] For complex G, compare with Theorem 5 of [26].

by generators and relations as follows: The generators are the simple reflections p_α, and the relations are $p_\alpha^2 = 1$ and $(p_\alpha p_\beta)^n = 1$ for $\alpha \neq \beta$, where n is an integer depending on α and β and is equal to 2, 3, 4, or 6.

For each simple α, let w_α be a member of M'_α such that $w_\alpha M = p_\alpha$, and let $m_{\alpha\beta} = (w_\alpha w_\beta)^n$. Shortly we shall apply Lemma 59 with $H = M$, $H' = M'$, and $s_\alpha = w_\alpha$.

The main step is to verify that the relations (18.5) are satisfied by the intertwining operators. The identities

$$\mathfrak{A}(1, \sigma, \lambda) = I \tag{18.6}$$

$$\mathfrak{A}(m_2^{-1}m_1^{-1}, m_1m_2\sigma, \lambda)\mathfrak{A}(m_1, m_2\sigma, m_2\lambda)\mathfrak{A}(m_2, \sigma, \lambda) = I \tag{18.7}$$

are immediate from (17.5). Next we show that

$$\begin{aligned}\mathfrak{A}\big(w_\alpha, mw_\alpha^{-1}(w_\alpha m^{-1}w_\alpha^{-1})(\sigma, \lambda)\big) \times \mathfrak{A}\big(m, w_\alpha^{-1}w_\alpha m^{-1}w_\alpha^{-1}(\sigma, \lambda)\big)\\ \times \mathfrak{A}\big(w_\alpha^{-1}, w_\alpha m^{-1}w_\alpha^{-1}(\sigma, \lambda)\big) \times \mathfrak{A}(w_\alpha m^{-1}w_\alpha^{-1}, \sigma, \lambda) = I\,.\end{aligned} \tag{18.8}$$

By (17.5) the left side of (18.8) is

$$\mathfrak{A}(w_\alpha, w_\alpha^{-1}\sigma, w_\alpha^{-1}\lambda)\sigma(w_\alpha m^{-1}w_\alpha^{-1})\mathfrak{A}\big(w_\alpha^{-1}, w_\alpha m^{-1}w_\alpha^{-1}(\sigma, \lambda)\big)\sigma(w_\alpha m w_\alpha^{-1})\,.$$

Applying (18.3) to the last three factors of this expression, we see that the left side of (18.8) reduces to

$$\mathfrak{A}(w_\alpha, w_\alpha^{-1}\sigma, w_\alpha^{-1}\lambda)\mathfrak{A}(w_\alpha^{-1}, \sigma, \lambda)\,.$$

Thus (18.8) follows if we show that

$$\mathfrak{A}(w_\alpha^{-1}, w_\alpha\sigma, w_\alpha\lambda)\mathfrak{A}(w_\alpha, \sigma, \lambda) = I\,. \tag{18.9}$$

By (17.8),

$$\begin{aligned}&\mathfrak{A}(w_\alpha^{-1}, w_\alpha\sigma, w_\alpha\lambda)\mathfrak{A}(w_\alpha, \sigma, \lambda)f(1)\\ &= \mathfrak{A}_\alpha(w_\alpha^{-1}, w_\alpha\sigma|_{M_\alpha}, w_\alpha\lambda|_{M_\alpha})[\mathfrak{A}(w_\alpha, \sigma, \lambda)f]_1(1)\,.\end{aligned} \tag{18.10}$$

Since $[\mathfrak{A}(w_\alpha, \sigma, \lambda)f]_1(k_\alpha) = \mathfrak{A}_\alpha(w, \sigma|_{M_\alpha}, \lambda|_{M_\alpha})f_{k_\alpha}(1)$ and since $\mathfrak{A}_\alpha$ commutes with right translation by k_α, the right side of (18.10) equals

$$\mathfrak{A}_\alpha(w_\alpha^{-1}, w_\alpha\sigma|_{M_\alpha}, w_\alpha\lambda|_{M_\alpha})\mathfrak{A}(w_\alpha, \sigma|_{M_\alpha}, \lambda|_{M_\alpha})f_1(1)\,,$$

and this is $f(1)$ by Proposition 38(i). Applying this result to the right translate of f by k and using (18.2) for x in K, we obtain (18.9). This completes the proof of (18.8).

We come to the main relations $m_{\alpha\beta}^{-1}(w_\alpha w_\beta)^n = 1$ with α possibly equal to β. In the product of $2n$ factors $(p_\alpha p_\beta)^n$, let us assume temporarily that the product of the first n factors (and therefore the product of the last n also) has length n. Let us write w_j for the j^{th} factor in $(w_\alpha w_\beta)^n$. By (17.3)

$$\mathfrak{A}\big(w_{n+1}, w_{n+2}\cdots w_{2n}(, \lambda)\big)\cdots\mathfrak{A}(w_{2n}, \sigma, \lambda) = \mathfrak{A}(w_{n+1}\cdots w_{2n}, \sigma, \lambda)$$

and

$$\mathfrak{A}(w_n^{-1}, w_{n-1}^{-1} \cdots m_{\alpha\beta}(\sigma, \lambda)) \cdots \mathfrak{A}(m_{\alpha\beta}, \sigma, \lambda) = \mathfrak{A}(w_n^{-1} \cdots w_1^{-1} m_{\alpha\beta}, \sigma, \lambda) .$$

