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Singular Integrals and the Principal Series, IV

(intertwining operators/semisimple Lie groups)

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ABSTRACT The intertwining operators that have been constructed for all the series of unitary representations appearing in the Plancherel formula of a connected real semisimple Lie group of matrices are given a new normalization and then applied in two ways. The first is to obtain dimension formulas for the commuting algebras of the unitary representations in question. The second is to establish the existence of complementary series. These bear the same relationship to the unitary representations under study that the complementary series found earlier bear to the principal series.

This announcement continues the applications of the theory developed in ref. 1 of intertwining operators for all the series of unitary representations appearing in the Plancherel formula of a connected real semisimple Lie group of matrices. For economy of presentation we retain the notation of ref. 1 and shall not repeat it here.

Our interest in this note is with questions of irreducibility of the unitary continuous series and existence of complementary series. The main result concerning irreducibility is that the dimension formula for the commuting algebra of a continuous series representation takes the same form as for the principal series (refs. 2 and 3). If the linear functional λ on $\mathfrak{a}$ is not purely imaginary, we say the representation $U_P(\xi,\lambda,\cdot)$ belongs to the *complementary series* if it is possible to redefine the inner product on the K-finite vectors for $U_P(\xi,\lambda,\cdot)$ so that $U_P(\xi,\lambda,\cdot)$ is infinitesimally unitary. We shall obtain, by the methods of ref. 4, sufficient conditions on the discrete series representation ξ of M for the existence of values of λ such that $U_P(\xi,\lambda,\cdot)$ is in the complementary series.

The methods used in obtaining the results below involve one additional twist beyond what was already announced in ref. 1. Recall from ref. 1 that the unnormalized intertwining operator $A(P_1:P:\xi:\lambda)$ was proved to exist and have an analytic continuation by using an imbedding of the discrete series ξ in a nonunitary principal series representation of M corresponding to some parameters (σ,λ_M). The procedure was to relate the operator to the intertwining operator for the nonunitary principal series representation of G with parameters $(\sigma,\lambda+\lambda_M)$; ultimately the operator $A(P_1:P:\xi:\lambda)$ is shown not to depend on the imbedding of ξ. However, we used the imbedding also to obtain a normalizing factor for $A(P_1:P:\xi:\lambda)$, and this factor *did* depend on the imbedding. These normalizing factors led us to some identities for the Plancherel measure, two of which were listed at the end of ref. 1. These normalizing factors were satisfactory for problems involving a single ξ, but when more than one discrete series ξ was involved, we made the assumption that the imbeddings of the discrete series representations satisfied a compatibility condition.

The additional twist is that questions of irreducibility and complementary series involve using a second normalizing factor. This new normalizing factor provides an analytic tool for handling the problems of the present note, but it does not suggest the identities for the Plancherel measure obtained in ref. 1. Thus the full theory of intertwining operators seems to require using *both* normalizing factors. We have not been able to determine whether the two factors can always be taken identical.

1. Normalization of intertwining operators

We begin by describing this new normalization of intertwining operators. First suppose that the parabolic subgroup $P = MAN$ has $\dim A = 1$. Using the earlier normalization and letting $\bar{P}$ be the opposite parabolic subgroup, we find that

$$A(P:\bar{P}:\xi:\lambda)A(\bar{P}:P:\xi:\lambda) = \eta(\bar{P}:P:\xi:\lambda)I$$

for a meromorphic scalar-valued function η of the one complex variable λ. Because of the last corollary of ref. 1, this function is even, is real on the real axis, and is ≥ 0 on the imaginary axis. We can therefore construct a normalizing factor $\gamma(\bar{P}:P:\xi:\lambda)$ for $A(\bar{P}:P:\xi:\lambda)$ by the technique of section 13 of ref. 4.

In the case for general $\dim A$, we are to normalize $A(P_2:P_1:\xi:\lambda)$. For any P_1-positive reduced root α of $\mathfrak{a}$ (see ref. 5 for definitions) let $N^{(\alpha)}$ be the analytic subgroup corresponding to the sum of the root spaces for the positive multiples of α, and let $V^{(\alpha)}$ be the opposite group. By a construction given in ref. 5, one can select parabolic groups $P^{(1)},\ldots,P^{(k)}$ such that $P^{(1)} = P_1$, $P^{(k)} = P_2$, and $V^{(i)} \cap N^{(i+1)} = V^{(\alpha_i)}$ for some P_1-positive root α_i of $\mathfrak{a}$. According to (*ii*) of Theorem 2 of ref. 1, we obtain a product decomposition

$$A(P_2:P_1:\xi:\lambda) = A(P^{(k)}:P^{(k-1)}:\xi:\lambda)\cdots A(P^{(2)}:P^{(1)}:\xi:\lambda).$$

Each of the operators on the right side can be identified with an operator of the kind described in the previous paragraph and so has a natural choice for normalizing factor. We take the normalizing factor $\gamma(P_2:P_1:\xi:\lambda)$ for $A(P_2:P_1:\xi:\lambda)$ to be the product of these normalizing factors. The normalized operator is $\mathfrak{A} = \gamma^{-1}A$.

