# Notes of a Course
# Functional Analysis
# Given by William Feller
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(Some lectures near the beginning were given by Christopher Anagnostakis.)

Personal Notes of A. W. Knapp

(Penciled comments in the margin for the first 15 pages were added probably by Benjamin Weiss.)

Functional analysis

Suggested reading:
A. E. Taylor, Functional analysis
Kelley, General topology

# Functional analysis

## Metric spaces

Definition:

A set $S$ of points $x$ and a function $\rho : S \times S \to R$ is a metric space if

1) $\rho(x,y) = 0$ if and only if $x = y$
2) $\rho(x,y) = \rho(y,x)$
3) $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$ triangle inequality

Remark:

$\rho(x,y) \geq 0$ follows from $z = x$ in (3), the fact that $\rho(x,x) = 0$, and the use of symmetry.

Examples:

1) Let $f$ be any strictly increasing function $f : R \to R$ and define a metric on $R$ by $\rho(x,y) = |f(x) - f(y)|$.
2) Discrete metric.
3) Hedgehog and the limiting case. The straight line metric is used across the circle.



Remarks: A subset of a metric space is a metric space under the induced metric. The sum of two metrics is a metric. Geodesics are paths where equality holds in the triangle inequality.

Definition:

Let $S$ and $T$ be metric spaces. Then $f : S \to T$ is continuous if for each $a \in S$ and for any $\epsilon > 0$ there exists a $\delta$ such that $\rho_T(f(x), f(a)) < \epsilon$ whenever $\rho_S(x,a) < \delta$.

Remark: For a function of several variables in $S$, one $\delta$ is required for each variable for the definition to be used.

Definition:

An open ball (or simply a ball about $p$) is a set of the form $\{x \in S \mid \rho(x,a) < r\}$, where $r > 0$.

Theorem:

$\rho(x,y)$ is a continuous function of two variables.

Proof:

Let $\epsilon > 0$ be given. If $\rho(x,a) < \frac{\epsilon}{2}$ and $\rho(y,b) < \frac{\epsilon}{2}$,

then $\rho(x,y) \le \rho(x,a) + \rho(a,b) + \rho(b,y)$

or $\rho(x,y) - \rho(a,b) < \epsilon$.

Similarly $\rho(a,b) - \rho(x,y) < \epsilon$.

Hence $|\rho(x,y) - \rho(a,b)| < \epsilon$. Q.E.D.

Definition:

$\{x_n\}$ is a Cauchy sequence if for any $\epsilon > 0$ there is an $N$ such that $\rho(x_n, x_m) < \epsilon$ whenever $n, m \ge N$. A metric space is complete if every Cauchy sequence has a limit.

The completion of a metric space is the set of all equivalence classes of Cauchy sequences with the natural metric and the canonical identification of $\{x\}$ with $x$. The completion is complete.

Definition:

Let $S$ be a metric space. Then $A \subset S$ is dense in $S$ if $\overline{A} = S$ or if every ball in $S$ contains a point of $A$. $A$ is dense in a subset $T$ of $S$ if $\overline{A} \supset T$.

Remark: If $A$ is dense in $S$, then $A$ is dense in every open subset of $S$. Hausdorff pg 154

Definition: $A$ is nowhere dense if $\overline{A}$ contains no open set. except $\phi$. The rationals are dense in R otherwise! nowhere dense in R.

Example:

Take the unit interval and remove an open interval of length $L_1$ from the middle. By induction remove at the $n$th step open intervals from each remaining interval in such a way that the points removed have length $L_n$. The limiting set is closed and has no interval of positive length; hence it is nowhere dense. Now, filling the blank spaces ^of equal size at any stage^ with replicas of the limiting set, we obtain inductively a set which is everywhere dense. It has measure 1 or 0 according as the limiting set had positive measure ~~less than one~~ or measure equal to ~~one~~ zero. We shall prove that in any case not every point of the unit interval is in the resulting set.

$L_n = \frac{1}{q^n}$

Outer measure $\frac{q-3}{q-2} = \frac{16}{17}$ if $q = 19$.

We insert disjoint mble sets — total measure $\frac{1}{q-2}$.

or according as $q \leq 3$ or $q > 3$.

Baire category theorems

Definition:

Let $S$ be a metric space. A subset $A$ of $S$ (possibly $S$ itself) is said to be of the first category in $S$ if $A$ is ^contained in^ the denumerable union of nowhere dense sets of $S$. Otherwise, $A$ is said to be of the second category.

Note: Subsets of sets of first category are of first category, and supersets of sets of second category are of second category.

BAIRE THEOREM

Theorem:

Hence the complement of a set of first category is dense.

Every neighborhood of a complete metric space is of second category in the space.

Proof:

By the note it is sufficient to prove the result for an open set $G$ in $S$. Suppose $G \subset \bigcup_{n=1}^{\infty} A_n$, where $A_n$ is nowhere dense for each $n$. $\bar{A}_1$ contains no ball of $S$ and hence no ball of $G$. Find a ball $S_1$ in $G - \bar{A}_1$ such that $\delta(S_1) < 1$. Since $\bar{A}_2$ does not contain $S_1$, there exists an $s_1 \in S_1$ not in $\bar{A}_2$. Let $S_2$ be a ball about $s_1$ such that $\bar{S}_2 \subset S_1 \cap (G - \bar{A}_2)$ and $\delta(S_2) < \frac{1}{2}$. Then $S_2 \cap (\bar{A}_1 \cup \bar{A}_2) = 0$. Proceding inductively we obtain a decreasing sequence of sets $S_1 \supset S_2 \supset \ldots$ and a Cauchy sequence of points $\{s_n\}$. Since $S_n \cap (\bar{A}_1 \cup \ldots \cup \bar{A}_n) = 0$, $(\cap S_n) \cap (\cup A_n) = 0$. But $\cup A_n \supset G$ and $G \supset \cap S_n$. Therefore $\cap S_n = 0$. Let $s$ be the limit of $\{s_n\}$. If $s \notin S_n$ for some $n$, then $s \notin \bar{S}_{n+1}$ and $s$ is not a limit point of $\bar{S}_{n+1}$. Hence $s \in S_n$ for every $n$, $\cap S_n \neq 0$, and the theorem is proved.

Lemma:

Let $X$ be any topological space ~~in which every~~ ? neighborhood is of second category. Let $A \subset X_1 \subset X$. If $A$ is of second category in $X$, then it is of second category in $X_1$.

Proof:

Write $A \subset \cup B_n$ and suppose that $B_n$ is nowhere dense in $X_1$ for every $n$. Then if $C_n = A \cap B_n$, $A = \cup(C_n)$ and $C_n$ is nowhere dense in $X_1$ for every $n$. Since $A$ is of second category in $X$, for some $k$ there exists a neighborhood (in $X$) $N \subset \bar{C}_n$, where the closure is taken in $X$. Now $N \cap X_1 = 0$, since otherwise $N$ is a (non-empty) neighborhood contained in the $X_1$ closure of $C_n$. Therefore, $N \cap C_n = 0$ because $C_n \subset A \subset X_1$. But $N \subset \bar{C}_n$ so that

$N \subset \bar{C}_n - C_n$. In other words, every point of $N$ is a limit point of $C_n$ and no point of $N$ is in $C_n$. Therefore no point of $N$ is an interior point (in $X$), and $N$ cannot be a neighborhood. Q.E.D.

Corollary:

If $A$ is of first category in $X_1$, then it also is of first category in $X$.

Corollary:

If $A$ is of second category in $X$, then it is of second category in itself.

Remark:

We may thus say that $A$ is of second category without ambiguity.

Examples:

1. Let $g(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ 1/q & \text{if } x = p/q, \text{ rational} \end{cases}$

   Then $g(x)$ is discontinuous at rational points and continuous at irrational points. It is the everywhere limit of continuous functions (construct spikes).

2. Let $k(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ 1 & \text{if } x \text{ is rational.} \end{cases}$

   Then
   $$k(x) = \lim_n \lim_k \left[\cos(\pi n!\, x)\right]^{2k}.$$
   We shall show shortly that $k$ is not obtainable as a single limit of continuous functions since it is discontinuous everywhere.

Remark:

The denumerable union of sets of first category is again of first category. Any set which is the complement of a set of first category (in a space in which every neighborhood is of second category) is a dense set.

Theorem:

Let $X$ be any topological space in which every neighborhood of the space is of second category. Suppose $f_n$ is a sequence of continuous functions with domain $X$ and range $R$ and with the property that $f_n$ converges to a function $f$ pointwise. Then the set of discontinuities of $f$ is of first category.

Remarks:

The oscillation of a function in a neighborhood is the difference of the supremum and the infimum. The oscillation of $f$ at $x$ is said to be greater than or equal to $\alpha$ if it is greater than or equal to $\alpha$ in every neighborhood of $x$. A function is discontinuous at a point if and only if it has positive oscillation at the point.

Proof of theorem:

Let $\epsilon > 0$ be any positive real number and let $N \subset X$ be any neighborhood of $X$. As a function of $\epsilon$ and $N$, set

$$A_{mn} = \{x \mid x \varepsilon N,\ |f_m(x) - f_n(x)| \le \epsilon/6\}.$$

Since $f_m - f_n$ is continuous, $A_{mn}$ is closed in $N$. Put

$$A_m = \bigcap_{n=m}^{\infty} A_{mn}.$$

Then $A_m \subseteq \{x \mid x \varepsilon N,\ |f_m(x) - f(x)| \le \epsilon/6\}$ and $A_m$ is closed, being the intersection of closed sets. By the

go to def'n of $A_m$.

definition of limit, $\cup A_m = N$. But $N$ is of second category so that not all of the $A_m$ are nowhere dense. Thus there exists a $k$ such that $\bar{A}_k \supset N_1$, where $N_1$ is a subneighborhood of $N$. Since $A_k$ is closed in $N$, $A_k \supset N_1$. Now for all $x \varepsilon N_1$ we have

$$|f_k(x) - f(x)| \le \epsilon/6.$$

Fix $x = x_0$. By continuity of $f_k$ there exists a subneighborhood $N_2$ of $N_1$ containing $x_0$ such that for all $y$ in $N_2$

$$|f_k(x_0) - f_k(y)| \le \epsilon/6 .$$

By the triangle inequality we have

$$|f(x_0) - f(y)| \le \epsilon/2$$

for all $y$ in $N_2$. Again by the triangle inequality we find that for all $x, y \in N_2$

$$|f(x) - f(y)| \le \epsilon .$$

Hence for any neighborhood $N$ in the space there is a subneighborhood $N_2$ in which the oscillation of $f$ is no greater than $\epsilon$. Alternatively the set $P_\epsilon$ of points in $X$ for which the oscillation of $f$ exceeds $\epsilon$ is nowhere dense. Since the set $P$ of points of discontinuity of $f$ satisfies $P = \bigcup_{n=1}^{\infty} P_{1/n}$,

$P$ is of first category. Q.E.D.

Lemma:

Let $X$ be a topological space in which every neighborhood is of second category, and let $E$ be a set of first category in $X$. Then every neighborhood of $\tilde{E}$ (with the induced topology) is of second category in $\tilde{E}$.

Proof:

Let $N'$ be a neighborhood in $\tilde{E}$. Then $N' = N - E$, where $N$ is a neighborhood in $X$. If $N'$ is of first category in $\tilde{E}$, then it is of first category in $X$ and $(N-E) \cup E$, being the union of two sets of first category in $X$, is of first category in $X$. Thus $N$ is of first category in $X$, which is impossible. Q.E.D.

Definition:

If $A \subset X$ and $f$ is defined on $X$, $f/A$ is the restriction of $f$ to $A$.

Definition:

A function $f$ has the Baire property if there exists a set $E$ of first category such that $f/\tilde{E}$ is continuous on $\tilde{E}$.

Remark:

The limit of continuous functions has the Baire property. So does the characteristic function of the rationals.

Let $\mathcal{B}$ be the collection of all functions $f$ with the Baire property.

Theorem:

Let $X$ be a topological space in which every neighborhood is of second category. Suppose $f_n \in \mathcal{B}$ and $f_n \to f$ pointwise. Then $f \in \mathcal{B}$.

Proof:

If $f_n \in \mathcal{B}$, then there exists a set $E_n$ of first category such that $f_n / \tilde{E}_n$ is continuous. Put $E = \cup E_n$. Then $E$ is of first category. Since $E \supset E_n$, $f_n / \tilde{E}$ is continuous for every $n$. By the lemma every neighborhood of $\tilde{E}$ is of second category in $\tilde{E}$. Thus by the preceding theorem the discontinuities of $f/\tilde{E}$ form a set of first category in $\tilde{E}$ (and hence in $X$). Call the set of discontinuities of $f/\tilde{E}$ $E_0$. Then $f/(\tilde{E} - E_0)$ is continuous, or equivalently $f/\widetilde{(E \cup E_0)}$ is continuous. Since $E \cup E_0$ is the union of two sets of first category, it is of first category and $f$ is in $\mathcal{B}$.

Q.E.D.

Corollary:

$\mathcal{B}$ is closed under sums, differences, products, multiplication by scalars, and limits.

Definition:

Let $A \Delta B$ denote $(A-B) \cup (B-A)$, the symmetric difference of $A$ and $B$. (A point is in $A \Delta B$ if and only if it belongs to exactly one of the component sets.)

Note:

$A \Delta B = C$ implies $A \Delta C = B$. For

$A \Delta C = A \Delta A \Delta B = 0 \Delta B = B$.

Definition:

A set $A$ in a space $X$ has the Baire property if there exists an open set $\Omega$ such that $A \Delta \Omega = B$ is of first category.

Remarks:

1. Equivalently $A$ has the Baire property if there is a set $B$ of first category such that $A \Delta B$ is open.
2. If $A$ is open, take $\Omega = A$. Then $A$ has the Baire property.
3. If $A$ is closed, then $A$ minus its interior is nowhere dense and is thus of first category. Take $\Omega$ to be the interior of $A$. We thus see that every closed set has the Baire property.

Theorem:

Let $\mathbb{B}$ be the class of all Baire sets in a topological space $X$ in which every neighborhood is of second category. If $A \in \mathbb{B}$, then $\tilde{A} \in \mathbb{B}$. If $A_n \in \mathbb{B}$, then $\bigcup_{n=1}^{\infty} A_n \in \mathbb{B}$. Thus $\mathbb{B}$ is a Borel field of sets.

Proof:

Complements:

Let $A \in \mathcal{B}$ and let $\Omega$ be an open set for which $A \Delta \Omega$ is of first category. Let $\Pi = \widetilde{(\overline{\Omega})}$. We shall show that $\Pi - \tilde{A}$ and $\tilde{A} - \Pi$ are of first category. First

$$\Pi - \tilde{A} = \Pi \cap A \subset \tilde{\Omega} \cap A = A - \Omega \subset A \Delta \Omega$$

and $A \Delta \Omega$ is of first category. Second

$$\begin{aligned} \tilde{A} - \Pi = \tilde{A} \cap \overline{\Omega} &= [\tilde{A} \cap (\overline{\Omega} - \Omega)] \cup [\tilde{A} \cap \Omega] \\ &\subset (\overline{\Omega} - \Omega) \cup (\Omega - A) \\ &\subset (\overline{\Omega} - \Omega) \cup (A \Delta \Omega) \end{aligned}$$

and each of these sets is of first category.

Unions:

Let $A_n \in \mathcal{B}$ and let $\Omega_n$ be open such that $A_n \Delta \Omega_n$ is of first category. Let $\Pi_n = \widetilde{(\overline{\Omega})}$ and put $\Pi = \cup \Pi_n$. Then

INCORRECT PROOF.

$$\Pi - A \subset \cup(\Pi_n - A_n)$$

and

$$A - \Pi \subset \cup(A_n - \Pi_n)$$

Since $\Pi_n - A_n$ and $A_n - \Pi_n$ are both of first category, the result follows. Q.E.D.

Remark:

It can be shown that $f \in \mathcal{B}$ if and only if $\{x \mid f(x) > a\} \in \mathcal{B}$ for every $a$.

Comments on point-set topology

Definition:

A topological space is a set $X$ with a class of open sets satisfying

1) $\emptyset$ and $X$ are open
2) $A \cap B$ is open whenever $A$ and $B$ are
3) $\cup A_\alpha$ is open whenever $A_\alpha$ is open.

Definition:

A space is Hausdorff if any two distinct points can be separated by open sets

Definition:

A family of open sets $\mathcal{B}$ forms a base at $p$ if $p \in B$ for every $B \in \mathcal{B}$ and if whenever $p \in U$ and $U$ is open, there is a $B \in \mathcal{B}$ such that $B \subset U$. A base for a topological space is a family of open sets such that there is a subfamily depending on $p$ which is a base at $p$.

Remark:

A class $\mathcal{B}$ of sets satisfying $\cup \mathcal{B} = X$ is a base if and only if for any pair $B_1, B_2 \in \mathcal{B}$ and for any $x \in B_1 \cap B_2$, there is a $B_0 \in \mathcal{B}$ such that $x \in B_0 \subset B_1 \cap B_2$.

Definition:

A space is compact if every covering by open sets has a finite subcovering.

Proposition:

In a Hausdorff space, any pair of compact sets can be separated by open sets.