Combining these expressions with (18.9) and the identity $m_{\alpha\beta}^{-1}(w_\alpha w_\beta)^n = 1$, we obtain

$$\begin{aligned} &\mathfrak{A}(m_{\alpha\beta}^{-1}, w_1 \cdots w_{2n}(\sigma, \lambda)) \cdots \mathfrak{A}(w_{2n}, \sigma, \lambda) \\ &= \mathfrak{A}(m_{\alpha\beta}, \sigma, \lambda)^{-1} \mathfrak{A}(w_1^{-1}, w_1(\sigma, \lambda))^{-1} \cdots \mathfrak{A}(w_n^{-1}, w_{n-1}^{-1} \cdots m_{\alpha\beta}(\sigma, \lambda)) \\ &\quad \times \mathfrak{A}(w_{n+1}, w_{n+2} \cdots w_{2n}(\sigma, \lambda)) \cdots \mathfrak{A}(w_{2n}, \sigma, \lambda) \\ &= \mathfrak{A}(w_{n+1}^{-1} \cdots w_1^{-1} m_{\alpha\beta}, \sigma, \lambda)^{-1} \mathfrak{A}(w_{n+1} \cdots w_{2n}, \sigma, \lambda) \\ &= \mathfrak{A}(w_{n+1} \cdots w_{2n}, \sigma, \lambda)^{-1} \mathfrak{A}(w_{n+1} \cdots w_{2n}, \sigma, \lambda) \\ &= I . \end{aligned} \tag{18.11}$$

Thus to complete the verification that the relations (18.5) are satisfied by the intertwining operators, we are to show that the product of first n factors of $(p_\alpha p_\beta)^n$ has length n. In doing so, we may assume that n is the smallest integer ≥ 1 for which $(p_\alpha p_\beta)^n = 1$, and we are to show that the product of the first n factors maps at least n restricted roots δ (such that $(1/2)\delta$ is not a restricted root) into negative restricted roots. (See [3, p. 158].) We do so case-by-case.

For $n = 1$, p_α maps α into $-\alpha$. For $n = 2$, $p_\alpha p_\beta$ maps α into $-\alpha$ and β into $-\beta$. For $n = 3$, we have

$$\frac{2\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} = \frac{2\langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle} = -1 ,$$

so that $p_\alpha \beta = \alpha + \beta = p_\beta \alpha$; then $\alpha + \beta$ is a restricted root, and $p_\alpha p_\beta p_\alpha$ maps α into $-\beta$, β into $-\alpha$, and $\alpha + \beta$ into $-\alpha - \beta$. For $n = 4$, we may assume

$$\frac{2\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} = -2, \qquad \frac{2\langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle} = -1 ,$$

so that $p_\alpha \beta = \beta + 2\alpha$ and $p_\beta \alpha = \alpha + \beta$; then $\alpha, \beta, \alpha + \beta$, and $2\alpha + \beta$ are restricted roots, and $p_\alpha p_\beta p_\alpha p_\beta$ maps each of them into its negative. For $n = 6$ we may assume

$$\frac{2\langle \alpha, \beta \rangle}{\langle \alpha, \beta \rangle} = -3, \quad \frac{2\langle \alpha, \beta \rangle}{\langle \alpha, \beta \rangle} = -1 ,$$

so that $p_\alpha \beta = 3\alpha + \beta$ and $p_\beta \alpha = \alpha + \beta$; then $\alpha, \beta, \alpha + \beta, 2\alpha + \beta, 3\alpha + \beta$ and $3\alpha + 2\beta$ are restricted roots, and $p_\alpha p_\beta p_\alpha p_\beta p_\alpha p_\beta$ maps each of them into its negative. This completes the proof of (18.11).

To finish the proof, we observe by (18.4) and (18.9) that

$$\mathfrak{A}(w, \sigma, \lambda)^{-1} = \mathfrak{A}(w^{-1}, w\sigma, w\lambda) . \tag{18.12}$$

Therefore we are to prove that

$$\mathfrak{A}(w_2^{-1}w_1^{-1}, w_1w_2\sigma, w_1w_2\lambda)\mathfrak{A}(w_1, w_2\sigma, w_2\lambda)\mathfrak{A}(w_2, \sigma, \lambda) = I .$$

Expanding each of the factors on the left side according to (17.3), we see that it is enough to prove that if $w_1 \cdots w_n = 1$ in M' with each w_j either lying in M or equal to some $w_\alpha^{\pm 1}$, then

$$\mathfrak{A}(w_1, w_2 \cdots w_n(\sigma, \lambda)) \cdots \mathfrak{A}(w_{n-1}, w_n\sigma, w_n\lambda)\mathfrak{A}(w_n, \sigma, \lambda) = I . \tag{18.13}$$