THEOREM 1. *The intertwining operators normalized with the new normalizing factor satisfy*

(*i*) $\mathfrak{A}(P_2:P_0:\xi:\lambda) = \mathfrak{A}(P_2:P_1:\xi:\lambda)\mathfrak{A}(P_1:P_0:\xi:\lambda)$.

(*ii*) $\mathfrak{A}(P_2:P_1:\xi:\lambda)^* = \mathfrak{A}(P_1:P_2:\xi:-\bar{\lambda})$ *if the adjoint is taken K-space by K-space.*

(*iii*) $\mathfrak{A}(P_2:P_1:\xi:\lambda)$ *is unitary for* λ *imaginary.*

(*iv*) *If w in K represents a member of the Weyl group* $W(\mathfrak{a})$, *then*

$$\mathfrak{A}(P_2:P_1:\xi:\lambda) = R(w^{-1})\mathfrak{A}(wP_2w^{-1}:wP_1w^{-1}:w\xi:w\lambda)R(w).$$

With the old normalizing factors, (*i*) was valid, but we could not prove (*ii*) and hence (*iii*). Conclusion (*iv*) for the old normalization used the assumption about compatibility of imbeddings.

For w in K representing a member of $W(\mathfrak{a})$, let

$$\mathfrak{A}_P(w,\xi,\lambda) = R(w)\mathfrak{A}(w^{-1}Pw:P:\xi:\lambda).$$

From (*i*) and (*iv*) we obtain the cocycle relation

$$\mathfrak{A}_P(w_1w_2,\xi,\lambda) = \mathfrak{A}_P(w_1,w_2\xi,w_2\lambda)\mathfrak{A}_P(w_2,\xi,\lambda).$$

From (*ii*) we find that

$$\mathfrak{A}_P(w,\xi,\lambda)^* = \mathfrak{A}_P(w^{-1},w\lambda,-w\bar{\lambda})$$

and hence that $\mathfrak{A}_P(w,\xi,\lambda)$ is unitary for λ imaginary. We shall use the operators $\mathfrak{A}_P(w,\xi,\lambda)$ in dealing with irreducibility and complementary series.

2. Irreducibility

Let ξ be a discrete series representation of M, and let $U_P(\xi,\lambda,\cdot)$ be the corresponding continuous series representations of G. Our problem is that of determining the algebra $\mathfrak{C}_P(\xi,\lambda)$ of operators that commute with the representation $U_P(\xi,\lambda,\cdot)$.

Let $[\xi]$ denote the equivalence class of the discrete series representation ξ of M, and let $W_{\xi,\lambda}$ be the subgroup of elements s in $W(\mathfrak{a})$ such that $s[\xi] = [\xi]$ and $s\lambda = \lambda$. If w is a representative in K of an s in $W(\mathfrak{a})$ and if $w\xi$ and ξ are equivalent, then one can define $\xi(w)$ in such a way that ξ extends to a representation of the smallest group containing M and w; the definition of $\xi(w)$ is unique up to a scalar factor equal to a root of unity. Then $\xi(w)\mathfrak{A}_P(w,\xi,\lambda)$ is independent of the representative w and intertwines $U_P(\xi,\lambda,\cdot)$ with $U_P(\xi,w\lambda,\cdot)$. The solution of our problem now involves two elements: first, the use of Harish-Chandra's theory of c-functions, together with their relation to the intertwining operators as given in Theorem 3′ of ref. 1; and second, the following lemma.

LEMMA. *The set of those operators* $\xi(s)\mathfrak{A}_P(s,\xi,\lambda)$ *with s in* $W_{\xi,\lambda}$ *whose normalizing factors are holomorphic at* λ *is linearly independent.*

To proceed further, we shall describe below a decomposition of $W_{\xi,\lambda}$ as a semi-direct product $W_{\xi,\lambda} = W'_{\xi,\lambda}R_{\xi,\lambda}$, where $R_{\xi,\lambda}$ normalizes $W'_{\xi,\lambda}$ and where $W'_{\xi,\lambda}$ is itself a Weyl group.

THEOREM 2. (*i*) *For s in* $W_{\xi,\lambda}$, $\xi(s)\mathfrak{A}_P(s,\xi,\lambda)$ *is scalar if and only if s is in* $W'_{\xi,\lambda}$.

(*ii*) *The operators* $\xi(r)\mathfrak{A}_P(r,\xi,\lambda)$ *for r in* $R_{\xi,\lambda}$ *are linearly independent and span the commuting algebra* $\mathfrak{C}_P(\xi,\lambda)$.

(*iii*) *Consequently, dim* $\mathfrak{C}_P(\xi,\lambda) = |R_{\xi,\lambda}|$.

To give the formula for the dimension of $\mathfrak{C}_P(\xi,\lambda)$ in terms of Plancherel measures, we write $\mu^s(\xi,\lambda) = \Pi\mu_{\xi,\beta}(\lambda)$, where the $\mu_{\xi,\beta}$ are the Plancherel measures of ref. 5 and the product defining μ^s is taken over all reduced $\beta > 0$ such that $s\beta < 0$.