Proof:

For each fixed $p$ in $A$, separate $p$ from each point in $B$, refine the covering of $B$, take the union of those sets as a set covering $B$ and the intersection of the corresponding sets as a set covering $p$. Doing so for every $p$ gives a covering of A which can be refined. Form the union of these sets and the intersection of the corresponding sets covering B. Then A and B are separated.

Equivalent defintions of continuity of $f: X \to Y$:

1) For any $x \in X$ and for any open set $V$ with $f(x) \in V$ there is a $U \subset X$ such that $x \in U$ and $f(U) \subset V$.
2) The inverse image of every open set is open.
3) The inverse image of every closed set is closed.

Remarks:

The continuous image of a compact set is compact. In a Hausdorff space a compact set is closed.

Definitions:

1) A space is sequentially compact if every sequence has a cluster point.
2) A space is sequentially separable if it has a countable dense set.
3) A space is separable if it has a countable base.
4) A space is locally compact if every point has a compact neighborhood.
5) A space in which every open covering has a countable subcovering is said to have the Lindelöf property.

Examples:

1) The hedgehog topology has a countable base at each point but not a countable base.
2) The topology on the real line which has intervals $[a, b)$ as a base is sequentially separable but not separable.

Theorem:

A $T_1$ space with the Lindelöf property is compact if and only if every denumerable set has a cluster point.

Proof:

Suppose $X$ is not compact. There exists an open covering $\{U_k\}$ which has no finite subcovering. We may assume $\phi \notin \{U_k\}$ and that the sequence of sets is irredundant. Choose $x_1 \in U_1$ and by induction $x_n \in U_n - (U_1 \cup U_2 \cup \dots \cup U_{n-1})$. Then $\{x_n\}$ has no cluster point. A similar argument establishes the converse.

Remark:

A space with a countable base is Lindelöf.

Theorem:

In a metric space, compact and sequentially compact are equivalent.

Proof:

We prove that a sequentially compact metric space has a countable base. By induction cover the space with as many disjoint spheres of radius $1/3$ as possible. Finitely many suffice by sequential separability. Expand the spheres to have radius 1; they then cover the space. Repeat the argument for radius $1/3n$ for every $n$. This procedure gives a countable base.

Remarks:

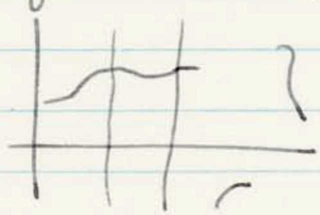
Every locally compact space has a one-point compactification in which the open neighborhoods of $\infty$ are the complements of the ~~finite~~ compact sets. Spaces can be compactified in other ways; for example, the real line has a natural two-point compactification.

Definition:

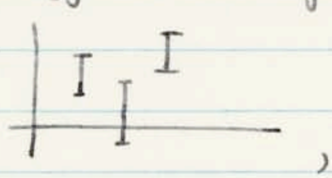
The Tychonoff topology for a Cartesian product of spaces is the topology with base consisting of all finite intersections of sets which are restricted only in one coordinate (and there the restriction is to an open set).

Standard probability space:

Let $\Omega$ be the Cartesian product of the real line with itself under an indexing by the set $\{t \mid t > 0\}$. Points $\omega$ in the space are functions $\omega(t)$. The result is a replica



of the real line for every $t > 0$. If intervals form a base for the topology of the real line, a base for the product topology consists of sets like



where the functions $\omega(t)$ are restricted to pass through the finite number of slits but are otherwise unrestricted.

Banach space

Linear space (over reals or complex numbers), strictly a linear set

Group under addition

Scalar mult. $\lambda x$

Rules $\lambda(x+y) = \lambda x + \lambda y$

$\lambda x = x \lambda$

$(\alpha+\beta)x = \alpha x + \beta x$

$1x = x$

Normed linear space

Topology to be formed will be preserved under $x \to x+a$. Metric, when present, is to be preserved.

Definition

$$\rho(x+a, y+a) = \rho(x,y) \iff \|x\| = \rho(x,0)$$

Normed linear space is a pre-Banach space

Banach space = Complete normed linear space

Example:

Continuous functions on a compact space

$\|f\| = \sup |f(s)|$

$\|f_n - f_m\| \to 0 \Rightarrow$ uniform convergence. Hence Banach space

Example:

Sequences $x = \{x_1, x_2, \dots\}$

$\|x\| = \sup |x_j|$

Linear functionals – linear mapping of space into scalars (now the reals)

Notation: $x^*x$

$$x^*(\alpha x + \beta y) = \alpha x^* x + \beta x^* y$$

$x^* + y^*$ and $cx^*$ defined in standard way

Norm on space

$$\|x^*\| = \sup_x \frac{|x^*x|}{\|x\|} \quad (x \text{ runs through unit sphere})$$

Gives pre-Banach space

Take $\{x_n^*\}$:

For fixed $x$ $x_n^* x$ is Cauchy and $\to f(x)$

Check that limit is a functional

Call limit $x^*x$

Prove $\|x^*\| = \lim \|x_n^*\|$ and space is Banach space

This space is called the dual space

Example:

If $\|x\| = \sup |x_j|$, ← finite-dimensional then $\|x^*\| = \Sigma |x_j^*|$. Spaces are not isometric, different metric

Dual space is space of measures, set functions

Typical functionals on $\{f\}$

$$\int_0^1 f(s)\,ds, \quad f\left(\frac{2}{\pi^2}\right)$$

↑ norm is one

General form will be $\int f\, d\mu$, fixed measure.

$2/\pi^2$ carries weight one, zero measure elsewhere.

Compact not separable: Cartesian product of reals with itself, function space

Take $A$ to be the set of functions which are 1 on all but a finite number of points, where they are zero. $f=0$ is acc. pt, but no sequence in $A$ converges to it.

Idea is that points are not $G_\delta$ sets.

Adjoint Banach space

$\|x^*\| = \dfrac{\sup|x^*x|}{\|x\|}$ Bounded linear functional has bounded norm

Existence of $x^*$ to be shown. This follows directly from

Hahn-Banach Theorem:

If $L \subset X$ is a closed linear (complete) space, and if $l^*$ a bounded linear functional on $L$, $\exists\, x^*$ such that $\|x^*\| = \|l^*\|$ and $x^* l = l^* l$ for $l \in L$.

Corollary:

For each $x \in X$ with $\|x\|=1$ there is an $x^*$ such that $x^* x = 1$, $\|x^*\| = 1$.

Corollary:

If $L \subset X$ is closed and $x \notin L$, there is a $y^* \in X^*$ such that $y^* L = 0$ and $y^* x \neq 0$.

Proof: Let $L_1 = \{z \mid z = l + \alpha x,\ l \in L,\ \alpha \text{ real}\}$. Define $l^* x = \alpha$. Then $l^*$ has the required properties. Extend it.

Codimension 1 means: $L_1 = X$ and functional is unique up to mult. by const.

$\{x \mid y^* x = 0\}$ is of codim 1. If $y^* a = \alpha$ and $y^* b = \beta$, then

$y^*(\beta a - \alpha b) = 0$. So $\beta a - \alpha b \in L$.

Value of functional gives linear manifold of codim 1 and everything in space is a translate of this manifold. Intersection of these things is as in finite spaces.

Note: Closed linear manifold A is dense if $y^*A=0$ implies $y^*=0$.

Remark: In Euclidean space, normal vector to hyperplane is really an element of the dual space. In an arbitrary Banach space no inner product exists; so functional interpretation is only possible one.

For any L the set $\{x^* \mid x^*L=0\}$ is the set of all "normals".

This set is called $L^\perp$.

When X and $X^*$ are identifiable, point in $X^*$ is normal. This is situation in Hilbert space

Example:

$x=(x_1,\dots,x_n)$ $\qquad \|x\|=\sup|x_j|$

$x^*=(x_1^*,\dots,x_n^*)$ $\qquad \|x\|^*=\sum|x_j^*|$ $\qquad$ X and $X^*$ not identifiable

More generally can take

$$\sum|x_j|^p=\|x\|^p, \quad p>1$$

$$\sum|x_j^*|^q=\|x^*\|^q, \quad \frac{1}{q}+\frac{1}{p}=1$$

$p=2$ is Euclidean space

Case above is limit as $p\to\infty$

In above example can take infinite sequences, finitely many non-zero, complete space. In $p, q$ space get obvious sequences. But in other case get null sequences just by looking at metric. If space is extended, null sequences form closed linear manifold.

If $\sum|x_j^*|<\infty$, define $x^*x=\sum x_j^*x_j$ and $|x^*x|\le\|x\|\sum|x_j^*|$ and $\|x^*\|\le\sum|x_j^*|$. To get equality take point with coordinates proportional to $x_j^*$

But these functionals are all defined on closed linear manifold

Hence $\exists$ functional vanishing on L and $\neq 0$ somewhere else.

Another example is all convergent sequences

We have $\mathcal{B}_0$ null seq

$\mathcal{C}$ convergent seq.

$\mathcal{B}$ whole space

$x^*$ projection onto $\sum x_j^* x_j$ finite vanishes on large set and is defined on smaller class than null sequences

$$\mathcal{B}_0 \subset \mathcal{C} \subset \mathcal{B}$$

Adjoint to $\mathcal{B}_0$ is as above because its value is determined on basis $(1, 0, - -)$

$(0, 1, - -)$

$$\mathcal{B}_0^* = \{ \sum |x_j^*| = \|x^*\| \}$$

$\mathcal{B}_0$ is space of codim one in $\mathcal{C}$ obviously

Adjoint to $\mathcal{C}$ is $\mathcal{B}_0^* + \{\text{limits}\}$

$x^*_\infty \; x_\infty$

measure limit

Adjoint to $\mathcal{B}$ exists by Hahn Banach theorem. No functional on whole space is known.

Adjoint to adjoint space $X^{**}$

$x^{**} x^*$ If $x$ is fixed and $x^*$ runs through $X^*$, get functional on $X^*$ $f(x, x^*) = x^* x$ Linearity is obvious

Every $x$ can be interpreted as an $x^{**}$ algebraically. In norm

$|f(x, x^*)| \le \|x^*\| \|x\|$. From preceding, $\exists$ $x, x^*$ unit vectors such that

$|f(x, x^*)| \ge 1 - \epsilon$. Hence norm is preserved. $X \subset X^{**}$

In $l^p$ spaces above $X = X^{**}$, reflexive space. But with $l^\infty$, $B = B_0^{**}$. Show this. Sequence

$$X \subset X^* \subset X^{**} \subset \dots$$

is strictly increasing.

Normed linear space

Lemma: $T$ additive: $X \to Y$ is continuous iff $T$ is continuous at $0$.

Proof:

$\Rightarrow$ trivial

$\Leftarrow$ $\epsilon > 0$ means $\exists \delta$ $\|x\| < \delta$ implies $\|Tx\| < \epsilon$.

Let $x_0 \in X$. If $\|x_0 - y\| < \delta$, then $\|T(x_0 - y)\| < \epsilon$

$\|T(x_0) - Ty\| < \epsilon$

so continuity at $x_0$

[This even gives uniform continuity]

Lemma: If $T$ homogeneous ($T(\alpha x) = \alpha T x$): $X \to Y$, then $T$ is continuous at $0$ iff $T$ is bounded.

($\exists A$ for which $\|Tx\| \le A\|x\|$.) $A \ge 0$

Proof:

If $T$ is bounded, take $\|x\| < \frac{\epsilon}{A}$; then $\|Tx\| < \epsilon$. (or $A = 0$)

Suppose $T$ continuous at $0$. If not bndd, can find of $x_n \in X$ such that $\|Tx_n\| \ge n\|x_n\|$. Then

$$\left\| T \frac{1}{n} \frac{x_n}{\|x_n\|} \right\| = \left\| \frac{1}{n\|x_n\|} Tx_n \right\|$$

$$= \frac{1}{n\|x_n\|} \|Tx_n\|$$

$$\ge 1.$$

But $\left\| \frac{1}{n} \frac{x_n}{\|x_n\|} \right\| = \frac{1}{n}$. So $T$ is not continuous at $0$.

Thm: $T$ linear: $X \to Y$, then $T$ is continuous iff $T$ is bounded

Def: $\|T\| = \inf \{A : \|Tx\| \le A\|x\| \text{ for } x \in X\}$ if $\|T\| < \infty$

$$= \sup_{x \ne 0} \frac{\|Tx\|}{\|x\|} = \sup_{\|x\| = 1} \|Tx\|$$

Let $X$ be a linear space. Suppose two norms $\| \|_1, \| \|_2$ induce the same topology. This means identity map

$$\underset{\| \|_1}{i : X} \to \underset{\| \|_2}{X} \quad \text{is continuous}$$

Cont. and linear means

$$\|x\|_2 \le A\|x_1\|$$

Symmetrically $\quad \|x_1\| \le B\|x_2\|$

This result requires linear structure

Dual spaces:

Dual is all bndd transformations of $X \to C$

$X^{**} \supseteq X$. linear map is one-one since $x^*(x) = 0 \ \forall x^* \Rightarrow x = 0$ by HBT.

$$(\varphi x)(x^*) = x^*(x)$$

Isometric: $\|\varphi x\| = \|x\|$

$$\|\varphi x\| = \sup_{\|x^*\|=1} \|\varphi x(x^*)\| = \sup_{\|x^*\|=1} \|x^*(x)\| \le \|x\|$$

Now $\exists\, y^* \in X^*$ for which

$y^*(x) = \|x\|$ and $\|y^*\| = 1$ by HBT.

$$\|\varphi x\| \ge \|y^*(x)\| = \|x\|$$

Hence $\varphi$ is an isometry.

$\varphi X$ is a closed subspace of $X^{**}$

Suppose $\varphi X \ni \varphi x_\nu \longrightarrow x_0 \in X^{**}$

$$\|\varphi x_\nu - \varphi x_\mu\| \to 0$$

Isometry and linear $\|x_\nu - x_\mu\|$

So $x_\nu$ is Cauchy and $x_\nu \to y_0 \in X$.

$\varphi$ is continuous so $\varphi x_\nu \to \varphi y_0$.. So $x_0 = \varphi y_0 \in \varphi X$.

Higher dual space

$$X \subset X^{**} \subset \dots \subset X^{2n*} \subset X^{2n+2*} \subset \dots$$

$$X^{*} \subset X^{***} \subset \dots \subset X^{2n+1*} \subset X^{2n+3*} \subset \dots$$

Theorem:
Either $X = X^{**}$ and hence $X = X^{2n*}$, $X^{*} = X^{2n+1*}$

or $X \subsetneq X^{**}$ and then $X^{2n*} \subsetneq X^{(2n+2)*}$, $X^{(2n+1)*} \subsetneq X^{(2n+3)*}$

Proof:
First half is easy. Suppose $X \neq X^{**}$. Claim $X^{*} \neq X^{***}$.

Suppose $\varphi_1$ is imbedding of $X^{*} \to X^{***}$, $\varphi$ is imbedding of $X$ in $X^{**}$

Find $a \neq 0$, lin. functional on $X^{**}$ such that $a\varphi X = 0$, $a \in X^{***}$. Will

show $a \notin \varphi_1 X^{*}$. Suppose

$$\varphi_1 b = a \quad \text{where } b \in X^{*}$$

For $x \in X$, $a(\varphi x) = 0$. Then

$$\begin{aligned} 0 = a(\varphi x) &= (\varphi_1 b)(\varphi x) \\ &= (\varphi x)(b) \\ &= b(x) \end{aligned}$$

So $b = 0$. Q.E.D.

Theorem of uniform boundedness:

$T_\alpha$ a family of continuous linear transformations $X \to Y$, $\alpha \in A$.

Suppose $\|T_\alpha x\| \leq N_x < \infty$ for $x \in X$. This means

$$\sup_\alpha \|T_\alpha x\| < \infty \text{ at any } x.$$

Then $\|T_\alpha\| \leq N < \infty$ for all $\alpha \in A$.

Proof:

Look at $\{ \|T_\alpha x\| \le n\|x\| \}$. This is closed for fixed $\alpha, n$.

Form intersection over $\alpha \in A$

Form union over $n$.

Given $x \neq 0$, choose $n$ such that $n\|x\| > N_x$. Then $x$ is in an intersection. And $0$ is in.

So $\bigcup_n \bigcap_{\alpha \in A} \{ \|T_\alpha x\| \le n\|x\| \} = X$.

By Baire category thm.,

$$\{x \mid \|x_0 - x\| \le r_0\} = \underset{r_0 \neq 0}{B(x_0, r_0)}$$

$$\subseteq \bigcap_{\alpha \in A} \{ \|T_\alpha x\| \le n_0\|x\| \}.$$

This means $\|x_0 - x\| \le r_0$ implies

$\|T_\alpha x\| \le n_0(\|x_0\| + r_0)$ for all $\alpha$

~~On~~ some open $\alpha$, transformations are uniformly bounded.