Consider the word $\bar{w}_1 \cdots \bar{w}_n$ in the free group on $\{\bar{m}, \bar{w}_\alpha\}$. Since $w_1 \cdots w_n = 1$ and since the map φ of Lemma 59 is one-to-one, $\bar{w}_1 \cdots \bar{w}_n$ is a relation. That is,

$$\bar{w}_1 \cdots \bar{w}_n = \textstyle\prod_j g_j R_j g_j^{-1}$$

as elements in the free group, where R_j is one of the relations (18.5) or its inverse. Two words in a free group that represent the same element can be transformed into one another by inserting and deleting pairs $\bar{w}\bar{w}^{-1}$ or $\bar{w}^{-1}\bar{w}$, for various generators $\bar{w}$. In view of (18.9), we can insert or delete corresponding pairs of operators in the left side of (18.12) so that the decomposition in (18.13) corresponds to $\prod_j(g_jR_jg_j^{-1})$. If $g = g^{(1)} \cdots g^{(m)}$ and $R = R^{(1)} \cdots R^{(p)}$, then the piece of the decomposition corresponding to gRg^{-1} is

$$\begin{aligned} &\mathfrak{A}(g^{(1)}, g^{(2)} \cdots g^{(m)}Rg^{-1}(\sigma', \lambda')) \cdots \mathfrak{A}(g^{(m)}, Rg^{-1}(\sigma', \lambda')) \\ &\times \mathfrak{A}(R^{(1)}, R^{(2)} \cdots R^{(p)}g^{-1}(\sigma', \lambda')) \cdots \mathfrak{A}(R^{(p)}, g^{-1}(\sigma', \lambda')) \\ &\times \mathfrak{A}(g^{(m)-1}, g^{(m-1)-1} \cdots g^{(1)-1}(\sigma', \lambda')) \cdots \mathfrak{A}(g^{(1)-1}, \sigma', \lambda') . \end{aligned} \tag{18.14}$$

The product of the middle factors, corresponding to R, is the identity by (18.6), (18.7), (18.8), or (18.11). Now $R = 1$ in M' and so the remaining factors of (18.14) match in pairs and contribute I, by (18.9). Thus (18.14) is the identity operator. The proof of the theorem is complete.

PROPOSITION 60. *If w is in M', then*

$$\mathfrak{A}(w, \sigma, \lambda)^* = \mathfrak{A}(w^{-1}, w\sigma, w\bar{\lambda}^{-1}) \tag{18.15}$$

on K-finite functions or smooth functions. Consequently

(i) *$\mathfrak{A}(w, \sigma, \lambda)$ is unitary if λ is unitary*

(ii) *$\mathfrak{A}(w, \sigma, \lambda)^* = \mathfrak{A}(w^{-1}, w\sigma, \lambda)$ if $w\lambda = \bar{\lambda}^{-1}$ and wM has order 2 in the Weyl group.*

Proof. Once (18.15) is proved, (i) follows immediately from Theorem 7 since $\bar{\lambda}^{-1} = \lambda$, and (ii) is just a special case of (18.15). To prove (18.15), we decompose w minimally as $w = m_0w_1 \cdots w_n$ with m_0 in M and each w_j in some M'_{α_j} for α_j simple. Let $|$ denote an appropriate restriction of a representation of M or A. In view of (17.6), we see that (18.15) will follow if we prove that

$$\begin{aligned} &\overline{\gamma_{w_2 \cdots w_n\sigma|}(w_2 \cdots w_n\lambda|)} \cdots \overline{\gamma_{w_n\sigma|}(w_n\lambda|)\gamma_{\sigma|}(\lambda|)} \\ &= \gamma_{w_n\sigma|}(w_n\bar{\lambda}^{-1}|)\gamma_{w_{n-1}w_n\sigma|}(w_{n-1}w_n\bar{\lambda}^{-1}|) \cdots \gamma_{w_1 \cdots w_n\sigma|}(w_1 \cdots w_n\bar{\lambda}^{-1}|) . \end{aligned} \tag{18.16}$$

There are n factors on each side of (18.16), and we claim that k^{th} factor on the left equals the $(n+1-k)^{\text{th}}$ factor on the right. That is, we claim that

$$\overline{\gamma_{w_{k+1}\cdots w_n\sigma|_{M_k}}(w_{k+1}\cdots w_n\lambda|_{A_k})} = \gamma_{w_k\cdots w_n\sigma|_{M_k}}(w_k\cdots w_n\bar{\lambda}^{-1}|_{A_k}) .$$

Since σ and λ are arbitrary, it is enough to show that

$$\overline{\gamma_{\sigma|_{M_k}}(\lambda|_{A_k})} = \gamma_{w_k\sigma|_{M_k}}(w_k\bar{\lambda}^{-1}|_{A_k}) .$$

But $w_k\bar{\lambda}^{-1}|_{A_k} = \bar{\lambda}|_{A_k}$, and so this identity follows from Proposition 37(iv). This proves (18.16) and the proposition.

19. Complementary series

Let $p = wM$ be an element of order 2 in the Weyl group M'/M, and let M_p be the subgroup of M' generated by M and w. Then M is of index 2 in M_p. If the irreducible representation σ of M is equivalent with $w\sigma$, then it follows from Lemma 18 that it is possible to define $\sigma(w)$ and thereby extend σ to a representation of M_p on E_σ; moreover, $\sigma(w)$ is determined up to a sign. Therefore we can form the operator

$$\sigma(w)\mathcal{A}(w, \sigma, 1) . \tag{19.1}$$

LEMMA 61. *The operator* $\sigma(w)\mathcal{A}(w, \sigma, 1)$ *is unitary and hermitian.*