COROLLARY. $\dim \mathfrak{C}_P(\xi,\lambda) = |\{s \in W_{\xi,\lambda} | \mu^s(\xi,\lambda) \neq 0\}|$.

Incidentally it follows from Theorem 3 and Corollary 2 of ref. 1 that μ^s is also given by

$$\mu^s(\xi,\lambda) = c_\xi p_\sigma{}^w(\lambda + \lambda_M),$$

where ξ is imbedded at (σ,λ_M) and w is the element of shortest length in $W(\mathfrak{a}_p)$ whose restriction to $\mathfrak{a}$ is s; $p_\sigma{}^w$ is the principal series Plancherel factor in ref. 2, p. 265, and the constant c_ξ depends on ξ and the imbedding.

We come now to the description of $W'_{\xi,\lambda}$ and $R_{\xi,\lambda}$. Simple examples show that the roots of $\mathfrak{a}$, which are nonzero restrictions to $\mathfrak{a}$ of roots of $\mathfrak{a}_p = \mathfrak{a} + \mathfrak{a}_M$, do not form a root system. There is a conjugation defined on linear functionals on $\mathfrak{a}_p$ as the identity on the $\mathfrak{a}$ part and minus the identity on the $\mathfrak{a}_M$ part. It follows from the fact that M has a discrete series that the conjugate $\bar{\alpha}$ of an $\mathfrak{a}_p$-root α is again an $\mathfrak{a}_p$-root. A root of $\mathfrak{a}$ will be called *useful* if it is the restriction to $\mathfrak{a}$ of some $\mathfrak{a}_p$-root α with $2\langle\alpha,\bar{\alpha}\rangle/|\alpha|^2 \neq 1$. We quote the main theorem of ref. 6.

THEOREM 3. *The useful roots of* $\mathfrak{a}$ *form a (possibly nonreduced) root system* Δ_0 *in a subspace of* $\mathfrak{a}$. *A reflection* p_β *of a root of* $\mathfrak{a}$ *is in* $W(\mathfrak{a})$ *if and only if* $t\beta$ *is useful for some* $t > 0$, *if and only if* β *itself is useful in case* $\mathfrak{g}$ *has no split* G_2 *factors. Moreover,* $W(\mathfrak{a})$ *coincides with the Weyl group of* Δ_0.

As in our work (ref. 3) in the case of minimal parabolics, we fix P and define for λ imaginary

$$\Delta' = \{\text{roots } \beta \text{ of } \mathfrak{a} | p_\beta \in W_{\xi,\lambda} \text{ and } \mu_{\xi,\beta}(\lambda) = 0\}$$

$$R_{\xi,\lambda} = \{p \in W_{\xi,\lambda} | p\beta > 0 \text{ for every } \beta > 0 \text{ in } \Delta'\}.$$

One can show that Δ' is the set of $\mathfrak{a}$-roots β such that $\xi(p_\beta)\mathfrak{A}_P(p_\beta,\xi,\lambda)$ is scalar. Then *Theorem 3* above and Theorem 2 of ref. 1 imply that Δ' is a root system. Let $W'_{\xi,\lambda}$ be its Weyl group. $R_{\xi,\lambda}$ is a group that normalizes $W'_{\xi,\lambda}$, and $W_{\xi,\lambda} = W'_{\xi,\lambda}R_{\xi,\lambda}$.

3. Complementary series

Let λ_0 be imaginary on $\mathfrak{a}$, and consider the question whether there are complementary series representations near the point (ξ,λ_0). Aside from an obvious symmetry condition, the obstruction to this possibility is the degree of reducibility of the representation $U_P(\xi,\lambda_0,\cdot)$; or, put another way, matters depend on the subgroup W'_{ξ,λ_0} discussed above. However, since it is not hard to see that $W'_{\xi,\lambda_0} \subseteq W'_{\xi,0}$, the whole question is subsumed by the case $\lambda_0 = 0$. Recall that $W'_{\xi,0}$ is a Weyl group, and hence as soon as it is nontrivial it contains elements of order 2.

THEOREM 4. *Let ξ be a discrete series representation of M, and suppose $W'_{\xi,0}$ is not the one-element group. If s is any element of order 2 in $W'_{\xi,0}$, then every complex λ that is not purely imaginary and satisfies*

(*i*) $s\lambda = -\bar{\lambda}$

(*ii*) $\left|Re\,\dfrac{\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle}\right| < 1/4$ *for every root α of $\mathfrak{a}$*

is such that $U_P(\xi,\lambda,\cdot)$ is in the complementary series.

Theorem 4 is proved by the technique of section 19 of ref. 4 after an investigation of the poles of the operators $A(\bar{P}:P:\xi:\lambda)$ and the zeros and poles of $\eta(\bar{P}:P:\xi:\lambda)$ for the case of $P = MAN$ with dim $A = 1$. The constant 1/4 can be replaced by 1/2 for the case that P is a minimal parabolic; in the minimal case this is the best possible universal constant. For earlier results in the direction of *Theorem 4*, see ref. 7, Theorem 8 of ref. 4, and Theorem 13 of ref. 5.

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