Now for $y \in X$, $y \neq 0$, then

$$T_\alpha y = \frac{\|y\|}{r_0} T_\alpha \frac{y}{\|y\|} r_0 \quad \text{by linearity}$$

$$= \frac{\|y\|}{r_0} T_\alpha \Big(\underbrace{x_0 + \frac{y}{\|y\|} r_0}_{\substack{\text{satisfies} \\ \|x_0 - x\| \le r_0}}\Big) - \frac{\|y\|}{r_0} T_\alpha x_0$$

$$\|T_\alpha y\| \le 2 \frac{\|y\|}{r_0} n_0(\|x_0\| + r_0)$$

fix $x_0$ also

$\|T_\alpha y\| \le \|y\|$ const. for all $y \in X$ and $\alpha$

and $\|T_\alpha\| \le$ const

Filters

Def 1: If $X$ is any set and $\mathcal{F} \subseteq PX$, then $\mathcal{F}$ is a filter if

1) $\phi \notin \mathcal{F}$, $\mathcal{F}$ is not empty.
2) If $A \in \mathcal{F}$ and $A \subseteq B$, then $B \in \mathcal{F}$.
3) If $A_\nu \in \mathcal{F}$ finitely many, then $\bigcap_{1 \le \nu \le m} A_\nu \in \mathcal{F}$

Examples 2:
1) Neighborhoods $\mathcal{U}_x$, where $x \in X$, a topological space.
2) All sets containing a fixed point
3) $X$ itself.

Def 3: If $\mathcal{F}_1$ and $\mathcal{F}_2$ are filters on $X$, then $\mathcal{F}_1$ is finer than $\mathcal{F}_2$ if $\mathcal{F}_1 \supset \mathcal{F}_2$.

Remarks: 1) The filters on $X$ are partially ordered.
2) If $\mathcal{F}_\alpha$ are filters, then $\bigcap_{\alpha \in A} \mathcal{F}_\alpha$ is a filter and this $\bigcap$ is $\inf_\alpha \mathcal{F}_\alpha$ under partial ordering

Lemma 4: $\mathcal{S} \subset PX$ is contained in a filter if and only if $\mathcal{S}$ has the finite intersection property.

Proof: Necessity is trivial. If $\mathcal{S}$ has f.i.p., let $\mathcal{F}$ be all sets $E$ such that $E$ contains a finite intersection of sets of $\mathcal{S}$; this is obviously a filter.

Corollary 5: If $\mathcal{F}$ is a filter on $X$ and $A \subseteq X$, then there exists a filter $\mathcal{F}'$ containing $\mathcal{F}' \supset \mathcal{F} \cup \{A\}$ if and only if $E \in \mathcal{F} \Rightarrow A \cap E \neq \phi$

Corollary 6: If $\mathcal{F}_\alpha$ is a filter for $\alpha \in A$, then there is a filter $\mathcal{F}'$ containing all $\mathcal{F}_\alpha$, $\alpha \in A$, iff $E_\nu \in \mathcal{F}_{\alpha_\nu} \Rightarrow \bigcap_{1 \le \nu \le m} E_\nu \neq 0$.

Theorem 7: The class of filters on a set $X$ is inductive.

Proof: Let $\mathcal{F}_\alpha$, $\alpha \in A$, be totally ordered. Apply Corollary 6.

Definition 8: If $\mathcal{F}$ is a filter and $\mathcal{B} \subset \mathcal{F}$, then $\mathcal{B}$ is a basis of $\mathcal{F}$ if $\mathcal{F} = \{E \mid E \supset B \text{ for some } B \in \mathcal{B}\}$.

Basis generates filter uniquely

<u>Lemma 9</u>: $\mathcal{B} \subset PX$ is a basis for some filter if and only if

1) $\phi \notin \mathcal{B}$, $\mathcal{B} \neq \phi$.

2) If $B_1, B_2 \in \mathcal{B}$ then $\exists\, B_3 \in \mathcal{B} \ni B_1 \cap B_2 \supset B_3$.

Proof: obvious

<u>Notation</u>: The filter generated by $\mathcal{B}$ will be called ${}^F\mathcal{B}$.

<u>Examples 10</u>:

1) Open neighborhoods of a point are a basis of $\mathcal{U}_x$.

2) Let $N = \{1, 2, \dots\}$. The sets $\{n, n+1, \dots\}$ form a filter, the complements of finite sets; called the Fréchet filter.

3) $\mathcal{F}$ is a basis of itself.

<u>Lemma 11</u>: If $\mathcal{F}_i$ is a filter with basis $\mathcal{B}_i$, $i = 1, 2$, then $\mathcal{F}_1 \supset \mathcal{F}_2$ if and only if to every set $B$ in $\mathcal{B}_2$ there exists a $B' \in \mathcal{B}_1$ such that $B' \subset B$.

Proof: obvious

<u>Definition 12</u>: An <u>ultra filter</u> on $X$ is a maximal filter.

Example: The class of sets containing some point $x_0 \in X$.

<u>Theorem 13</u>: Every filter $\mathcal{F}$ is contained in an ultra filter.

Proof: Let $\mathcal{C}$ be the class of filters containing $\mathcal{F}$; this class is inductively ordered. By Zorn's Lemma, there exists a maximal element; this maximal element is a maximal filter.

<u>Lemma 14</u>: If $\mathcal{F}$ is an ultra filter and $A \cup B \in \mathcal{F}$, then $A \in \mathcal{F}$ or $B \in \mathcal{F}$.

Proof: Suppose $A, B \notin \mathcal{F}$. Let $\mathcal{G} = \{E \mid A \cup E \in \mathcal{F}\}$. Then $\mathcal{G}$ is a filter, $B \in \mathcal{G}$, $\mathcal{F} \subset \mathcal{G}$, $B \notin \mathcal{F}$. Contradiction.

Lemma 15: If $\mathcal{B}$ is a basis for a filter $\mathcal{F}$ over $X$ and if $A \subseteq X$ implies $A \in \mathcal{B}$ or $\tilde{A} \in \mathcal{B}$, then $\mathcal{F}$ is an ultrafilter.

Proof:

Suppose $\mathcal{F}' \supset \mathcal{B}$ is a filter. If $A \in \mathcal{F}'$, then $\tilde{A} \notin \mathcal{F}'$. Hence $A \in \mathcal{B}$. So $\mathcal{F}' = \mathcal{B}$, $\mathcal{F} = \mathcal{B}$, and $\mathcal{F}$ is maximal.

Note: Image of filter need not be a filter

Lemma 16: Let $\mathcal{B}$ be a filterbasis on $X$ and let $f: X \to Y$. Then $f(\mathcal{B})$ is a filter basis.

Proof:

1) $\phi \notin f(\mathcal{B}) \neq \phi$ obviously

2) $f(A_1), f(A_2) \in f(\mathcal{B})$

Find $A_3 \subseteq A_1 \cap A_2$. Then $f(A_3) \subset f(A_1) \cap f(A_2)$.

Lemma 17: If $\mathcal{B}$ is a basis of an ultrafilter $\mathcal{F}$ over $X$ and if $f: X \to Y$, then $f(\mathcal{B})$ is a basis of an ultra filter.

Proof: Let $\mathcal{F}'$ be generated by $f(\mathcal{B})$. If $A \subseteq Y$, then

$f^{-1}A \cup f^{-1}\tilde{A} = X \in \mathcal{F}$. So $f^{-1}A \supset B_1 \in \mathcal{B}$ or $f^{-1}\tilde{A} \supset B_2 \in \mathcal{B}$.

Hence $A \supset fB_1$ or $\tilde{A} \supset fB_2$. Thus $A$ or $\tilde{A} \in \mathcal{F}'$ and by 15, $\mathcal{F}'$ is an ultrafilter.

Definition 19: Let $X$ be a topological space. If $\mathcal{F}$ is a filter on $X$ and $x \in X$, then $\mathcal{F}$ converges to $X$ if $\mathcal{F} \supset \mathcal{U}_x$

Lemma 20: If $\mathcal{B}$ is a basis for $\mathcal{F}$, then $\mathcal{F}$ converges to $x$ if and only if $U \in \mathcal{U}_x$ implies that there is a $B \in \mathcal{B}$ with $B \subset U$.

Proof: By 11.

Theorem 21: $X$ is Hausdorff if and only every filter converges to at most one point.

Proof: Let $X$ be Hausdorff. If $x \neq y$ and $\mathcal{F} \supset \mathcal{U}_x$. Find $U \in \mathcal{U}_x$, $V \in \mathcal{U}_y$, $U \cap V = \phi$. If $\mathcal{F} \supset \mathcal{U}_y$, then $\phi \in \mathcal{F}$. Conversely suppose every filter has at most one point to which it converges. If there is a non-Hausdorff pair, suppose $U \in \mathcal{U}_{x_1}$, $V \in \mathcal{U}_{x_2}$ and $U \cap V \neq 0$. Then $\mathcal{U}_{x_1} \cup \mathcal{U}_{x_2}$ has finite intersection property. By 6, $\exists\ \mathcal{F} \supset \mathcal{U}_{x_1}, \mathcal{U}_{x_2}$, a contradiction.

Definition 22: Let $\mathcal{B}$ be a filter a basis on $X$. If $y \in X$, then $y$ is an accumulation point of $\mathcal{B}$ if and only $y \in \overline{B}$, $B \in \mathcal{B}$.

Lemma 23: If $\mathcal{B}$ is a basis for $\mathcal{F}$, then $y$ is an accumulation point of $\mathcal{B}$ if and only if $y$ is an accumulation point of $\mathcal{F}$.

Proof: exercise

Lemma 24: $x$ is an accumulation point of a filter $\mathcal{F}$ if and only if there is an $\mathcal{F}' \supset \mathcal{F}$ such that $\mathcal{F}'$ converges to $x$.

Proof: If $\mathcal{F}'$ exists, suppose $F \in \mathcal{F}$. Now $\mathcal{F}' \supset \mathcal{U}_x, \mathcal{F}$. Hence $F \cap U \neq 0$ for $U \in \mathcal{U}_x$. Conversely $U \cap F \neq 0$ for $U \in \mathcal{U}_x$ and $F \in \mathcal{F}$. Apply 4 to $\mathcal{U}_x \cup \mathcal{F}$.

Corollary 25: If $x$ is an accumulation point of an ultra filter $\mathcal{F}$, then $\mathcal{F}$ converges to $x$.

Definition 26: Let $\mathcal{F}$ be a filter on $X$ and suppose $f: X \to Y$. Then $L_{\mathcal{F}} f = y$ (limit along $\mathcal{F}$ of $f$) if $F(f\mathcal{F})$ converges to $y$

Proposition 27: Let $f: X \to Y$. Then $f$ is continuous at $a$ if and only if $L_{\mathcal{U}_a} f = f(a)$

Proof: Use 20 and 26.

Corollary 28: $f$ cont. at $a$ iff $\mathcal{F}$ converging to $a$ implies ${}^F(f\mathcal{F})$ converges to $f(a)$.

<u>Theorem 29</u>: Let $\prod_{\alpha\in A} X_\alpha = X$ be a topological space and $\mathcal{F}$ be a filter on $X$. Then $\mathcal{F}$ converges to $\{x_\alpha\}$ if and only if ${}^F(p_\alpha \mathcal{F})$ converges to $x_\alpha$ for $\alpha \in A$.

Proof: $\Rightarrow$ from 28 since projections are continuous. Conversely let $V = \prod_{\alpha\in A} V_\alpha$, $V_\alpha \in \mathcal{U}_{x_\alpha}$ with $V_\alpha = X_\alpha$ for all but a finite number of $\alpha$. $V = \cap\, p_{\alpha_\nu}^{-1} V_{\alpha_\nu}$, $V_\alpha \in {}^F(p_\alpha \mathcal{F})$. Find $A_\alpha \in \mathcal{F}$ such that $p_\alpha A_\alpha \subset V_\alpha$. $V = \cap\, p_{\alpha_\nu}^{-1} V_{\alpha_\nu}$ and $A_{\alpha_\nu} \subset p_{\alpha_\nu}^{-1} V_{\alpha_\nu}$; hence $V \supset \cap A_{\alpha_\nu} \in \mathcal{F}$. So $\mathcal{F} \supset \mathcal{U}_{\{x_\alpha\}}$.

<u>Theorem 30</u>: A topological space $X$ is compact if and only if every ultra filter on $X$ converges (a) (or filter has acc. pt. or f.i.p.) (b).

Proof:

(a) $\Leftrightarrow$ (b) by 25, 13, 24.

Other by standard argument using 4.

Theorem 31: Let $f: X \to Y$ be continuous and $X$ be compact. Then $fX$ is compact.

Theorem 32 (Tychonoff): The product of topological spaces is compact if and only if all the factors are compact.

Proof: $X = \prod_{\alpha \in A} X_\alpha$. If $X$ is compact, use $X_\alpha = p_\alpha X$ and Theorem 31.
Let $X_\alpha$ be compact, $\alpha \in A$. Let $\mathcal{U}$ be an ultrafilter on $X$. $F(p_\alpha \mathcal{U})$ is an ultrafilter on $X_\alpha$ by 17. Let $F(p_\alpha \mathcal{U})$ converge to $x_\alpha \in X_\alpha$.
By 29 $\mathcal{U}$ converges to $(x_\alpha) = x$. Q.E.D.

Theorem 33: Let $X = \prod_{\alpha \in A} X_\alpha$ be a product space and suppose infinitely many $X_\alpha$ are non-compact. Then any compact set $B$ in $X$ has empty interior.

Proof:
Suppose $B^\circ \neq 0$ and $B$ is compact. Find a basic open set in $B^\circ$ such that $B^\circ \supset \bigcap_{1 \le \nu \le n} p_{\alpha_\nu}^{-1} U_\nu$. Now $p_\alpha B \supset p_\alpha B^\circ \supset p_\alpha \bigcap p_\alpha^{-1} U_\nu$
$= X_\alpha$ for $\alpha \neq \alpha_1, \ldots, \alpha_n$, which is impossible.

## Nets

Definition: A partially ordered set $D$ is directed ($\vec{\le}$) if
$\alpha \in D, \beta \in D \Rightarrow \exists \gamma \in D$ such that $\gamma \ge \alpha, \gamma \ge \beta$.

Definition: A net is a mapping $S$ of a directed set $D$ into a set $X$.

Notation: $S(\alpha) = S_\alpha$

Suppose $X$ is topological

Definition: A net $S: D \to X$ converges to $a \in X$ if for each $U \in \mathcal{U}_a$, there is an $\alpha_0 \in D$ for which $S_\alpha \in U$ if $\alpha \ge \alpha_0$

A point $a$ is an accumulation point of a net $S = D \to X$ if to each $U \in \mathcal{U}_\alpha$, $\alpha \in D$, there exists a $\beta \geq \alpha$ such that $S_\beta \in U$.

Definition: $C \subset D$ is cofinal if for each $\alpha \in D$ there is a $\beta$ in $C$ with $\alpha \leq \beta$.

(A pt is an acc. pt. if to each $U \in \mathcal{U}_\alpha$ there is a cofinal set mapped into $U$)

Filters and nets

If $D$ is directed, then the class $\{D_\alpha \mid \alpha \in D\}$ is a filter basis.

Take $D_{\alpha_1}, D_{\alpha_2}$. If $\alpha_3 \geq \begin{smallmatrix}\alpha_1\\ \alpha_2\end{smallmatrix}$, then $D_{\alpha_3} \subset \begin{smallmatrix}D_{\alpha_1}\\ D_{\alpha_2}\end{smallmatrix}$

$$D_\alpha = \{\beta \in D \mid \beta \geq \alpha\}$$

${}^F S = {}^F(S(\mathcal{B}D)$ is the filter associated with a net.

${}^F S$ is the class of supersets of any set of the form $\{S_\beta \mid \beta \geq \alpha\}$

$= \{S D_\alpha\}$

Lemma 35: Let $S: D \to X$ topological and $a \in X$. Then $S_\alpha \to a$ ($S_\alpha$ converges to $a$) if and only ${}^F S$ converges to $a$. Furthermore $a$ is an accumulation point of $S$ iff $a$ is an accumulation point of ${}^F S$.

Proof: Let $S_\alpha \to a$. Let $U \in \mathcal{U}_a$. Then for some $\alpha$, $SD_\alpha \subseteq U$. Hence $U \in {}^F S$ and $\mathcal{U}_a \subseteq {}^F S$. Conversely let ${}^F S \to a$. Let $V \in \mathcal{U}_a$. Since $V \in {}^F S$, find a set of which it is a superset. For the second half let $a$ be an accumulation point of $S$. Let $E \in {}^F S$; $E \supset SD_\alpha$ for some $\alpha \in D$. Take $U \in \mathcal{U}_a$ and let $\beta \geq \alpha$ be such that $S_\beta \in U$. But then $E \cap U \neq \phi$. Since $U$ is arbitrary, $a \in \bar{E}$. Conversely, suppose $a$ is an accumulation point of ${}^F S$. Let $\alpha \in D$ and $U \in \mathcal{U}_a$. Now $SD_\alpha \in {}^F S$, so that $U \cap SD_\alpha \neq \phi$ because $a \in \bar{E}$ for $E \in {}^F S$. Pick $\beta$ such that $S_\beta \in U \cap SD_\alpha$.

Let $\mathcal{F}$ be a filter on $X$; $\mathcal{F}$ is directed by $\overrightarrow{\supset}$ : $E, F \in \mathcal{F} \Rightarrow {}^E_F \supset E \cap F \in \mathcal{F}$. Any map $S: E \in \mathcal{F} \to x_E \in X$ is a net. Assume also that $x_E \in E$. Let $D_E = \{F \mid F \in \mathcal{F} \text{ and } F \subset E\}$. Form $SD_E$. If $F \in D_E$, then $S_F \in F \subset E$. Hence $SD_E \subset E$. Therefore ${}^F S \supset \mathcal{F}$ is a refinement of $\mathcal{F}$.