Proof. It is unitary by Proposition 60(i). It is hermitian because

$$\begin{aligned}
[\sigma(w)\mathcal{A}(w, \sigma, 1)]^* &= \mathcal{A}(w, \sigma, 1)^*\sigma(w)^{-1} \\
&= \mathcal{A}(w^{-1}, w\sigma, 1)\sigma(w)^{-1} && \text{by Proposition 60(ii)} \\
&= \sigma(w^{-1})\mathcal{A}(w^{-1}, \sigma, 1) && \text{by (18.3)} \\
&= \sigma(w^{-1})\mathcal{A}(w^{-2}w, \sigma, 1) \\
&= \sigma(w^{-1})\sigma(w^2)\mathcal{A}(w, \sigma, 1) && \text{by Theorem 7 and (17.5)} \\
&= \sigma(w)\mathcal{A}(w, \sigma, 1) .
\end{aligned}$$

The critical question about the intertwining operators is whether (19.1) *is scalar.* If it is scalar, then there will be complementary series associated to w and σ, in the sense described below. If it is not scalar, then there will be reducible representations of the principal series. The rest of Part III will be devoted to verifying these conclusions and to producing techniques for deciding whether (19.1) is scalar. In the present section we deal with the alternative that (19.1) is scalar.

As in § 9, we say that the representation $U(\sigma, \lambda, x)$ is a member of the *complementary series* if there is a positive definite continuous inner product $\langle\cdot, \cdot\rangle$ on $C^\infty(\sigma) \times C^\infty(\sigma)$ with respect to which $U(\sigma, \lambda, x)$ is unitary.

LEMMA 62. *If* $p = wM$ *is an element of order* 2 *in the Weyl group such that* $w\sigma$ *is equivalent with* σ *and* $p\lambda = \bar{\lambda}^{-1}$, *then the operator*

$$\sigma(w)\mathfrak{A}(w, \sigma, \lambda) \tag{19.2}$$

is hermitian. If, in addition, (19.2) *is finite and positive or negative definite on the K-finite space* $\sum H_D^g$, *then* $U(\sigma, \lambda, x)$ *is in the complementary series.*

Remark. For the extent to which the existence of p as in the lemma is necessary for $U(\sigma, \lambda, x)$ to be in the complementary series, see [4, p. 196].

Proof. Since $p^2 = 1$ and $w\sigma$ is equivalent with σ and $p\lambda = \bar{\lambda}^{-1}$, we can conclude from a computation similar to the one in Lemma 61 that (19.2) is hermitian. If it is finite and definite on $\sum H_D^g$, then its (continuous) extension to $C^\infty(\sigma)$ will be finite and definite, by the argument in the remarks with Proposition 25. Thus we can define a positive or negative definite continuous inner product on $C^\infty(\sigma)$ by

$$\langle f, g\rangle = \big(\sigma(w)\mathfrak{A}(w, \sigma, \lambda)f, g\big) .$$

By Lemma 20 of [26] and by (18.6),

$$\begin{aligned}\langle U(\sigma, \lambda, x)f, U(\sigma, \lambda, x)g\rangle &= \big(\sigma(w)\mathfrak{A}(w, \sigma, \lambda)U(\sigma, \lambda, x)f, U(\sigma, \lambda, x)g\big)\\ &= \big(U(\sigma, \bar{\lambda}^{-1}, x^{-1})\sigma(w)U(w\sigma, w\lambda, x)\mathfrak{A}(w, \sigma, \lambda)f, g\big)\\ &= \big(U(\sigma, w\lambda, x^{-1})U(\sigma, w\lambda, x)\sigma(w)\mathfrak{A}(w, \sigma, \lambda)f, g\big)\\ &= \big(\sigma(w)\mathfrak{A}(w, \sigma, \lambda)f, g\big)\\ &= \langle f, g\rangle .\end{aligned}$$

THEOREM 8. *Let* $p = wM$ *be an element of order* 2 *in the Weyl group. If* λ *is any charactor of* A *sufficiently close to* 1 *such that* $p\lambda = \bar{\lambda}^{-1}$ *and if* σ *is any irreducible representation of* M *with* σ *equivalent to* $w\sigma$ *and with* (19.1) *scalar, then* $U(\sigma, \lambda, x)$ *is in the complementary series.*

Remark. We need prove only these two statements: (1) If λ_0 can be connected to 1 through characters λ with $p\lambda = \bar{\lambda}^{-1}$ and with (19.2) finite and nonsingular, then $U(\sigma, \lambda_0, x)$ is in the complementary series. (2) There is a fixed neighborhood of the trivial character of A such that $\mathfrak{A}(w, \sigma, \lambda)$ is nonsingular for all λ in the neighborhood and for all σ.

Proof of (1). The operator (19.1) is unitary, hermitian, and scalar, hence equal to $\pm I$. By redefining $\sigma(w)$ if necessary, we may assume (19.1) is $+I$. The operator (19.2) on each H_D^g is then a continuous function into nonsingular matrices that is positive definite for $\lambda = 1$. By Lemma 40, $\sigma(w)\mathfrak{A}(w, \sigma, \lambda_0)$ is positive definite, and by Lemma 62, $U(\sigma, \lambda, x)$ is in the complementary series.