Lemma 36: If $\mathcal{F}$ converges to $a$, then $S$ converges to $a$.

Proof: ${}^F S \supset \mathcal{F} \supset \mathcal{U}_a$ and by 35.

Lemma 37: Let $A \subset X$. Then $a \in \bar{A}$ if and only if there exists some filter $\mathcal{F}$ converging to $a$ and $A \in \mathcal{F}$.

Proof: $a \in \bar{A} \Leftrightarrow U \in \mathcal{U}_a \Rightarrow U \cap A \neq \phi$. By 5 this is equivalent to the existence of a filter $\mathcal{F}$ such that $\mathcal{F} \supset \mathcal{U}_a$ and $\mathcal{F} \to a$.

Q.E.D.

Lemma 37': Let $A \subset X$, $a \in X$. Then $a \in \overline{A}$ if and only if there exists a net $S: D \to X$ such that $a$ is a limit of $S$ and $SD \subset A$ ($S_\alpha \in A$ if $\alpha \in D$).

Proof:

Suppose $SD \subset A$, $S_\alpha \to a$. Let $U \in \mathcal{U}_a$. $\exists\, \beta_U$ such that $SD_{\beta_U} \subset U$ but $SD_{\beta_U} \subset A$, $SD_{\beta_U} \subset A \cap U$, $A \cap U \neq \phi$.

Conversely, let $a \in \overline{A}$. Let $D = \mathcal{U}_a$ $(\vec{\supset})$. Now $U \in D \to S_U \in U \cap A$, where $S_U$ is any point in $U \cap A$, which is non-empty. Now $\mathcal{U}_a \to a$. By 36 $S_U \to a$.

Lemma 38: Let $f: X \to Y$ be continuous and $S: D \to X$ be a net with $S_\alpha \to a \in X$. Then the composition map $f \circ S$ is a net and $(f \circ S)_\alpha \to f(a)$.

Proof:

Let $V \in \mathcal{U}_{f(a)}$. Let $U \in \mathcal{U}_a$ be such that $f(U) \subset V$. By convergence, $\exists\, \alpha$ such that $SD_\alpha \subset U$. Apply $f$

$$(f \circ S) D_\alpha = f(SD_\alpha) \subset f(U) \subset V$$

Lemma 39: In a Hausdorff a net can converge to at most one point.

Proof: Immediate from directedness.

Examples:

1) Sequences

2) $\mathbb{R}^{\mathbb{R}}$, all functions from $\mathbb{R}$ to $\mathbb{R}$

$A = \{f \in \mathbb{R}^{\mathbb{R}} \mid f(x) = 1$ except for finitely many $x \in \mathbb{R}$, where $f = 0\}$

Let $f_1 \leq f_2$ if and only $f_1(x) \leq f_2(x)$ for $x \in \mathbb{R}$. Directed under $\geqq$. Let $S$ be the identity map. Then $S_f \to 0$.

To see this we note

$$U = \{f \mid |f(x_1)|, \dots, |f(x_m)| < \epsilon,\ \epsilon > 0, 1 \le m,\ x_i \in \mathbb{R}\}$$

Let $g(x_1) = \dots = g(x_m) = 0$ and $g \in A$. Then $g' \le g$ implies and net converges.

Banach spaces

Theorem: Let $X$ be a normed linear space, $X^*$ its dual. Let $B^* = \{x^* \in X^* \mid \|x^*\| \le 1\}$. Then $B^*$ is compact in the weak-star topology

Remark:
$X$, dual $X^*$
The smallest (coarsest) topology on $X^*$ which makes every mapping $X^* \ni y^* \to y^* x \in$ field, $x \in X$, continuous is the weak-star topology. Note continuity in norm topology.
Topology on $X^*$ is topology induced by product topology on $(\text{field})^X$

Proof:
Let $C = \prod_{x \in X} [-\|x\|, \|x\|] \subset (\text{field})^X$, since

$$\prod_{x \in X} \text{field} = \text{field}^X$$

Now $C$ is compact and $C \supset B^*$. Let $x^* \in B^*$. Form $|p_y x^*|, y \in X, = |x^* y| \le \|x^*\| \|y\| \le \|y\|$. Thus
$p_y x^* \in [-\|y\|, \|y\|]$ for every $y$.

Hence $B^* \subset C$. We shall show $B^*$ is closed. Suppose $x^* \in \overline{B^*}$. Let $x_\alpha^*$ be a net in $B^*$ which converges to $x^*$. Now

$$x^*(ax+by) = pr_{ax+by}\, x^*$$

is limit of $pr_{ax+by}\, x^*_\alpha$

$= x^*_\alpha(ax+by)$

$= a x^*_\alpha x + b x^*_\alpha y$

$= a\, pr_x x^*_\alpha + b\, pr_y x^*_\alpha$

$\to a\, pr_x x^* + b\, pr_y x^*$

$= a x^* x + b x^* y.$

two limits, but product of Hausdorff spaces is Hausdorff

So $x^*$ is linear.

We now prove $\|x^*\| \le 1$.

Let $\|x\| = 1$. $\displaystyle \|x^*\| = \sup_{\|x\|=1} |x^* x|$.

$$|x^* x| = |pr_x x^*| \leftarrow |pr_x x^*_\alpha|$$
$$= |x^*_\alpha x|$$
$$\le \|x^*_\alpha\| \|x\|$$
$$\le 1 \quad \text{because } \|x^*_\alpha\| \le 1 \text{ in net.}$$

Hence $\|x^*\| \le 1$.

Thus $B^*$ is closed in the weak-star topology; hence it is compact.

Notation:

Field $K = \mathbb{R}$ or $\mathbb{C}$.

Definition: Let $X$ be a vector space over $K$. A function $p: X \to \mathbb{R}$ is called a seminorm if

(1) $p(\alpha x) = |\alpha| p(x)$

(2) $p(x+y) \le p(x) + p(y)$

Remarks:

$p(0) = 0$, $p \ge 0$.

It is not necessarily true that $p(x) = 0$ implies $x = 0$.

<u>Theorem</u>: Let $X$ be a vector space over $\mathbb{R}$, $p$ a semi-norm on $X$, $L$ a linear subspace of $X$, $f$ a linear functional $f: L \to \mathbb{R}$, and $|f| \le p$ on $L$, $x_0 \in X - L$. Then there exists a linear $F: L + \mathbb{R}x_0 \to \mathbb{R}$ such that $|F| \le p$ on $L + \mathbb{R}x_0$, $F|L = f$.

Proof:

A functional is determined by its value on $L$ and $x_0$

$$F|L = f$$
$$F(x_0) = a_0 \in \mathbb{R}$$

Then $F$ is uniquely defined on $L + \mathbb{R}x_0$ by linearity.

Must have $|F(l + \alpha x_0)| \le p(l + \alpha x_0)$ for $l \in L, \alpha \in \mathbb{R}$

$$|fl + \alpha F x_0|$$

$$-p(l + \alpha x_0) \le fl + \alpha F x_0 \le p(l + \alpha x_0)$$

$$-p(l + \alpha x_0) - fl \le \alpha F x_0 \le p(l + \alpha x_0) - fl$$

If $\alpha > 0$, $-\frac{1}{\alpha}\left(p(l + \alpha x_0) + fl\right) \le F x_0 \le \frac{1}{\alpha}\left(p(l + \alpha x_0) - fl\right)$

$\alpha < 0$, $-\frac{1}{\alpha}\left(p(l + \alpha x_0 + fl\right) \ge F x_0 \ge \frac{1}{\alpha}\left(p(l + \alpha x_0) - fl\right)$

In either case $-p\left(\frac{l}{\alpha} + x_0\right) - f\left(\frac{l}{\alpha}\right) \le F x_0 \le p\left(\frac{l}{\alpha} + x_0\right) - f\left(\frac{l}{\alpha}\right)$

This is necessary and sufficient. It remains to produce an $Fx_0$

$$|f(x-y)| \le p(x-y) \quad \text{for } x, y \in L$$

and $f(x) - f(y) = f(x-y) \le |f(x-y)|$

So $f(x) - f(y) \le p(x - x_0) + p(y - x_0)$

$$f(x) - p(x - x_0) \le f(y) + p(y - x_0) \quad \text{for } x, y \in L$$

Take supremum on left

$$s = \sup_{x \in L} \left(f(x) - p(x - x_0)\right)$$

$t = \inf\left(fy + p(y - x_0)\right)$; both are finite.

Hence $s \le i$

Let $s \le F(x_0) \le i$. (Let $F(x_0) = s$, for example)

Then $f(x) - p(x - x_0) \le Fx_0 \le f(y) + p(y - x_0)$ for $x, y \in L$.

And the condition is satisfied.

Q.E.D.

Theorem: (Hahn-Banach)

Let $X$ be a vector space over $\mathbb{R}$, $p$ a semi-norm on $X$, $L$ a sublinear space of $X$, $f : L \to \mathbb{R}$ linear, $|f| \le p$ on $L$. Then there exists an $F$ linear $F : X \to \mathbb{R}$ such that $|F| \le p$ on $X$ and

$F|L = f$.

Proof:

Let $\mathcal{C}$ be the class of all linear extensions of $f$ on linear subspaces of $X$ satisfying the norm condition. Order by

$$(L', g) \ge (M, h) \quad \text{if and only if} \quad L' \supset M,\ g|M = h$$

This is inductive. Find maximal element. Apply lemma.

Q.E.D.

For normed linear spaces

If $L \subset X$ and $|f(x)| \le M\|x\|$ on $L$. Then

$$p(x) = M\|x\| \text{ is a semi-norm on } X.$$

$$\text{with } |Fx| \le p(x) = M\|x\|$$

Complete the results.

Complex spaces

Lemma: Let $X$ be a vector space over $\mathbb{C}$. Suppose $f$ is $\mathbb{R}$-linear and $f : X \to \mathbb{R}$. Then $F(x) = f(x) - i f(ix)$ is $\mathbb{C}$-linear and Real $F = f$.

Proof: as exercise

Lemma: Let $X$ be a vector space over $\mathbb{C}$, $f: X \to \mathbb{C}$ $\mathbb{C}$ linear.

Let $f_1 = \frac{f + \bar{f}}{2} = \text{Real } f$. Then $f_1$ is $\mathbb{R}$ linear $: X \to \mathbb{R}$

and $f(x) = f_1(x) - i f_1(ix)$

For H B Thm: Take $f$, form $f_1$, extend, apply first lemma.

<u>Theorem</u>: Let $X$ be a vector space over $\mathbb{C}$, $Y$ a subspace of $X$, $f$ linear from $Y$ to $\mathbb{C}$, $p$ a seminorm on $X$, $|f| \le p$ on $Y$; then there exists an $F$ linear from $X \to \mathbb{C}$ and $|F| \le p$ on $X$ and $F|Y = f$.

Proof:

Let $f = f_1 - i f_1(i\,\cdot)$, where $f_1 = \mathrm{Re}\, f = \frac{f + \bar{f}}{2}$

$f_1$ is $\mathbb{R}$-linear and $|f_1| \le |f| \le p$ on $Y$

Extend $f_1$ to $F_1 : X \to \mathbb{R}$ with $|F_1| \le p$. Take

$$F = F_1 - i F_1(i\,\cdot)$$

$F$ is $\mathbb{C}$ linear from $X$ to $\mathbb{C}$

$F|Y = f$ since $F|Y = (F_1 - iF_1(i\,\cdot))|Y = f_1 - i f_1(i\,\cdot) = f$

$$|F(x)| = e^{i\varphi} F(x) = F(e^{i\varphi} x) = F_1(e^{i\varphi} x) \quad \text{since } F(e^{i\varphi}x) \text{ is real}$$

$$\le p(e^{i\varphi} x)$$

$$= |e^{i\varphi}|\, p(x)$$

$$= p(x)$$

On a normed linear space $X$

$Y \subset X$, norm $\| \ \|$

If $f : Y \to K$ is continuous

$$|fx| \le \|x\|\,\|f\| \leftarrow \text{norm of } f \text{ on } Y$$

$$\text{Set } p(x) = \|x\|\,\|f\|$$

Extend $f$ to $F$ $\quad F : X \to K$

$$|F(x)| \le p(x) = \|f\|\,\|x\|$$

and so $\|F\| = \|f\|$

Corollary: Let $X$ be normed linear over $K$ $(=\mathbb{R}$ or $\mathbb{C})$; let $x \in X$. Then there exists a continuous linear functional $y^* \in X^*$ on $X$ such that $\|y^*\| = 1$ and $y^* x = \|x\|$

Proof:

Take $Y = Kx$. This is a closed linear subspace of $X$. Let

$$f(\alpha x) = \alpha \|x\| \quad \text{for } \alpha \in K$$

$\|f\|_Y = 1$ because $|f(\alpha x)| = |\alpha|\,\|x\|$. If $\|\alpha x\| = 1$, then $|f(\alpha x)| = 1$. Extend $f$ to $y^* \in X^*$ with $\|y^*\| = 1$.

Corollary: Let $Y$ be a closed linear proper subspace of $X$ and $x_0 \notin Y$. (Then $d = \inf_{y \in Y} \|x_0 - y\| > 0$). Then there exists a $y^* \in X^*$ such that

$y^*|Y = 0$, $y^* x_0 = 1$, and $\|y^*\| = 1/d$.

Proof:

Consider $Y + Kx_0$. The sum is direct: $Y \oplus Kx_0$. Define $l(y + \alpha x_0) = \alpha$, well-defined because sum is direct. Now

$$\sup_{y+\alpha x_0 \neq 0} \frac{|l(y+\alpha x_0)|}{\|y+\alpha x_0\|} = \sup \frac{|\alpha|\,|l(y/\alpha + x_0)|}{|\alpha|\,\|y/\alpha + x_0\|}$$

$$= \sup \frac{|l(y/\alpha + x_0)|}{\|y/\alpha + x_0\|}$$

$$= \sup \frac{1}{\|y/\alpha + x_0\|}$$

$$= 1/d$$

Extend $l$ by Hahn-Banach Theorem.

Lemma: Let $X$ be normed linear over $\mathbb{K}$. If $L$ is a closed maximal proper subspace of $X$, then there exists a linear function $l \neq 0 \in X^*$ such that $\ker l = L$, and conversely.

Proof:

If $L$ is closed, maximal, and proper. Take any $x_0 \notin L$ and apply preceding corollary to $L + Kx_0$. Conversely let $l^* \in X^*$, $l^* \neq 0$. Let $L = \ker L^*$. $L$ is closed because $l^*$ is continuous and $L = l^{-1*}(0)$. $L$ is proper because $l^* \neq 0$. Maximal: $a \notin L$, to show $X = L + Ka$

Let $\alpha \neq 0$ Then $l^*\left(\alpha \frac{a}{l^*a} + L\right) = \alpha$. If $l^*b = \alpha$, then

$$l^*\left(b - \alpha \frac{a}{l^*a}\right) = 0$$

So $\quad b - \alpha \frac{a}{l^*a} \in L$

and $\quad b \in \alpha \frac{a}{l^*a} + L.$

$$l^{*-1}(\alpha) = \alpha \frac{a}{l^*a} + L$$

$$X = \bigcup_{\alpha \in K} l^{*-1}(\alpha) = \bigcup_\alpha \left(\alpha \frac{a}{l^*a} + L\right) = K \frac{a}{l^*a} + L.$$

Hence $L$ is maximal.

Theorem: Let $X$ be normed linear. Then $X^*$ is separable implies $X$ separable.

Proof:

Let $\{x_\mu^* \mid 1 \leq \mu\}$ be dense in $X^*$

$$\|x_\mu^*\| = \sup_{\|x\|=1} |x_\mu^* x|$$

To each $\mu \geq 1$ take an $x_\mu$ $\quad |x_\mu^* x_\mu| \geq \|x_\mu^*\|/2 \quad \|x_\mu\| = 1$

Take $\overline{\text{lin span}\{x_\mu\}}$ ; we shall show this $= X$, and this is sufficient (take rational finite linear combinations of $x_\mu$. Suppose this fails. Then there exists $x^* \in X^*$ such that $x^* \neq 0$ and $x^* x_\mu = 0$

Let $x^*_{n_\mu} \to x^*$. Then $0 \leftarrow \|x^* - x^*_{n_\mu}\|$

$$= \|x^* - x^*_{n_\mu}\| \|x_{n_\mu}\|$$

$$\geq |(x^* - x^*_{n_\mu})\, x_{n_\mu}|$$

$$= |x^*_{n_\mu}\, x_{n_\mu}|$$

$$\geq \|x^*_{n_\mu}\|/2$$

So $\|x^*_{n_\mu}\| \to 0$ and $x^*_{n_\mu} \to 0$. So $x^* = 0$, contradiction.

Theorem: (Interior Mapping Principle)
Let $T$ be continuous and linear from $X$ to $Y$, $X$ and $Y$ both Banach spaces. If $T$ is onto, then $T$ is open.