Proof of (2). Decompose p as a product $\prod p_\alpha$ of simple reflections in as short a fashion as possible. Let w_α be an element of M'_σ with $p_\alpha = w_\alpha M$, and write $w = m_0(\prod w_\alpha)$ for some m_0 in M. Applying Theorem 7 to this decomposition and using the continuity of the action of the Weyl group on the characters of A, we see that it is enough to show that $\mathcal{A}(w_\alpha, \sigma, \lambda)$ is finite and nonsingular if λ is sufficiently close to 1 (with the neighborhood independent of σ). But $\mathcal{A}(w_\alpha, \sigma, \lambda)$ has the same behavior as the real-rank one operator

$$\mathcal{A}(w_\alpha, \sigma|_{M_\alpha}, \lambda|_{A_\alpha}) ,$$

which is known from § 14 to be finite and nonsingular if $|\mathrm{Re}\log \lambda_{|\mathfrak{a}_\alpha}|$ is closer to 0 than $(1/2)\alpha$ is. This completes the proof of the theorem.

The next lemma is useful for concluding in some cases that (19.1) is scalar. In Theorem 9 we shall apply the lemma to complex semisimple G.

LEMMA 63. *Each element p of order 2 in the Weyl group decomposes as a product*

$$p = p_{\alpha_1} \cdots p_{\alpha_n}$$

of distinct commuting reflections, where each α_i is a (not necessarily simple) restricted root. Furthermore, any character λ of A for which $p\lambda = \lambda$ has $p_{\alpha_j}\lambda = \lambda$ for $1 \leqq j \leqq n$.

Proof. Since p operates orthogonally as an involution on $\mathfrak{a}$, we have the orthogonal decomposition $\mathfrak{a} = P_p \oplus N_p$, where P_p and N_p are the eigenspaces for the eigenvalues $+1$ and -1, respectively. The proof will be by induction on the dimension of N_p. If $N_p = 0$, then $p = 1$ and the lemma is trivial.

In the general case when $\dim N_p \geqq 1$, we claim that $\ker\alpha \supseteqq P_p$ for some restricted root α. If not, then $P_p - \ker\alpha$ is an open dense subset of P_p for each α. Consequently the intersection $\bigcap_\alpha (P_p - \ker\alpha)$ is not empty. Every member of the intersection is a regular element of $\mathfrak{a}$ and is left fixed by p. Hence $p = 1$ or we have a contradiction. Thus $\ker\alpha \supseteqq P_p$ for some α if $\dim N_p \geqq 1$.

For this α, form $p_\alpha p$. The elements p_α and p commute because $\ker\alpha \supseteqq P_p$, and thus $p_\alpha p$ has order 2. Also $P_{p_\alpha} \supseteqq P_p$ says that the orthogonal complements satisfy $N_{p_\alpha} \subseteqq N_p$. Let N be the orthogonal complement of N_{p_α} in N_p, and let P be the orthogonal complement of N in $\mathfrak{a}$. Then p_α is 1 on N and p is -1 on N, whence $N \subseteqq N_{p_\alpha p}$. Since $P_{p_\alpha} \supseteqq P_p$, $P_{p_\alpha p} \supseteqq P_p$; also $P_{p_\alpha p} \supseteqq N_{p_\alpha} \cap N_p = N_{p_\alpha}$. Thus $P_{p_\alpha p} \supseteqq P_p \oplus N_{p_\alpha} = P$. That is, $N = N_{p_\alpha p}$ and $P = P_{p_\alpha p}$, so that

$$N_p = N_{p_\alpha} \oplus N_{p_\alpha p} .$$

Consequently $\dim N_{p_\alpha p} < \dim N_{p_\alpha}$, and by induction $p_\alpha p = p_{\alpha_1} \cdots p_{\alpha_{n-1}}$ with the N spaces for the p_{α_j} orthogonal and contained in $N_{p_\alpha p}$. Then

$$p = p_{\alpha_1} \cdots p_{\alpha_{n-1}} p_\alpha$$

with the N spaces for the elements on the right orthogonal and contained in

N_p. This establishes the decomposition.

Finally let $p\lambda = \lambda$, and let $\lambda = \exp \Lambda$. Using the identification of $\mathfrak{a}$ with its dual by means of the Killing form, we can regard Λ as an element of $\mathfrak{a}$. Then $p\Lambda = \Lambda$, Λ is in P_p, $\Lambda \perp N_p$, $\Lambda \perp N_{p_{\alpha_j}}$ for each j, Λ is in $P_{p_{\alpha_j}}$ for each j, and $p_{\alpha_j}\Lambda = \Lambda$ for each j. This proves the lemma.

THEOREM 9. *Let G be a complex semisimple group, $p = wM$ an element of order 2 of the Weyl group, σ an irreducible representation (necessarily one-dimensional) of M, and $\lambda = \exp \Lambda$ a character of A. Suppose that the symmetry conditions $p\sigma = \sigma$ and $p\Lambda = -\bar{\Lambda}$ are satisfied. If*

$$\frac{|\langle \operatorname{Re} \Lambda, \alpha \rangle|}{\langle \alpha, \alpha \rangle} < 1 \tag{19.3}$$

for every root α, then $U(\sigma, \lambda, x)$ is in the complementary series.

Remarks. 1. Since G is complex, we have $\mathfrak{m} = i\mathfrak{a}$ and $M = \exp \mathfrak{m}$. Thus M is abelian and its irreducible representations are one-dimensional. For one-dimensional representations, equivalence becomes identity, and we can replace unambiguously the usual condition "σ is equivalent with $w\sigma$" by the condition "$p\sigma = \sigma$."