Proof:

1) To show if $U \in \mathcal{U}_0^X$, system at $0$ in $X$, then $\overline{TU} \in \mathcal{U}_0^Y$

Let $W$ be an open neighborhood of $0$ in $X$ such that $W - W \subseteq U$

Write $X = \bigcup_{n \geq 1} nW$

$$Y = TX = \bigcup_{n\geq 1} Tn(W) = \bigcup_{n \geq 1} nTW = \bigcup_{n\geq 1} n\overline{TW}$$

By Baire category theorem $\exists\, n_0$ such that

$$\overline{n_0 TW} = n_0 \overline{TW} \supseteq W' \text{ open and non-empty} \quad \left(\text{since } x \to nx \text{ is homeomorphism}\right)$$

So $\qquad \overline{TW} \supseteq \frac{1}{n_0} W'$

$$\overline{TU} \supseteq \overline{T(W-W)} = \overline{TW - TW} \supseteq \overline{TW} - \overline{TW} \supseteq \frac{1}{n_0}W' - \frac{1}{n_0}W' \in \mathcal{U}_0^Y,$$

in fact is open

2) If $\epsilon_0 > 0$, then there exists an $\eta_0$ such that $\overline{TB(0, 2\epsilon_0)} \supseteq B(0, \eta_0)$

Take $\epsilon_1, \epsilon_2, \ldots > 0$ and $\sum_{\nu \geq 1} \epsilon_\nu < \epsilon_0$. Choose $\eta_\nu$ for $0 \leq \nu$ such that $\overline{TB(0, \epsilon_\nu)} \supseteq B(0, \eta_\nu)$ for $\nu \geq 0$ and such that $\eta_\nu \to 0$.

Take $y \in B(0, \eta_0) \subseteq \overline{TB(0, \epsilon_0)}$. Let $x_0 \in B(0, \epsilon_0)$ such that $\|Tx_0 - y\| < \eta_1$. But $Tx_0 - y \in B(0, \eta_1) \subseteq \overline{TB(0, \epsilon_1)}$.

Find $x_1 \in B(0, \epsilon_1)$ and continue: $\|x_1\| < \epsilon_1$ and $\|Tx_1 + (Tx_0 - y)\| < \eta_2$

$$\|Tx_0 + \cdots + Tx_m - y\| < \eta_{m+1} \quad \text{and} \quad \|x_\nu\| < \epsilon_\nu$$

Now $\| \sum_{n \leq \nu \leq m} x_\nu \| < \sum_n^m \epsilon_\nu$, and the latter is Cauchy. Let

$$x = \lim_\nu \sum_0^m x_\nu$$

By continuity of norm,

$$\|x\| \leq \sum_{\nu \geq 0} \epsilon_\nu < 2\epsilon_0 \quad \text{and}$$

$$Tx = \lim \sum_0^n Tx_\nu \quad \text{by continuity of } T$$
$$= y. \quad \text{This proves (2)}$$

Hence if $U$ is any neighborhood of $0$ in $X$, then $TU$ is a neighborhood of $0$ in $Y$.

3) Let $O$ be any open set. Let $y \in TO$, $y = Tx$, $x \in O$. Find $U \in \mathcal{U}_0^X$ such that $x + U \subseteq O$. Then

$$Tx + TU = T(x+U) \subseteq TO$$

$TU$ is a nbd of $0$ in $TX$ and $Tx + TU$ is nbd of $Tx$ in $Y$. Hence $TO$ is open. Q.E.D.

Corollary:

Let $T$ be continuous linear one-one and onto such that $X \to Y$, $X$ and $Y$ are Banach spaces. Then $T^{-1}\, Y \to X$ is continuous and linear.

Proof: $T^{-1}$ exists and is linear. To show inverse image of open set is open, which is the theorem.

Corollary. (Closed graph theorem):

Remark: If $T: X \to Y$, then the graph of $T$, $\Gamma_T$, is $\{x, Tx\} \subseteq X \times Y$.

Statement:

If $X$ and $Y$ are Banach spaces and $T$ linear $: X \to Y$, then $\Gamma_T$ closed implies $T$ continuous.

Proof:

Projection to $X$ is continuous is linear one-one onto. So inverse is continuous. Second projection is continuous and composition $T$ is cont.

(Closed subset of $X \times Y$ is Banach with $\|(x,y)\| = \|x\| + \|y\|$.)

Corollary:

If $X$ is a Banach space under $\| \ \|_1$ and $\| \ \|_2$ and if $\| \ \|_1 \leq \alpha \| \ \|_2$ (the identity from 2 to 1 is continuous), then there exists a $\beta$ such that $\|x_2\| \leq \beta \|x_1\|$ for $x \in X$.

Proof:

Identity is continuous, linear, one-one, onto. So inverse is continuous.

Examples:

1) Let $X$ be an infinite dimensional normed linear space over $K$. Let $B$ be a basis of $X$ over $K$. Let basis be such that $\|b\| = 1$ for $b \in B$. We shall define a linear functional on $B$ and extend by linearity. Take countable subset and set $f b_n = n$; define $f b = 0$ for the rest. Then $\|f\| = \sup_{\|x\|=1} |fx| \geq |f b_n| = n$.

So linear functional is unbounded and therefore not continuous. Graph is a subspace and is not closed. This is a non-closed subspace.

2) Separable space with non-separable dual.

Let $X = l_1 = \{\alpha_1, \alpha_2, \ldots\}$, $\Sigma|\alpha_\nu| < \infty$, $\alpha_\nu \in K$.

$\|\alpha\| = \Sigma|\alpha_\nu|$

$X$ is a separable Banach space

Fundamental set is countable $= (0, \ldots 0, 1, 0, \ldots)$; closed span in $X$.

Space with countable fundamental set is separable

$X^*$ is the space of all bounded sequences.

To see this look at effect of $f$ on fundamental sequence. Then define a linear functional to go the other way. Can show norm is sup.

Theorem: A normed linear space is locally compact if and only if it is finite-dimensional.

Proof:

In a ~~Banach~~ space, $\overline{B(0,1)} = \{\|x\| \leq 1\}$. We shall show $\{\|x\|=1\}$ is not compact. Define inductively

$$\|x_1\|=1$$

$$\vdots$$

$$\|x_\nu\|=1 \text{ for } 1 \leq \nu \leq n \text{ and } \|x_\nu - x_\mu\| > \tfrac{1}{2} \text{ for } \nu \neq \mu$$

Take $y \notin$ lin span $\{x_1, \ldots, x_n\} = V_n = \overline{V_n}$. Then

$$d(y, V_n) > 0.$$

Make $d(y, v) < 2d(y, V_n)$

$$\|y - v\|, \quad y - v \notin V_n$$

Take $x_{n+1} = \dfrac{y-v}{\|y-v\|} \notin V_n$, $\|x_{n+1}\|=1$.

Now let $w \in V_n$ and then $\|x_{n+1} - w\| = \left\|\dfrac{y-v}{\|y-v\|} - w\right\|$

$$= \|y - \underbrace{v - \|y-v\| w}_{\in V_n}\| \cdot \frac{1}{\|y-v\|}$$

$$\geq d(y, V_n) \frac{1}{\|y-v\|}$$

$$> 1/2.$$

This sequence has no limit point. Q.E.D.

Space which is compact but not sequentially compact. Take

$$X = \{0,1\}^{\{0,1\}^{\aleph_0}}, \text{ product topology}$$

X is set of functions of the set of countable sequences of 0's and 1's into 0,1.

$f^n(s) = s_n$, $s_n$ = nth coordinate, $f^n$ does not converge (Take sequence $\left(\ldots, \frac{1+(-1)^n}{2}, \ldots\right)$)

**Lemma:**

Let $X$ be a vector space over a field $F$ and suppose $X = A \oplus B$. Then there exist linear maps onto: $\varphi : X \to A$

$\psi : X \to B$

such that $\varphi y + \psi y = y$ for all $y \in X$. $(\varphi + \psi = 1)$

Proof:

Write $y = y_1 + y_2$ uniquely. Set $\varphi y = y_1$, $\psi y = y_2$. Etc.

**Remark:**

Suppose $X = A \oplus C$. If we define $\varphi'$ and $\chi$, then $\varphi \neq \varphi'$ in general.

**Proposition:**

Let $X$ be a Banach space.

Suppose $P \in \mathcal{L}X = \{$bounded linear maps of $X$ into $X\}$ and suppose $P^2 = P$. Then image $P$ is a closed linear subspace of $X$; and $(I-P)^2 = I - P \in \mathcal{L}X$. $X = \text{image } P \oplus \text{image } (I-P)$

Proof:

Image $P$ is linear. Assert image $P = \ker(I-P)$, which is closed. Let

$$y = Px$$

$Py = P \cdot Px = Px = y$. Hence $y - Py = 0$, $(I-P)y = 0$, and $y \in \ker(I-P)$. And conversely. Also $(I-P)^2 = I - P - P + P^2 = I - P$ by expanding. Finally for $X = \text{image } P \oplus \text{image } (I-P)$. If $x$ is in intersection, $Px - x = 0$ and $Px = 0$. Hence $x = 0$. For spanning, $x = Px + x - Px$ $= Px + (I-P)x$.

Q.E.D.

**Proposition:**

Let $X$ be a Banach space.

Suppose $X = A \oplus B$, $A$ and $B$ closed. Then there exists a $P \in \mathcal{L}X$ such that $P^2 = P$ and image $P = A$, image $(I-P) = B$.

Proof: Let $P = \varphi$ in lemma. Use closed graph theorem.

Let graph of $P : X \to A$ be $\Gamma_P$. Let $(x_\nu, Px_\nu) \in \Gamma_P \to (y, z) \in X \times A$

Then $x_\nu \to y$, $x_\nu = Px_\nu + (I-P)x_\nu$

$$\begin{array}{ccc} \downarrow & \downarrow & \downarrow \text{ so that} \\ y & z & w \end{array} \qquad y = z + w$$

Now $Px_\nu \in A$ and $(I-P)x_\nu \in B$. Since $A$ and $B$ are closed, $z \in A$ and $w \in B$. By uniqueness in lemma $z = Py$. Hence graph is closed and $P$ is continuous.

Let $T$ be bounded linear : $X \to Y$ over $\mathbb{K}$. Then $T$ induces

$T^* : f \in Y^* \to f \circ T \in X^*$ with

$$X \xrightarrow{T} Y \xrightarrow{f} K. \quad \text{Hence } T^* : Y^* \to X^*$$

$T^*$ is linear and is bounded

$$\|T^* f\| = \|f \circ T\| = \sup_{\|x\|=1} |(f\circ T)x| = \sup_{\|x\|=1} |f(Tx)|$$

$$\leq \sup_{\|x\|=1} \|f\| \, \|Tx\|$$

$$= \|f\| \sup_{\|x\|=1} \|Tx\|$$

$$= \|f\| \, \|T\|$$

Hence $\quad \|T^*\| \leq \|T\|$

<u>Proposition</u>:

$\|T^*\| = \|T\|$

Proof:

Look at $T^{**} : X^{**} \to Y^{**}$ and $\|T^{**}\| \leq \|T^*\| \leq \|T\|$. We shall show $\|T\| \leq \|T^{**}\|$.

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "\varphi"'] & Y \arrow[d, "\psi"] \\ X^{**} \arrow[r, "T^{**}"] & Y^{**} \end{tikzcd}$$

We show the diagram commutes; this is sufficient.

$$(T^{**} \varphi x) g = \ ?$$

$g \in Y^*$

$$Y^* \xrightarrow{T^*} X^* \xrightarrow{\varphi x} K$$

$$\begin{aligned}(T^{**} \varphi x)\, g &= (\varphi x \circ T^*) g \\ &= \varphi x\,(T^* g) \\ &= (T^* g)\, x \\ &= (g \circ T)\, x \\ &= g(Tx) \\ &= [\psi(Tx)]\, g\end{aligned}$$

$$T^{**}(\varphi x) = \psi(Tx)$$

$$T^{**} \circ \varphi = \psi \circ T.$$

Now $\|Tx\| = \|\psi(Tx)\| = \|T^{**}(\varphi x)\| \le \|T^{**}\| \, \|\varphi x\|$
$= \|T^{**}\| \, \|x\|$. Q.E.D.

<u>Proposition</u>:

Let $X$ be normed linear over $K$ and $L$ a closed subspace of $X$. In $X/L$ set $\|x + L\| = \inf_{y \in x+L} \|y\|$. Then $X/L$ is a normed linear space over $K$. $X$ complete implies $X/L$ complete.

Proof:

Norm is $= \inf_{z \in L} \|x + z\|$.

1) $\|\alpha x + L\| = \|\alpha(x+L)\| = \inf_{z \in L} \|\alpha x + z\|$, which for $\alpha \neq 0$

$$= \inf_{z \in L} \|\alpha x + \alpha z\|$$

$$= |\alpha| \inf \|x + z\| = |\alpha| \, \|x + L\| .$$

For $\alpha = 0$, norm is zero.

2) Triangle inequality is okay.

3) If $0 = \|x + L\|$, take $x + y_n \to 0$. Then

$L \ni -y_n \to x$ and result $x \in L$ comes from closure.

In $X/L$, $\|x+L\| = \inf_{y\in L} \|x+y\|$

1) $\varphi : X \to X/L$ is continuous

2) $\|\varphi\| \le 1$

3) If $X$ is complete, $X/L$ is complete.

Suppose $\|(x_\nu + L) - (x_\mu + L)\| \to 0$

Then $\|y_\nu - y_\mu\| \le \frac{1}{k}$

Choose a subsequence $n_\nu$ such that $y_\nu \in x_{n_\nu} + L$ and

$$\|y_\nu - y_{\nu+1}\| < \frac{1}{2^\nu}$$

So $\{y_\nu\} \to y_0$, $y_\nu + L \to y_0 + L$ by (1)

$$x_{n_\nu} + L \to y_0 + L$$

Subsequence converges so whole sequence converges.

If $X$ is normed linear and $A \subseteq X$, then $A^\perp = \{ f \mid f \in X^* \text{ and } f|A = 0\}$.

$A^\perp$ is a <u>closed</u> linear subspace of $X^*$

Proposition:

If $X$ is a normed linear space and $L$ closed in $X$, then $(X/L)^*$ is isometric to $L^\perp$. ?

Proof.

Let $\varphi : X \to X/L$ and $f \in (X/L)^*$.

Now $f \xrightarrow{\chi} f\circ\varphi \in L^\perp$

since $f\circ\varphi = f(\varphi l) = f(0) = 0$ $\quad l \in L$.

$\chi$ is linear. $\chi$ is one-one. Let

$0 = \chi f = f\circ\varphi$ and $f = 0$ since $\varphi$ is onto.

Hence $\chi$ is one-one. For onto let $l \in L^\perp$. Then

$l : X \to K$ with kernel $l \supseteq L$

Factor map:
$$\begin{array}{ccc} X & \xrightarrow{\;l\;} & K \\ & {}_{\varphi}\searrow & \nearrow_{l'} \\ & X/L & \end{array}$$
$l = l'\circ\varphi$ since $L \subseteq \ker l$.

②

$l'$ is continuous:

Let $x_\nu + L \to 0$. Find $y_\nu \in x_\nu + L$ with $\|y_\nu\| \to 0$.

Then $0 \leftarrow l\, y_\nu = l' \circ \varphi\, y_\nu = l'(x_\nu + L)$

and $l' \in (X/L)^*$.

Hence $X$ is onto.

Let $f \in (X/L)^*$. Then $\|f\| = \sup\limits_{x \notin L} \dfrac{|f(x+L)|}{\|x+L\|}$

$$= \sup_{x \notin L} \frac{|f(x+L)|}{\inf\limits_{y \in x+L} \|y\|} \qquad f(x+L) = f(y+L)$$

$$= \sup_{x \notin L} \left\{ f(y+L) \sup_{y \in x+L} \frac{1}{\|y\|} \right\}$$

$$= \sup_{x \notin L} \sup_{y \in x+L} \frac{\|f(y+L)|}{\|y\|} \qquad \text{monotone limits}$$

$$= \sup_{y \notin L} \frac{|f(y+L)|}{\|y\|}$$

$$= \sup_{y \notin L} \frac{|f \circ \varphi(y)|}{\|y\|}$$

$$= \sup_{y \neq 0} \frac{|f \circ \varphi(y)|}{\|y\|}$$

$$= \|f \circ \varphi\|$$

$$= \|X f\|$$

$L^*$ is isomorphic to $X^*/L^\perp$:

$f \in X^* \xrightarrow{\rho} f|L \in L^*$

$X^* \underset{\text{onto}}{\xrightarrow{\rho}} L^* \to 0$ is exact by Hahn-Banach.

Hence $L^* \cong X^*/\ker \rho = X^*/L^\perp$

Bounded linear transformations on X form Banach space. If X is range, result is an algebra, called a Banach algebra, in which also $\|AB\| \leq \|A\| \cdot \|B\|$

Lemma: Let X be a normed linear space over $\mathbb{C}$, in which

$$\|x+y\|^2 + \|x-y\|^2 = 2\left[\|x\|^2 + \|y\|^2\right].$$

Let $(x,y) = \sum_{\sigma=0}^{3} \frac{i^\sigma}{4}\|x+i^\sigma y\|^2$. Then $(x,y)$ is a sesquilinear form (linear in first, conjugate linear in second), $(x,y) = \overline{(y,x)}$, and $(x,x) = \|x\|^2$.

Proof:

$$\alpha = 2\left(\|x+y+z\|^2 + \|y-z\|^2\right) = \|x+2y\|^2 + \|x+2z\|^2$$

If $z=0$,

$$2\left(\|x+y\|^2 + \|y\|^2\right) = \|x+2y\|^2 + \|x\|^2$$

and similarly for $y=0$

Substitute

$$\alpha = 2\left(\|x+y\|^2 + \|y\|^2\right) - \|x\|^2$$
$$+ 2\left(\|x+z\|^2 + \|z\|^2\right) - \|x\|^2$$

$$\|x+y+z\|^2 = \|x+y\|^2 + \|x+z\|^2 + \|y+z\|^2$$
$$-\|x\|^2 - \|y\|^2 - \|z\|^2$$

For $(x,y) = \overline{(y,x)}$, just write things down:

$$(y,x) = \sum \frac{i^\sigma}{4}\|y+i^\sigma x\|^2 = \sum \frac{i^\sigma}{4}\left\|\frac{1}{i^\sigma}y + x\right\|^2$$
$$= \overline{\sum \frac{1}{i^\sigma 4}\left\|\frac{1}{i^\sigma}y + x\right\|^2}$$
$$= \overline{(x,y)}$$

Easy for $(x,x) = \|x\|^2$

Prove that $-1, i$ can be factored out of $(x, y)$

By additivity $n(x,y) = (nx, y)$ for integers

Then $\left(\frac{1}{n}z, y\right) = \frac{1}{n}(z, y)$

and get for all rationals, then for all Gaussian rationals.

$$0 \le \|x - \rho y\|^2 = (x - \rho y, x - \rho y) \qquad \rho \text{ Gaussian rational}$$

$$= \|x\|^2 - \rho(y, x) - \bar{\rho}(x, y) + \rho\bar{\rho}\|y\|^2$$

Let $\rho \to (x,y)/\|y\|^2$ through rationals,

and $|(x,y)| \le \|x\|\,\|y\|$

If $\rho_\nu \to \alpha$

$$|(\rho_\nu x - \alpha x, y)| \le \|(\rho_\nu - \alpha)x\|\,\|y\|$$

$$= |\rho_\nu - \alpha|\,\|x\|\,\|y\| \to 0$$

$$\| \qquad |\rho_\nu(x,y) - (\alpha x, y)|$$

$$\downarrow$$

$\alpha(x,y)$ Hence constant multiples by

Schwarz's inequality.

Converse is also known.

Definition: A Hilbert space over $\mathbb{C}$ is a Banach space over $\mathbb{C}$ is a Banach space $H$ over $\mathbb{C}$ with a positive symmetric sesquilinear form such that $\|x\|^2 = (x, x)$.

Schwarz inequality: $|(x,y)| \le \|x\|\|y\|$

Theorem: Let $A$ be a non-empty closed convex subset of $H$. Then there is a unique $v_0 \in A$ such that

$$\|v_0\| \le \|v\| \quad \text{for } v \in A.$$

Remark: Need $x, y \in A \Rightarrow \frac{x+y}{2} \in A$.

Proof:

$$\inf_{v \in A} \|v\| = d \ge 0$$

Take $v_\nu \in A$ such that $\|v_\nu\| \searrow d$.

$$\|v_\nu - v_\mu\|^2 = 2(\|v_\nu\|^2 + \|v_\mu\|^2) - \|v_\nu + v_\mu\|^2$$

$$= 2\left(\|v_\nu\|^2 + \|v_\mu\|^2 - 2\left\|\underbrace{\frac{v_\nu + v_\mu}{2}}_{\in A}\right\|^2\right)$$

$$\le 2(\|v_\nu\|^2 + \|v_\mu\|^2 - 2d^2)$$

$$\to 2(d^2 + d^2 - 2d^2) = 0.$$

So $\{v_\nu\}$ is Cauchy with limit $v_0$, which is in $A$ since $A$ is closed. $\|v_0\| = d$ by continuity.

Suppose $\|v_1\| \le \|v\|$ for $v \in A$.

$$\left\|\frac{v_0 + v_1}{2}\right\| \ge \frac{\|v_0\|}{\|v_1\|}$$

$$\|v_0 - v_1\|^2 + \|v_0 + v_1\|^2 = 2\|v_0\|^2 + 2\|v_1\|^2$$
$$\le 4\left\|\frac{v_0 + v_1}{2}\right\|^2 = \|v_0 + v_1\|^2$$ Hence uniqueness.

Corollary:

Let $A \neq \emptyset$ be closed and convex in $H$. Then there exists exactly one $v_x \in A$ such that $\|x - v_x\| \leq \|x - v\|$ for $v \in A$

Proof:

Just translate to zero.

Perpendicularity:

Definition: $X \perp Y$ if $(x, y) = 0$

Definition is symmetric.

Proposition:

If $M$ is a closed subspace of $H$ and $x \notin M$, then there is a unique $u \in M$ such that $x - u \perp M$ and $\|x-u\| \leq \|x - v\|$ for $v \in M$.

Remark:

If $A \subseteq H$, then $A^{\perp} = \{x \mid x \perp y$ for all $y \in A\}$.

Proof:

$M$ is convex. Let $u \in M$ with $\|x - u\| \leq \|x - v\|$ for $v \in M$ and $u$ is unique. If not $\perp$, find $y \in M$ with

$$(x-u, y) \neq 0.$$

Make $\|y\| = 1$. Project $x - u$ onto $y$ and look at

$$x - u - \underbrace{(x-u, y) y}_{\in M}$$

$$\| x - u - (x-u, y) y \|^2 = \|x - u\|^2 - (x-u, y)(y, x-u) - \overline{(x-u, y)}\,(x-u, y) + |(x-u, y)|^2$$

$$= \|x-u\|^2 - |(x-u,y)|^2$$
$$< \|x-u\|^2, \text{ contradiction}. \qquad \text{Q.E.D.}$$

$A^\perp$ is a closed linear subspace always

<u>Lemma</u>: 1) If $A \subseteq B$, then $A^\perp \supseteq B^\perp$.

2) $A^{\perp\perp} \supseteq A$

3) $A^{\perp\perp} = A$ if and only if $A$ is a closed subspace.

4) $A^{\perp\perp}$ is the smallest closed subspace containing $A$.

Proof:

3) One direction trivial. Suppose conversely that $A$ is closed and $z \in A^{\perp\perp}$, $z \notin A$. Take $x \in A$ with $z - x \in A^\perp$.

$z \perp (z-x) \perp x$. Hence $(z, z-x) = 0$ and $(x, z-x) = 0$

and $\|z-x\|^2 = 0$. Hence $z = x$.

4) $A \subseteq \text{lin span } A \subseteq \overline{\text{lin span } A} = B$. $A^\perp \supseteq B^\perp$. Then $A^\perp = B^\perp$ and $A^{\perp\perp} = B^{\perp\perp} = B$.

<u>Lemma</u>: Let $M$ and $N$ be closed linear subspaces with $M \perp N$ (or $M \subseteq N^\perp$ or $N \subseteq M^\perp$). Then $M+N$ is a closed linear subspace and $M+N = M \oplus N$.

Proof:

If $x \in M \cap N$, then $x \perp x$ and $x = 0$. Take sequence in $M \oplus N$, $x_\nu + y_\nu \to z$. It is Cauchy.

$\| x_\nu + y_\nu - (x_\mu + y_\mu)\|^2 \to 0$. Reorder and distribute.

$$\|x_\nu + y_\nu - (x_\mu + y_\mu)\|^2 = \|x_\nu - x_\mu\|^2 + \|y_\nu - y_\mu\|^2$$

$$x_\nu \to x_0$$

$$y_\nu \to y_0$$

Hence $x_0 + y_0 \in M+N$ since $M$ and $N$ are each closed. Q.E.D

Lemma: Suppose $x \in H$ and $\hat{x} = y \in H \to (y, x)$. Then $\hat{x} \in H^*$ and $\|\hat{x}\| = \|x\|$

Proof: $\hat{x}$ is linear and is bounded because

$$|\hat{x}\, y| = |(y, x)| \leq \|x\| \|y\|$$

And $\|\hat{x}\| \leq \|x\|$. But

$$\|x\|^2 = (x, x) = |\hat{x}\, x| \leq \|\hat{x}\| \|x\|, \quad \|x\| \leq \|\hat{x}\|.$$

Every linear functional arises in this manner, to be proved.

Theorem (Riesz):

Let $l \in H^*$. Then there exists a unique $x_l \in H$ such that $l(x) = (x, x_l)$ for $x \in H$.

Proof:

Uniqueness Suppose $(x, x_l) = (x, y)$ for all $x$. Then $(x, x_l - y) = 0$.

Hence $x_l - y = 0$.

Existence: If $l = 0$, take $x_l = 0$. If $l \neq 0$, then ker $l$ is a closed maximal proper subspace of $H$. Take vector not in kernel and take perpendicular to it. Make it have norm 1. Thus find $x \in H$ with

$$x \perp \ker l \text{ and } \|x\| = 1.$$

Then $H = \ker l \oplus \mathbb{C} x_l$

Let $x_l = \overline{l(x)}\, x$.

$y \in$ kernel

Take $z \in H$, $z = y + \alpha x$. Then

$$\begin{aligned} l(z) = l(y+\alpha x_l) &= \alpha\, l(x_l) \\ &= \alpha\, \overline{l(x)}\, l(x) \\ &= \alpha\, \overline{l(x)}\, l(x)\, (x,x) \\ &= \alpha\, (\overline{l(x)}\, x, \overline{l(x)}\, x) \\ &= (\alpha x_l, x_l) \\ &= (y + \alpha x_l, x_l) \\ &= (z, x_l) \end{aligned}$$

Q.E.D.

Map $x \to \hat{x}$ is conjugate linear, is onto by Riesz, and is norm preserving. Hence $H$ is homeomorphic to $H^*$.

Weak topologies are also homeomorphic.

$L^* \cong X^*/L^\perp$ is an isometry

$f + L^\perp \xrightarrow{\rho} f|L$, where $f \in X^*$.

Norm preservation:

$$\|f+L^\perp\| = \inf_{g \in L^\perp} \|f+g\|$$

$$(f+g)|L = f|L, \quad \|f+g\| \geq \|(f+g)|L\|_L$$
$$= \|f|L\|_L$$

If $l \in L^*$, by HB there is an $f \in X^*$ such that

$$f|L = l \text{ and } \|f\| = \|l\|$$

Then $f \in \rho^{-1}\{l\}$. Hence equality holds and

$$\|f+L^\perp\| = \|f|L\|_L$$

Theorem: If $M$ is a closed subspace of a Hilbert space $H$, then

$$M \oplus M^\perp = H.$$

Proof:

We know $N = M \oplus M^\perp$ is closed. , $N \subseteq M$, $N \subseteq M^\perp$

$$N^\perp \subseteq \begin{matrix} M^\perp \\ M \end{matrix}$$

$$N^\perp \subseteq M \cap M^\perp = \{0\}$$

So $N = N^{\perp\perp} = H$

Example:

$$l_2 = \{\alpha = (\alpha_1, \alpha_2, \dots) \mid \alpha_\nu \in \mathbb{C}, \ \sum |\alpha_\nu|^2 < \infty\}$$

$$(\alpha, \beta) = \sum \alpha_\nu \overline{\beta_\nu}$$

Definition: A subset $\{e_\alpha \mid \alpha \in A\}$ of $H$ is called orthonormal if and only if $(e_\alpha, e_\beta) = \delta_{\alpha\beta}$. A basis of $H$ is a maximal orthonormal subset.

<u>Theorem</u>: Any orthonormal set is contained in some basis of H.

Proof: By Zorn's Lemma.

<u>Corollary</u>: Every Hilbert space has a basis.

<u>Theorem</u>: Let $\{e_\alpha \mid \alpha \in A\}$ be a basis of H. Then for $x \in H$, at most a countable number of $(x, e_\alpha)$'s are not zero.

Let $(\alpha_1, \alpha_2, \alpha_3, \dots)$ be such that $(x, e_\alpha) = 0$ for $\alpha \notin \{\alpha_1, \alpha_2, \dots\}$

Then $x = \sum_{1 \le \nu} (x, e_{\alpha_\nu}) e_{\alpha_\nu}$ and

$$\|x\|^2 = \sum |(x, e_{\alpha_\nu})|^2, \quad \text{Parseval}$$

Proof:

Let F be finite in A. Then

$$0 \le \Big\| x - \sum_{\alpha \in F} (x, e_\alpha) e_\alpha \Big\|^2$$

$$= (x, x) - \sum \overline{(x, e_\alpha)} (x, e_\alpha) - \sum (x, e_\alpha)(e_\alpha, x) + \sum |(x, e_\alpha)|^2$$

$$= \|x\|^2 - \sum |(x, e_\alpha)|^2$$

Fourier coefficients of $x$

Hence $\sum_F |(x, e_\alpha)|^2 \le \|x\|^2$

This means that at most a countable number of $(x, e_\alpha)$'s are non-zero. Let these be contained in $(x, e_{\alpha_\nu})$ for $1 \le \nu$.

Then $\sum_{1 \le \nu \le m} |(x, e_{\alpha_\nu})|^2 \le \|x\|^2$

Hence $\sum_{1 \le \nu} |(x, e_{\alpha_\nu})|^2 \le \|x\|^2$, Bessel's inequality,

which holds for any sequence of orthonormal elements.

Lemma: If $\{e_\nu \mid 1 \le \nu)$ is orthonormal in $H$ and $\sum |\alpha_\nu|^2 < \infty$, then $\sum \alpha_\nu e_\nu$ exists and $\|\sum \alpha_\nu e_\nu\|^2 = \sum |\alpha_\nu|^2$.

Proof:

Look at $\|\sum_{\nu \le n} \alpha_\nu e_\nu - \sum_{\nu \le n+p} \alpha_\nu e_\nu\|^2$

$$= \|\sum_{n+1 \le \nu \le n+p} \alpha_\nu e_\nu\|^2$$

$$= \sum_{n+1 \le \nu \le n+p} |\alpha_\nu|^2 \longrightarrow 0.$$

Hence $\sum_{\nu \le n} \alpha_\nu e_\nu$ converges, say to $\sum \alpha_\nu x_\nu$. (Completeness)

Now $\|\sum_{1 \le \nu \le n} \alpha_\nu e_\nu\|^2 = \sum_1^n |\alpha_\nu|^2$

By continuity norm of left converges to norm of right.

End of lemma.

So form $x' = \sum_{1 \le \nu} (x, e_{\alpha_\nu}) e_{\alpha_\nu}$ and suppose $x' \ne x$.

$$(x' - x, e_\alpha) = \begin{cases} 0 & \text{for } \alpha \ne \alpha_\nu \\ (x', e_{\alpha_\nu}) - (x, e_{\alpha_\nu}) = 0 & \text{for } \alpha = \alpha_\nu. \end{cases}$$

$(x', e_{\alpha_\nu}) = (x, e_{\alpha_\nu})$ from sum by continuity of inner product

Now $\left\{\frac{x' - x}{\|x' - x\|}\right\} \cup B$ is orthonormal.

By maximality of $B$, $x' = x$.

Parseval follows from relation for $x'$. Q.E.D.

Let $\mathcal{L}H = \{$bounded linear operators of $H$ into $H\}$, a Banach algebra.

In $H$: $\|x\| = \sup_{\|y\|=1} |(x,y)|$ by Riesz theorem together with HB

If $A$ is an operator,

$$\|A\| = \sup_{\|x\|=1=\|y\|} |(Ax,y)| \quad \text{(shown easily)}$$

Theorem: If $A \varepsilon \mathcal{L}H$, there is exactly one $A^* \varepsilon \mathcal{L}H$ and

$$(Ax, y) = (x, A^*y) \quad \text{for } x, y \,\varepsilon\, H.$$

Proof:

For fixed $y$ let $x \varepsilon H \longrightarrow (Ax, y)$.

This is linear and bounded: $|(Ax,y)| \leq \|A\|\|x\|\|y\|$

By Riesz, map is $x \rightarrow (x, A^*y)$, uniquely

For each $y$ we have $(Ax,y) = (x, A^*y)$ for all $x$.

Now

$$
\begin{aligned}
(x, A^*(\alpha y+z)) &= (Ax, \alpha y+z) \\
&= \bar{\alpha}(Ax,y) + (Ax,z) \\
&= \bar{\alpha}(x, A^*y) + (x, A^*z) \\
&= (x, \alpha A^*y) + (x, A^*z) \\
&= (x, \alpha A^*y + A^*z) \quad \text{for all } x
\end{aligned}
$$

So $A^*(\alpha y + z) = \alpha A^*y + A^*z$. Hence linearity.

For boundedness

$$
\begin{aligned}
\|A^*x\| &= \sup_{\|y\|=1} |(A^*x, y)| \quad \text{from above} \\
&= \sup_{\|y\|=1} |(y, A^*y)| \\
&= \sup_{\|y\|=1} |(Ay, x)| \leq \|A\|\|x\|\|y\| = \|A\|\|x\|
\end{aligned}
$$

Thus $\|A^*\| \le \|A\|$

Now $A^{**} = A$ trivially. Hence

$$\|A\| = \|A^*\|.$$ Q.E.D.

$A^*$ is called the adjoint of $A$.

Identities

1) $(\alpha A + \beta B)^* = \bar{\alpha} A^* + \bar{\beta} B^*$

2) $(AB)^* = B^* A^*$

$A$ is called self-adjoint if and only if $A = A^*$

Note: For any $A$, $A = B + iC$ with $B$ and $C$ self-adjoint

$C = \dfrac{A - A^*}{2i}$ and $B = \dfrac{A + A^*}{2}$ are self-adjoint.