2. Roots in the statement of the theorem are understood relative to the Cartan subalgebra $\mathfrak{a} + i\mathfrak{a}$, and the inner product is the one induced by the Killing form.

3. For some earlier results related to Theorem 9, see Kunze [25]. Kostant [24] dealt with the case that σ is trivial and G is real.[8]

Proof. By Lemma 63, decompose p as the product $p_{\alpha_1} \cdots p_{\alpha_n}$ of distinct commuting reflections. Since $\mathfrak{m} = i\mathfrak{a}$, the second half of the lemma shows that $p\sigma = \sigma$ implies $p_{\alpha_j}\sigma = \sigma$ for each j. Write $w = \prod w_{\alpha_j}$ with $w_{\alpha_j}M = p_{\alpha_j}$. By Theorem 7,

$$\begin{aligned}\mathcal{A}(w, \sigma, 1) &= \mathcal{A}(w_{\alpha_1}, p_{\alpha_2} \cdots p_{\alpha_n}\sigma, 1) \cdots \mathcal{A}(w_{\alpha_n}, \sigma, 1) \\ &= \mathcal{A}(w_{\alpha_1}, \sigma, 1) \cdots \mathcal{A}(w_{\alpha_n}, \sigma, 1) .\end{aligned} \tag{19.4}$$

We shall show each factor in (19.4) is scalar. In fact, consider the factor $\mathcal{A}(w_\alpha, \sigma, 1)$ with $w_\alpha\sigma = \sigma$. Here α is conjugate to a simple restricted root β, and it follows that $p_\alpha = s^{-1}p_\beta s$ for some s in the Weyl group. Let $w_\beta M = p_\beta$ with w_β in M'_β, and let $w_0M = s$. Then $w_\alpha = m_0w_0^{-1}w_\beta w_0$ for some m_0 in M, and $w_\alpha\sigma = \sigma$ implies $w_\beta(w_0\sigma) = w_0\sigma$ since $\sigma(m_0)$ is scalar. By Theorem 7

[8] The result for G real and σ trivial given in [24] concerning the existence of the complementary series can also be derived by the use of the present methods, as an examination of the proof of Theorem 9 shows.

$$\begin{aligned}\mathcal{A}(w_\alpha, \sigma, 1) &= \sigma(m_0)^{-1}\mathcal{A}(w_0^{-1}, w_\beta w_0\sigma, 1)\mathcal{A}(w_\beta, w_0\sigma, 1)\mathcal{A}(w_0, \sigma, 1)\\ &= \sigma(m_0)^{-1}\mathcal{A}(w_0^{-1}, w_0\sigma, 1)\mathcal{A}(w_\beta, w_0\sigma, 1)\mathcal{A}(w_0, \sigma, 1) .\end{aligned}$$

The factor $\mathcal{A}(w_\beta, w_0\sigma, 1)$ is essentially an operator for the real-rank one group G_β, which is locally isomorphic to $SL(2, \mathrm{C})$ since G is complex. Since $w_\beta(w_0\sigma) = w_0\sigma$, it follows from facts about $SL(2, \mathrm{C})$ that $w_0\sigma$ is trivial on M_β. Then $\mathcal{A}(w_\beta, w_0\sigma, 1)$ is scalar by Proposition 49. But $\mathcal{A}(w_0^{-1}, w_0\sigma, 1)\mathcal{A}(w_0, \sigma, 1) = I$ by Theorem 7, and $\sigma(m_0)^{-1}$ is scalar since σ is one-dimensional. Thus $\mathcal{A}(w_\alpha, \sigma, 1)$ is scalar. By (19.4), $\mathcal{A}(w, \sigma, 1)$ is scalar, and the one-dimensionality of σ implies that (19.1) is scalar.

Therefore by Theorem 8 and its remarks, the proof of Theorem 9 will be complete if we show that $\mathcal{A}(w, \sigma, \exp\Lambda)$ is finite and nonsingular as long as (19.3) holds. Write w as the product of an element of M with a product of elements w_α, where α is a simple restricted root and w_α is in M'_α, and decompose $\mathcal{A}(w, \sigma, \exp\Lambda)$ accordingly, as in (18.4). Since the condition (19.3) on Λ is invariant under the Weyl group, a typical factor in the decomposition is

$$\mathcal{A}(w_\alpha, \sigma', \exp\Lambda') , \tag{19.5}$$

where Λ' satisfies (19.3). Now (19.5) has the same behavior as the operator for $SL(2, \mathrm{C})$ given by

$$\mathcal{A}_\alpha(w_\alpha, \sigma'|_{M_\alpha}, \exp(\Lambda'|_{\mathfrak{a}_\alpha})) .$$

This operator is finite and nonsingular, by Proposition 48, provided

$$|\langle \mathrm{Re}\,\Lambda'|_{\mathfrak{a}_\alpha}, \rho_\alpha\rangle_\alpha| < \langle\rho_\alpha, \rho_\alpha\rangle_\alpha . \tag{19.6}$$

Thus the proof will be complete if we show that any Λ' satisfying (19.3) satisfies (19.6). But $\langle\cdot, \cdot\rangle_\alpha$ is proportional to the restriction of $\langle\cdot, \cdot\rangle$, and $\rho_\alpha = \alpha$ since α as a restricted root has multiplicity two. Thus (19.6) follows from (19.3) applied to the simple root α. The proof of the theorem is complete.

20. Reducible principal series

We turn to a discussion of the alternative that the unitary hermitian operator $\sigma(w)\mathcal{A}(w, \sigma, 1)$ of (19.1) is not scalar.

THEOREM 10. *Let $p = wM$ be an element of order 2 in the Weyl group, and let σ be an irreducible representation of M with σ equivalent to $w\sigma$. If* (19.1) *is not scalar, then the principal series representation $U(\sigma, \lambda, x)$ is reducible for every unitary λ such that $p\lambda = \lambda$.*

Proof. By (18.2),

$$\begin{aligned}\sigma(w)\mathcal{A}(w, \sigma, \lambda)\, U(\sigma, \lambda, x) &= \sigma(w)\, U(w\sigma, p\lambda, x)\mathcal{A}(w, \sigma, \lambda)\\ &= \sigma(w)\, U(w\sigma, \lambda, x)\mathcal{A}(w, \sigma, \lambda)\\ &= U(\sigma, \lambda, x)\sigma(w)\mathcal{A}(w, \sigma, \lambda) .\end{aligned}$$

Therefore it is enough to prove that (19.2) is not scalar. Now (19.2) is in any event hermitian and unitary by an argument similar to the one in Lemma 61, since $p\lambda = \lambda = \bar{\lambda}^{-1}$. If (19.2) is scalar for some $\lambda = \lambda_0$, then it is $\pm I$, and we may take it to be $+I$ by adjusting $\sigma(w)$.

The restriction of (19.2) to any H_D^g is a continuously varying hermitian unitary operator that is I at λ_0. Its characteristic polynomial has coefficients that are integers (because hermitian unitary matrices have only ± 1 as eigenvalues) varying continuously with λ, hence constant. That is, the characteristic polynomial is $(x-1)^n$ for all λ with $p\lambda = \lambda$, where n is the dimension of H_D^g. Since (19.2) is hermitian, this characteristic polynomial determines the restriction of (19.2) as the identity. Hence (19.2) is scalar for all λ with $p\lambda = \lambda$, in contradiction with the assumption about $\lambda = 1$. This contradiction proves the theorem.

The next lemma is useful for concluding in some cases that (19.1) is not scalar. After the lemma, we shall apply the result to $SL(2n, \mathbf{R})$, and we shall comment on some other examples.

LEMMA 64. *Let $p = wM$ be an element of the Weyl group, let $p = p_{\alpha_1} \cdots p_{\alpha_n}$ be a minimal decomposition as the product of (not necessarily commuting) simple reflections, and let $w = w_1 \cdots w_n$ be a corresponding decomposition in M'. If there is no j such that*

$$\mathcal{A}(w_j, w_{j+1} \cdots w_n\sigma, 1) \tag{20.1}$$

is a constant operator (i.e., *a constant matrix operating on E_σ*) *then $\mathcal{A}(w, \sigma, 1)$ is not a constant operator.*

Remark. If σ is equivalent with w_σ, the lemma gives a sufficient condition for $\sigma(w)\mathcal{A}(w, \sigma, 1)$ not to be scalar. The condition is one that can be verified easily, since (20.1) is essentially an operator in the real-rank one case. By Theorem 5, (20.1) will be nonconstant if and only if the Plancherel density $p_\tau(z)$ for the representation $\tau = w_{j+1} \cdots w_n\sigma|_{M_{\alpha_j}}$ is nonzero at 0.

Proof. The assumption is that the normalizing factor for each operator $\mathcal{A}(w_j, w_{j+1} \cdots w_n\sigma, w_{j+1} \cdots w_n\lambda)$ has no pole at $\lambda = 1$. Consequently the normalizing factor for $\mathcal{A}(w, \sigma, \lambda)$ is regular at $\lambda = 1$, and $\mathcal{A}(w, \sigma, 1)$ is a finite constant multiple of $A(w, \sigma, 1)$. It is therefore enough to exhibit a function f in $C^\infty(\sigma)$ such that $f(1) = 0$ and $A(w, \sigma, 1)f(1) \neq 0$.

Let $\varepsilon > 0$. Then $\mu^{\varepsilon/2} = \exp(\varepsilon\rho)$ and $\langle \varepsilon\rho, \alpha\rangle > 0$ for every restricted root α with $\alpha > 0$ and $w\alpha < 0$. From Schiffmann's results quoted in § 17, we know that the integral defining $A(w, \sigma, \mu^{\varepsilon/2})$ is absolutely convergent. Moreover since $A(w, \sigma, \lambda)$ is regular at $\lambda = 1$,

$$A(w, \sigma, 1)f = \lim_{\varepsilon \downarrow 0} A(w, \sigma, \mu^{\varepsilon/2})f \tag{20.2}$$

for all f in $C^\infty(\sigma)$. Also

$$A(w, \sigma, \mu^{\varepsilon/2})f(k) = \int_{V \cap w^{-1}Nw} e^{(1+\varepsilon)\rho H(v)} f(\kappa(v)w^{-1}k)dv \ . \tag{20.3}$$

The mapping of V into $M\backslash K$ given by $v \to M\kappa(v)$ is a diffeomorphism onto an open set. Choose a compact neighborhood C of the identity coset that lies in the image, and let φ be a smooth function on $M\backslash K$ such that $\varphi = 0$ off Cw^{-1} and $\varphi(Mw^{-1}) \neq 0$. Write φ also for the lift of this function to a left M-invariant function on K.