$U$ is unitary if $UU^* = 1 = U^*U$

Discussion of adjoint spaces

Consider space of continuous functions on compact space with norm $\sup |f(x)|$.

Problem reduces to discussion of measures.

Valuation - additive set function $\mu$ defined on a kind of class of sets to be specified. Characteristic functions of these sets give positive linear functionals $\chi_A(x) = 1$ or $0$. for simple functions

Extend by if $\underline{\sigma} \leq f \leq \overline{\sigma}$

and if $E(\overline{\sigma} - \underline{\sigma}) < \epsilon$, then $E(f)$ is defined uniquely

We call this $E(f) = \int f(x)\, \mu(dx)$

Given a functional

Take $f$ and let $A = \{x \mid f(x) > 0\}$

$f^{1/n} \to \chi_A$

Also $E(f^{1/n}) \nearrow \chi_A$ and $\chi_A$ will lead to a functional. Then integrate.

Measures lead to functionals.

Let space be locally compact

In a compact space $E(1)$ is finite, so that $0 \leq f \leq \|f\| 1$ and

$E(f) \leq \|f\| E(1)$

If postulate that compact sets have finite measure, take a sequence of sets increasing to the whole space, and problem is easy. We get all $\sigma$-finite measures this way

Another example

Bounded harmonic functions in the disk. NASC is $u(P) = \underset{O}{\text{Average}}\ u$

Under sup norm, they form a Banach space

Completeness: Uniform limit is continuous and

$u = \lim u_n = \lim \text{aver } u_n = \text{av } u$

Uniform limit of bounded harmonic functions is bounded harmonic.

Subspace of bnded cont fcns on space. We do not get an algebra; product of harmonic functions is harmonic. So no $f''^{n}$ and $\chi_A$ does not have to be a monotone limit.

For cont. fcns, $f \vee g$ and $f \wedge g$ are in. (locally max and min).

Cont. fcns with $\vee$ and $\wedge$ form a "Riesz space" (linear lattice).

To every two functions there is a unique "maximum".

For harmonic fcns $f \vee g$ is the unique smallest harmonic fcn greater than $f$ and $g$. Similarly for $f \wedge g$. We get a linear lattice. Ordering

$$f \geq g \text{ if } f \wedge g = g.$$

Measure theory (including passing to limits) can be done from lattice point of view. Get Daniell integral.

Connection in methods indicates that some lattices are isomorphic to continuous functions on a compact space. (Thing done is actually proper measure theory on this unknown compact space).

Poisson integral: positive cont. fcn on boundary $\leftrightarrow$ unique bounded harmonic fcn in disk, for example.

Probability problem is reverse: given harmonic fcns in set and obtain a topological boundary. $f \vee g$ inside corresponds to $\max(f, g)$ on boundary. Problem is obtaining proper boundary.

Denumerable space

Natural topology

Function is a column vector infinite in both directions

Attach to $n$ two positive numbers $p_n + q_n = 1$. Neighbors are $n+1$ and $n-1$.

Average operator is $y_n = p_n x_{n+1} + q_n x_{n-1}$, or $y = Tx$

Look at functions $x$ such that $x = Tx$; these are harmonic functions.

The function $1$ is a solution. There are no more than two solutions since values at two points lead to a recursion equation.

We consider only bounded solutions, which we may assume non-negative.

Random walk

Path space is a function space

We know how to assign probabilities to intervals; products of $p$'s and $q$'s.

One of the problems is to ask for probabilities on a larger space.

Situations

1) $p_n < q_n$ for $n > 0$, drift from + integers toward zero
$p_{-n} > q_{-n}$ also for $n > 0$, drift toward origin. If drift is strong in some sense, there is no boundary and $1$ is only solution

2) drift in one direction

3) drift outward, probabilities associated with going to $+$ and $-\infty$.

Let $i$ be any point $> 0$.
What is probability$_i$ that process never reaches origin. Call it $y_i$, defined for $i > 0$

$$y_i = p_i y_{i+1} + q_i y_{i-1} \quad \text{for } i > 1$$

$$y_1 = p_1 y_2.$$

Is there a non-trivial solution to the system? If there is no solution $y \neq 0$, then there is no drift. If solution exists it is unique by recursion.

$y$'s are monotonically increasing. If non-trivial solution, make limit one provided it is bounded.

$$\frac{y_{i+1}-y_i}{y_i-y_{i-1}} = \frac{q_i}{p_i}$$

By induction

$$y_{i+1}-y_i = y_1\,\frac{q_1}{p_1}\,\frac{q_2}{p_2}\cdots\frac{q_i}{p_i}$$

For boundedness let

$\rho_k = q_k/p_k$, question is whether

$\sum_{k=1}^{\infty} \rho_1\,\rho_2\cdots\rho_k$ converges. NASC

Product with $y_1$ should be 1 if convergence.

Shifting indices gives $p_0$ never reach $-i$. Pass to limit to get $-\infty$ value. Set $r_k = \frac{p_k}{q_k}$ and get symmetric condition

Let $\sigma_k = \rho_k + \rho_k\rho_{k+1} + \rho_k\rho_{k+1}\rho_{k+2} + \cdots$

$S_{k+1} = 1 + r_k + r_{k-1} + r_{k-1}r_{k-2} + \cdots$

If both series converge let

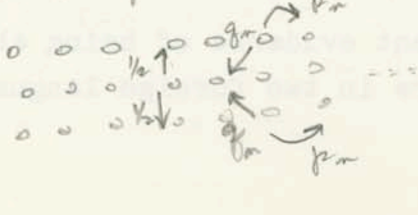
$$y_k = \frac{S_k}{S_k + \sigma_k}$$

Get $S_{k+1} + \sigma_{k+1} = r_k(S_k + \sigma_k)$ and this is a solution.

Probability of drift to right is harmonic, 1 − it is the probability in the other direction

Lattice is linear combinations of $y$ and $1-y$. Attach weights to right and left points. Like Poisson integral; harmonic functions come from boundary values.

Another example: three lines




$a_n, b_n, c_n$

If $\sum q_n < \infty$, then only finitely many steps backward and hence process stays on top or bottom line from some point on. And conversely. Boundary will be one or two points. But this is not a compactification if there is oscillation (then one pt)

Harmonic functions in disk
Defined by integral over largest circle divided by area. Uniform probability in disk. Single out functions which are 1 on one arc, 0 elsewhere; associate arc with function. Build up boundary from it.

Continuous functions on a space
$\cap$ $\cup$, pointwise minimum and maximum
Properties (abstract lattice): I

1) $x \cap x = x$ idempotent
2) $x \cap y = y \cap x$ commutative
3) $x \cap (y \cap z) = (x \cap y) \cap z$ associative — and dually
4) $x \cap (x \cup y) = x$ absorption law

Partial ordering properties II

$x \leq y$ means $x \cap y = x$ and $x \cup y = y$

Riesz space - a linear lattice (over the reals)

III Compatibility conditions

$$x \geq 0, \alpha > 0 \Rightarrow \alpha x \geq 0$$
$$x \geq y \Rightarrow x + z \geq y + z$$

Properties derived

1) $(x+z) \cup (y+z) = (x \cup y) + z$ and $\cap$ for $\alpha > 0$
2) $\alpha x \cup \alpha y = \alpha(x \cup y)$
3) $x \geq y$ implies $-x \leq -y$
4) $(-x) \cup (-y) = -(x \cap y)$

IV. $x+y = x\cup y + x\cap y$

$x\cap(y\cup z) = (x\cap y)\cup(x\cap z)$

$x\cup(y\cap z) = (x\cup y)\cap(x\cup z)$

Proof:

1) $x+(y-x\cap y) \geq x$

$y+(x-x\cap y) \geq y$ and hence the maximum $x\cup y$

$x-(x\cup y - y) \leq \begin{matrix} x \\ y \end{matrix}$ and hence the minimum $x\cap y$.

Hence (1) holds.

2) and (3) are dual

$(x\cap y)\cup(x\cap z) \leq \begin{matrix} x \\ y\cup z \end{matrix}$

Hence $(x\cap y)\cup(x\cap z) \leq x\cap(y\cup z)$

$(x\cap y)\cup(x\cap z) = (x+y-x\cup y)\cup(x+z-x\cup z)$ by (1)

$\geq (x+y-x\cup y\cup z)\cup(x+z-x\cup y\cup z)$

$= y\cup z + x - x\cup y\cup z$

$= x\cap(y\cup z)$ by (1)

Hence (2)

In a Riesz space define $x^+ = x\cup 0$, $x^- = -x\cap 0$

$|x| = x^+ + x^-$

Properties

1) $x = x^+ - x^-$

$x = x+0 = x\cup 0 + x\cap 0 = x^+ - x^-$

2) $x^+\cap x^- = 0$

$x^+\cap x^- = (x+x^-)\cap x^- = (x\cap 0) + x^- = -x^- + x^- = 0$

3) $x\cup x^+ = x^+$

$x\cap x^+ = x$

4) $x\geq 0$ implies $x = x^+ = |x|$, $|x+y| \leq |x|+|y|$

5) $x^+ \cup x = x^+ + x^- = |x|$

Proof of $\triangle$ inequality

$$|x| + |y| \geq (x^+ + y^+) \cup (x^- + y^-) = |x+y|$$

Positive elements form a cone $C$

We have shown $X = C + (-C)$

A Riesz space is a Banach lattice if it is normed and complete and such that

1) if $0 \leq x \leq y$, then $\|x\| \leq \|y\|$

2) $\|(|x|)\| = \|x\|$

Let $X$ be a Riesz space. A linear functional is a real-valued $f$ such that $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$. $f$ is positive if $x \geq 0$ implies $f(x) \geq 0$.

A functional is lattice-bounded if $\sup_{0 \leq y \leq x} |f(y)| < M_x$

Theorem: If $f$ is lattice-bounded, then $f = f^+ - f^-$, where $f^+ \geq 0$, $f^- \geq 0$. Conversely if such a decomposition exists, $f$ is lattice bounded.

Proof: If $f \geq 0$, $f$ is bounded. $y \leq x$ implies $x = y + z$ and $f(x) \geq f(y)$. If decomposition exists, $f$ is lattice bounded. Conversely, define

$$f^+(x) = \sup_{0 \leq y \leq x} f(y)$$ and similarly for $f^-$

$$f^+(x_1 + x_2) = \sup_{0 \leq y \leq x_1 + x_2} f(y) \geq \sup_{\substack{y_1 + y_2 \\ 0 \leq y_1 \leq x_1 \\ 0 \leq y_2 \leq x_2}} f(y_1 + y_2)$$

$$= f^+(x_1) + f^+(x_2)$$

For reverse inequality, for $0 \le y \le x$ define

$$y_1 = y \cap x_1$$
$$y_2 = y - y_1$$

$$y_2 = y - (y \cap x_1) = x_1 \cup y - x_1$$

since $x_1 + y = x_1 \cap y + x_1 \cup y$

$$\le x_1 \cup (x_1 + x_2) - x_1$$
$$= x_2 \quad \text{and } y_2 \le x_2$$

Now $f^+(x_1 + x_2) = \sup\limits_{0 \le y \le x_1 + x_2} (f(y_1) + f(y_2))$

$$\le \sup_{\substack{y_1 \le x_1 \\ y_2 \le x_2}} (f(y_1) + f(y_2)) \text{, etc.}$$

Hence linearity. Etc.

This definition gives the minimal decomposition

Lattice-bounded functionals therefore themselves form a Riesz space, the adjoint space.

Norm on adjoint space

If $x \geq 0$ and $x^* \geq 0$ and $0 \leq y \leq x$, then $x^* y \leq x^* x$ for lattice bounded functionals

Lemma:

If $x_n \to x$ in the norm of the Banach space, then $x_n \cap a \to x \to a$

Corollary:

If $x_n \geq 0$ and $x_n \to x$, then $x \geq 0$.

Proof of lemma:

$|x-y| = x \cup y - x \cap y$ because

$|x-y| = (x-y) \cup 0 - (x-y) \cap 0$

$= (x \cup y) - y - (x \cap y - y)$ by lifting property

Then $|x \cup y - y \cup a| + |x \cap a - y \cap a| = |x-y|$ by distributive law

Hence $|x-y| \geq |x \cap a - y \cap a|$

So $|x_n \cap a - x \cap a| \leq |x_n - x|$

$\|(|x_n \cap a - x \cap a|)\| \leq \|(|x_n - x|)\|$

$= \|x_n - x\|$

$\to 0$

Other norm condition on $x^* = f$

$\|f\| = \|(|f|)\|$

$w = f^+ + f^-$

Want $\|f\| = \|w\|$

$|f(x)| \leq f^+(|x|) + f^-(|x|) = w(|x|) \leq \|w\| \|(|x|)\|$

$= \|w\| \|x\|$

So $\|f\| \leq \|w\|$

Other direction:

Take $\|x\| = 1$ and $x$ such that $w(x) \geq \|w\| - \epsilon$.

Assume $x > 0$ without loss of generality. Then there is a $y$, $0 < y < x$ such that $f(y) > f^+(x) - \epsilon$ by def. of $f^+$

Write $f = f^+ - f^-$, so that

$$f(x-y) \leq -f^-(x) + \epsilon$$

$$f(2y-x) \geq w(x) - 2\epsilon$$

Now $(2y-x)^+ < x$

and $(2y-x)^- < x$

so that $|2y-x| < x$

$$\|(2y-x)\| \leq \|x\| = 1$$

$$|f(2y-x)| \leq \|f\| \|2y-x\| \leq \|f\|$$

Hence $\|f\| \geq w(x) - 2\epsilon \geq \|w\| - 3\epsilon$ Q.E.D.

Continuous functions on a compact space X: want functionals on set

Let $u$ be cont. on X. Want to prove exists measure such that

$$x^*(u) = \int u(p)\mu(dp) \qquad \text{for } x^* \geq 0$$

Will extend class of functions in domain of $x^*$ and show there are many characteristic functions in set. Then we define $\mu(A) = x^*(\chi_A)$

On simple functions $x^*\left(\Sigma c_i \chi_{A_i}\right) = \Sigma c_i\, \mu(A_i)$.

Extension will work for functions $u$ such that

$$\underline{\sigma} < u < \overline{\sigma} \quad \text{arbitrarily closely.}$$

We shall start with a family of functions $\mathcal{B}$ and consider smallest family closed under $+, \times$, limits.

But we may want to enlarge class for certain fixed $x^*$'s so that if $0 \leq f \leq g$ and $x^*(g) = 0$, then $x^* f = 0$

Cantor function has property that any uncountable set is the image of a subset of a set of measure zero.

A family $\mathcal{F}$ of subsets of a set $X$ is an algebra if it is closed under finite unions and complements and if $X \in \mathcal{F}$, $\phi \in \mathcal{F}$.

$\mathcal{F}$ is a monotone class if together with any monotone (ascending) sequence of sets it contains the limit.

If $\mathcal{F}$ is an algebra and $B_n \in \mathcal{F}$ implies $\cup B_n \in \mathcal{F}$, then $\mathcal{F}$ is a $\sigma$-algebra.

$\overline{\lim}\, B_n$ = set of points in $B_j$ infinitely often $= \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} B_k$

If $R_n = B_n \cup B_{n+1} \cup \cdots$, then $R_n \downarrow \overline{\lim}\, B_n$

$\underline{\lim}\, B_n$ $\quad D_n = B_n \cap B_{n+1} \cap \cdots$. Then $\cup D_n \uparrow \underline{\lim}\, B_n$

$B_n \to B$ if $\underline{\lim}\, B_n = \overline{\lim}\, B_n$

If $B$ is a family of functions, $B$ is a monotone class if it is closed under pointwise limits. $B_0$ is the smallest $\sigma$-class containing $B$

$(X, \mathcal{F}_\sigma, \mu)$: For $A \in \mathcal{F}_\sigma$, $\mu(A) \geq 0$. Whenever $A_i \cap A_k = \phi$ for $i \neq k$, $\sum_{i=1}^{\infty} \mu(A_i) = \mu(\bigcup_{i=1}^{\infty} A_i)$.

Example: discrete space and measure

Monotone condition is equivalent to complete additivity

Completion $\overline{\mathcal{F}}_\sigma$. Adjoin all sets $B \Delta \Gamma$, where $\Gamma \subset A$, $A, B \in \mathcal{F}_\sigma$, $\mu(A) = 0$.

The null sets form a $\sigma$-ring. $\overline{\mathcal{F}}_\sigma$ is a $\sigma$-algebra $(X, \overline{\mathcal{F}}_\sigma, \mu)$

Consider the product space $R^{[0,1]}$.

All basic sets containing $0$ are assigned measure 1; others are assigned measure 0.

The $\sigma$-algebra consists of those sets describable by countably many coordinates.

The set $\{f=0\}$ is not in the $\sigma$-algebra, even after completion. It is not a $\mathcal{G}_\delta$ and Baire measures are insufficient.

The measure assigning weight one to $f=0$ and extended is a Borel measure.

For a finite algebra $B_i \in \mathcal{F}$ and $B_i \uparrow B$ if $B \in \mathcal{F}$. Then $\mu(B_i) \to \mu(B)$.
Every measure on an algebra can be extended to the smallest $\sigma$-algebra containing the algebra.