Fix F in $C^\infty(\sigma)$ and let f be any smooth left M- invariant scalar function on K. Then $f\varphi F$ is in $C^\infty(\sigma)$, and

$$f(\kappa(v)w^{-1})\varphi(\kappa(v)w^{-1})F(\kappa(v)w^{-1})$$

is a function of compact support on V. By (20.2), (20.3), and dominated convergence, we have

$$\begin{aligned} A(w, \sigma, 1)f\varphi F(1) &= \lim_{\varepsilon \downarrow 0} A(w, \sigma, \mu^{\varepsilon/2})f\varphi F(1) \\ &= \lim_{\varepsilon \downarrow 0} \int_{V \cap w^{-1}Nw} e^{(1+\varepsilon)\rho H(v)} f\varphi F(\kappa(v)w^{-1})dv \\ &= \int_{V \cap w^{-1}Nw} e^{\rho H(v)} f(\kappa(v)w^{-1})\varphi F(\kappa(v)w^{-1})dv \ . \end{aligned} \tag{20.4}$$

If the lemma were false, then the equality $F(1) = 0$ would imply that both sides of (20.4) were 0. Now the facts that $v \to M\kappa(v)$ is a diffeomorphism onto an open set and that $V \cap w^{-1}Nw$ is smoothly imbedded as a euclidean subspace of V imply that $f(\kappa(v)w^{-1})$ can be an arbitrary function of v (on $V \cap w^{-1}Nw$) that is supported near $v = 1$. Therefore if the lemma were false, we would have $\varphi(\kappa(v)w^{-1})F(\kappa(v)w^{-1}) = 0$ whenever $F(1) = 0$. Since $\varphi(w^{-1}) \neq 0$, this means that $F(w^{-1}) = 0$ whenever $F(1) = 0$.

Thus the lemma will be proved if we produce F in $C^\infty(\sigma)$ with $F(1) = 0$ and $F(w^{-1}) \neq 0$. Let $b(k)$ be a member of $C^\infty(\sigma)$ with $b(k_0) \neq 0$, say. Then $U(\sigma, 1, wk_0)b$ is not 0 at w^{-1}. Since $Mw^{-1} \neq M$, we can choose a function on $M\backslash K$ that vanishes at M but not at Mw^{-1}; let χ be the lift to K of such a function. Then $\chi U(\sigma, 1, wk_0)b$ is a member of $C^\infty(\sigma)$ that vanishes at 1 but not at w^{-1}. The proof of the lemma is complete.

We shall apply the results of this section to the group $SL(2n, \mathbf{R})$ of all real $2n$-by-$2n$ matrices of determinant one to obtain reducible members of the principal series. Compare this assertion with the paper [10] by Gelfand and Graev, where the contrary is suggested.

PROPOSITION 65.[9] *In $SL(2n, \mathbf{R})$ choose M as the group of diagonal matrices with diagonal entries $\varepsilon_i = \pm 1$. Define*

$$\sigma(\varepsilon_1, \cdots, \varepsilon_{2n}) = \varepsilon_1\varepsilon_3\varepsilon_5 \cdots \varepsilon_{2n-1} ,$$

and let p be the permutation

$$p = (1\,2)(3\,4)(5\,6) \cdots (2n-1, 2n) . \tag{20.5}$$

Then the principal series representation $U(\sigma, \lambda, x)$ is reducible for every unitary character λ of A such that $p\lambda = \lambda$.

Proof. The element p is in the Weyl group, and $p\sigma$ is the product of the even-numbered entries. Since the members of M have determinant one, $p\sigma = \sigma$. Let $p = wM$. Since $\sigma(w)$ is scalar, Theorem 10 shows that it is enough to prove that $\mathcal{A}(w, \sigma, 1)$ is not scalar.

Now (20.5) is a minimal decomposition of p as a product of simple reflections, and Lemma 64 shows that $\mathcal{A}(w, \sigma, 1)$ is not scalar if each operator

$$\mathcal{A}\big((2j-1, 2j), (2j+1, 2j+2) \cdots (2n-1, 2n)\sigma, 1\big) \tag{20.6}$$

is not scalar. The associated real-rank one group for this operator is the copy of $SL(2, \mathbf{R})$ obtained from the $2j - 1^{\text{st}}$ and $2j^{\text{th}}$ entries. Also the representation of M in (20.6) is

$$\varepsilon_1\varepsilon_3 \cdots \varepsilon_{2j-1}\varepsilon_{2j+2}\varepsilon_{2j+4}\, \varepsilon_{2n} ,$$

and its restriction to the M of this SL(2, $\mathbf{R}$) is ε_{2j-1}, i.e., the nontrivial representation of the M. Thus the real-rank one operator associated to (20.6) is a Hilbert transform and is not scalar. The proof is complete.

By similar methods one can exhibit reducible representations in the principal series for all the groups $Sp(n, \mathbf{R})$ for $n \geqq 1$, $SO(n, n)$ for $n \geqq 2$, and $SO(2n+1, 2n)$ for $n \geqq 1$.

* (Added in Proof.) J. Arthur has shown that a strong enough version of the asympotic expansion can be obtained by combining the proofs given by Harish-Chandra in Amer. J. Math. **80** (1958), 576 and 582, and Acta Math. **116** (1966), 71.

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