$(X, \mathcal{F}_0, \mu)$. For $A \in \mathcal{F}_\sigma$ set

$$\mu_e(\Omega) = \inf_{A \supset \Omega} \mu(A)$$

$$\mu_i(\Omega) = \sup_{A \subset \Omega} \mu(A)$$

The sets in $\overline{\mathcal{F}}_\sigma$ are those for which $\mu_e = \mu_i$

If $A_i \in \mathcal{F}_\sigma$ define a simple function to be one satisfying $f = \sum_{i=1}^{N} c_i \chi_{A_i}$ and set

$$E(f) = \sum c_i \mu(A_i)$$

Functions which can be approximated by simple functions are those which for all $a$, $\{x \mid f(x) > a\} \in \mathcal{F}_\sigma$.

For $f \geq 0$ let $A_n = \{x \mid (n-1)\epsilon \leq f(x) < n\epsilon\}$ Then $\cup A_n = X$ and $f$ is bounded iff only finitely many $A_n$ are non-empty. Let

$$\overline{\sigma} = \sum n\epsilon \chi_{A_n}$$

$$\underline{\sigma} = \sum (n-1)\epsilon \chi_{A_n}$$

Class of these functions is monotone containing the indicator functions

$$C_n = \{x \mid f_n(x) > a\} \in \mathcal{F}_\sigma$$

$$C_n \to C$$

It is the smallest such class.

Let $\mathcal{L}$ be the set of positive linear functionals on a bounded positive class of functions. $\mathcal{L}$ is closed under $\cap, \cup, +, \times$, and $1 \in \mathcal{L}$. Note that $E \in \mathcal{L}^*$, that $f(1) = 1$, and that if $f_n \downarrow 0$, then $E(f_n) \downarrow 0$.

With the multiplication in $\mathcal{L}$, $f^{1/n}$ is defined or can be approximated if $f \in \mathcal{L}$. Let $L = \{x \mid f(x) > 0$ for some $f \in \mathcal{B}\}$

$f^{1/n} \to$ indicator $\to 1$, so that measure can be defined since the class is monotone.

Consider the class of continuous functions on a compact space $X$.

$E(f^{1/m}) \rightarrow$ measure of $\{x \mid f(x) > 0\}$

But some open sets are not representable in the form $\{x \mid f(x) > 0\}$, as we have seen.

X compact. Additive functional E on cont. fcns, assumed non-negative.
$E(1) = 1$.

Contour sets: $\{x \mid f(x) > 0$, some $f$ with $0 \le f \le 1\}$

Union of denumerably many

$A_n$ defined by $f_n$

$\cup A_n$ defined by $\sum \frac{1}{2^n} f_n$

Every one of them is open, but the class is not closed under non-denum. unions.
In a countable-base space, one can talk just about open sets.

Baire sets: smallest σ-algebra generated by contour sets
Borel sets: -- -- -- open sets

Define non-negative set function

$$\mu(A) = \sup_{\substack{f \text{ carried} \\ \text{by } A,\ 0 \le f \le 1;\ f(x) = 0,\ x \notin A}} E(f)$$

(f vanishes outside A)

If $f > 0$ on A, $\mu(A) = \lim_{n \to \infty} E(f^{1/n})$

Lemma: If $f_n \uparrow \phi$, $g_n \uparrow \phi$, $f_n$ and $g_n$ continuous, then $\lim E(f_n)$ exists and $\lim E(f_n) = \lim E(g_n)$

Remark: This shows the remark above.

Proof: $g_m$ fixed, $f_n \cap g_m \uparrow \phi \cap g_m = g_m$. So automatically uniform convergence. Hence $E(f_n \cap g_m) \to E(g_m)$ and

$$\lim E(f_n) \ge E(g_m) \text{ for every } m.$$

Then $$\lim E(f_n) = \lim E(g_m)$$

Lemma: If $A = \cup A_n$, $A_n$ contour sets, then $\mu(A) \le \sum_1^\infty \mu(A_n)$

Equality holds if sets are disjoint.

Proof:
To $A_n$ fix $f_n \le 1$, $0$ outside $A_n$. Define $f = \Sigma \frac{1}{2^n} f_n$.
If series $\Sigma \mu(A_n)$, we are done. Otherwise
Now $a^{1/n} + b^{1/n} \ge (a+b)^{1/n}$ so that

$$f^{1/n} \le \Sigma \left(\frac{1}{2^n} f_n\right)^{1/n} \longrightarrow \Sigma \mu(A_n)$$

use finite partial sums

For non-overlapping sets, the reverse inequality is easy.

Null set: a set which can be covered by a contour set of measure less than $\epsilon$, for any $\epsilon$.

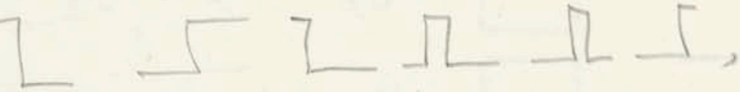
Union of denumerably many null sets is null.

Egoroff's theorem:
Let $f_n$ be a sequence of cont. fcns $\ge 0$ converging to $\phi$ finite. The convergence is uniform except on a set contained in a contour set of measure $< \epsilon$.

Introduce $\|f\|_1 = E(|f|)$ and make continuous functions a normed space. $L^1$ norm. Take Cauchy sequence in $L^1$ norm, call it $f_n$. We want the abstract limit to be interpreted as function. There exists a subsequence $f_{n_k} \to \phi$ converging uniformly except on a contour set of measure less than $\epsilon$. If $\|f_n\|_1 \to 0$, then $\phi \equiv 0$ on that set. (Convergence in measure implies subsequence converging pointwise on set of measure $1-\epsilon$)

Remark: All representatives $\phi$ agree except on null sets.
Identify limit with this class. Conversely, if convergence is as above, $E(\text{limit})$ is uniquely defined.

Example:

 , etc.

Converges in norm, not pointwise

Proof:

Consider $A = \{x \mid f(x) > a\}$; this is a contour set. If associated $g \leq 1$, then $g \leq \frac{1}{a} f$ and hence

$$\mu(A) \leq \frac{1}{a} E(f).$$

Fix an $N$.

Pick $\nu_n$ such that $\|f_n - f_m\| < \frac{1}{N^n}$, $n, m \geq \nu_n$, $\nu_0$ arbitrary

(Makes convergence geometric) Look at

$$f_{\nu_0} + (f_{\nu_1} - f_{\nu_0}) + \dots + (f_{\nu_n} - f_{\nu_{n-1}}) = f_{\nu_n}$$

Take $A_n = \{x \mid |f_{\nu_n}(x) - f_{\nu_{n-1}}(x)| > \frac{1}{M^n}\}$

$$\mu(A_n) \leq M^n E(|f_{\nu_n} - f_{\nu_{n-1}}|)$$ indices off by 1

$$\leq \frac{M^n}{N^{n-1}}$$

Outside $\bigcup_p^\infty A_n$, the series $\sum_1^\infty (f_{\nu_n} - f_{\nu_{n-1}})$ converges uniformly by M-test. Difference bounded by $\Sigma \frac{1}{M^n}$ except for finitely many terms. And this completes the proof; the set $\bigcup_p^\infty A_n$ has measure arbitrarily small. If $\|f_n\| \to 0$, then $\phi \equiv 0$ on the big set.

$\|f\| = E(|f|)$, $f_n$ Cauchy in norm. Then for any $\epsilon$, $\exists$ $A$ with $\mu(A) < \epsilon$ and $f_{n_k} \to f$ uniformly.

Take $\epsilon_n \downarrow 0$. Find $A_n$ and use subsequences and diagonals to get subsequence that converges a.e. pointwise. But uniform convergence is stronger result.

Norm plays only a small role. We need only "convergence in measure". Look at $\{x \mid |f_n(x) - f_m(x)| > \epsilon\}$. If $\mu$ of this $\to 0$, then $f_n$ forms a Cauchy sequence in measure. Integral norm convergence $L^p$ implies this convergence. We have convergence in measure implying subsequence converges a.e. pointwise.

Any two subsequences converge to equivalent $f$'s by corollary.

Similarly if $\{f_n\}$ and $\{g_n\}$ are equivalent.

So every Cauchy sequence has an equivalence class of $f$'s with it.

Example:

Take $[0,1]$. Write $x = .\epsilon_1\epsilon_2 \ldots$ with $\epsilon_k = 0$ or $1$. Have sequence of fcns representing number of zeroes $- n/2$. Call them $f_n$. Look at $f_n/n$. Pick an $N$. For $\epsilon$ small, the set $\left|\frac{f_N}{N}\right| > \epsilon$ is small. Strong law on other hand gives a.e. convergence. From CLT

$$\frac{f_n}{\sqrt{n}\, a_n} \to 0 \text{ in measure if } a_n \to \infty$$

$\Pr\left(\dfrac{f_n}{\sqrt{n}} > \epsilon\right)$ is small. But lim sup $\frac{f_n}{\sqrt{n}}$ pointwise oscillates and convergence is pointwise almost nowhere. $\lim \dfrac{f_n}{\sqrt{n \log\log n}} = \sqrt{2}$ at most pts.

We have a large class of fcns and we can associate $E(f) = \lim E(f_n)$.
If $f_n$ is Cauchy, then $f_n^+$ and $f_n^-$ are Cauchy since

$$|f(x) - g(x)| \geq |f^+(x) - g^+(x)| \quad \text{in all cases}$$

Suppose $f_n \uparrow f$. If $E(f_n)$ is bounded, then $f_n$ is Cauchy. If $f_n \downarrow$, then $E(f_n)$ is Cauchy.

$E(f) \geq E(g)$ if $f \geq g$

Let $f_n$ arbitrary $\geq 0$ converge to $f(x)$ pointwise. Form

$$f_n \cap f_{n+1} \cap f_{n+2} \cap \cdots \cap f_{n+r}$$

As $r \to \infty$, this decreases (and $f \cap g$ is in class when $f$ and $g$ are).
Hence $f_n \cap \cdots$ exists and is in class. Set

$$t_n = f_n \cap \cdots .$$

Then $t_n \uparrow$ and $\lim t_n(x) = \underline{\lim}\, f_n(x) = f(x)$
Now look at $f_n \cup f_{n+1} \cup \cdots = T_n$, which is in class if all $E$'s are bounded. Suppose that $f_n < F$ with $E(F) < \infty$.

Then $T_n$ is in class and

$$E(t_n) \leq E(f_n) \leq E(T_n)$$

$$\lim E(t_n) \leq \underline{\lim}\, E(f_n) \leq \overline{\lim}\, E(f_n) \leq \lim E(T_n) = E(\overline{\lim}\, f_n)$$

$\leq E(\underline{\lim}\, f_n)$

Hence if $f_n \to f$ and $f_n \leq F$, then $E(f_n) \to E(f)$
(Dominated Convergence)

Always $\underline{\lim}\, E(f_n) \geq E(f)$ (Fatou)

$\overline{B}$, the class we have constructed, is a conditionally monotone class (bounded limits are in class)

Egoroff:

$f_n \varepsilon \overline{B}$ and suppose $f_n \to f$, $f$ finite. Throw out a contour set of measure $\epsilon$ and the convergence of a subsequence is uniform. Proof: If $f$ is bnded, $f_n$ is Cauchy and can be replaced by cont. fcns. If $f$ is not bnded, take $f \cap a$. For every $a$ apply result to the truncation

$\overline{B}$ is conditionally monotone, closed under $\cap$ and $\cup$.

Discussion of measurable sets, those such that $\chi_A \varepsilon \overline{B}$:

$$\chi_A \cap \chi_B = \chi_{A \cap B}$$

Limits of functions $\cap$, $\cap$ are in so that family contains contour sets and is closed under countable unions and intersections. Measurable sets form $\sigma$-algebra. Define $\mu(A) = E(\chi_A)$. Every simple function is in, so every uniform limit of them is in.

If $f \varepsilon B$, then $\{x | f(x) > a\}$ is measurable since it is limit of $\{x | f_n(x) > a,\ f_n \text{ cont.}\}$. Take sets where

$$\{x \,|\, (n-1)\epsilon \le f(x) < n\epsilon\} = A_n$$

$$\overline{\sigma} = \sum (n+1)\epsilon \chi_{A_n}$$

$$\underline{\sigma} = \sum n\epsilon \chi_{A_n}$$

And $f$ is limit of these.

$$E(f) = \int_X f(x)\mu(dx)$$

This is the Riesz theorem.

Class of functions is complete.

If $f \geq 0$, $E(f) = 0$, $0 \leq g \leq f$, then $E(g) = 0$.

Restrict and extend $\overline{B}$ to $B_\sigma$, the smallest monotone $\sigma$-class containing cont. functions: These are Baire fcns.

Look at $u(f,g)$, $f, g \in B_\sigma$, $u$ defined in a region of the plane, $u$ taken as a Baire fcns.

<u>Theorem</u>: $u(f,g) \in B_\sigma$. (Expectation need not be finite, of course)

Proof:
Take smallest $\sigma$-algebra $\mathcal{F}$ containing contour sets. $f \in B_\sigma$ iff $\{x \mid f(x) > a\} \in \mathcal{F}$. To prove result suppose $u$ and $f$ are continuous. Take $Z$, family of all $g$ for which this is true. Trivial for cont $g$. Monotone limits are in class. We use sets $U(f(x), g_n(x)) > a$. So statement is true for every Baire function. Fix $u$ cont, $g$ arbitrary, let $f$ be anything. Repeat. Etc for $u$.

Can extend any fcnl $E$ to class $B_\sigma$. Can complete class for any fixed $E$.

For any $A$ with $\chi_A \in \overline{B}$, there is $\Omega$ contour $A \subset \Omega$ with $\mu(\Omega_n) < \mu(A) + \epsilon_n$. Then $\mu(\cap \Omega_n) = \mu(A)$. Every set has a $G_\delta$ bigger than it, or $F_\sigma$ smaller than it, both with same measure.

Start with cont. fcns.

Basic family $\mathcal{B}_\sigma$ = smallest σ-algebra containing cont. fcns.

For $E$, enlarge cont fcns. to $\overline{\mathcal{B}}$ by throwing in Cauchy sequences and limits, $\overline{\mathcal{B}} \supset \mathcal{B}_\sigma$

$\overline{\mathcal{B}}$ is the completion of $\mathcal{B}_\sigma$ with respect to $E$.

In sets

$\Sigma_\sigma$ = smallest σ-algebra of sets containing contour sets

$\overline{\Sigma}$ = sets represented by indicators in $\overline{\mathcal{B}}$.

$\overline{\Sigma} \supset \Sigma_\sigma$

Continuous functions give contour sets

Take monotone limits once up and down to get everything in $\overline{\Sigma}$ (except sets of measure zero)

For every $n$ $\exists$ $\Omega_n \supset A$ such that $\mu(\Omega_n) \le \mu(A) + \epsilon$. If $\Omega_n \downarrow \Omega$, then $\mu(\Omega) = \mu(A)$. To prove first statement we use original theorem.

An exceptional set $E$ for a seq. of cont. fcns can be subtracted so that $f_n \Rightarrow \chi_A$ uniformly outside $E$. So suppose

$$|f(x) - \chi_A(x)| < \epsilon < \tfrac{1}{2} \text{ on } E$$

Take $B = \{x \in E' \mid f(x) > \frac{1}{2}\}$. Is contour set.

Measure $\mu(B) \le \mu(A) + \epsilon$ (whole space has measure one)

because $1 \ge f(x) > 1 - \epsilon$ for $x \in A \cap E'$

$0 \le f(x) < \epsilon$ for $x \in A' \cap E'$

$$E(f) \le 1 \cdot \mu(E) + 1 \cdot \mu(A \cap E') + \epsilon \mu(A' \cap E')$$

$$\le 2\epsilon + \mu(A)$$

$E_1 = \{x \mid \frac{1}{1-\epsilon} f(x) > 1\} \le \mu(A) +$ something. Confusion.

Form $E_1 \cup E$, contour set containing $A$. Measure is as close to $A$ as wanted. So limit of monotone sequence has same measure of $A$.

②

Start with $C$ = cont. fcns.

Form $C_\sigma$ and $C_\delta$, monotone limits up and down.

Note $C_\sigma = C_{\sigma\sigma}$

Form $C_{\sigma\delta}$ and continue.

$C_{\delta\sigma}$

"Young classification". This process does not end but exhausts $B_\sigma$

$C_\sigma$ corresponds to contour sets, $C_\delta$ complements.

To any fcn $f$ in $B_\sigma$, there is one fcn in $C_{\sigma\delta}$ and one in $C_{\delta\sigma}$ such that integrals are same as $f$ and $f$ is squeezed between the two fcns.

Baire classification

$C_1$ cont fcns = first class

$C_2$ = arb. limits = second class

Etc by transfinite induction

One limit is two monotone limits; $C_{\sigma\delta}$ and $C_{\delta\sigma}$ are in $C_2$.

We defined measure for contour sets, null sets, took Cauchy sequences

Bigger class is Borel class; other were Baire sets.

Start with open sets $\Omega$

$\Omega$ contains many contour sets.

Define $\mu(\Omega) = \sup\limits_{\Lambda \subset \Omega} \mu(\Lambda)$. Must be $\geqslant$; no earthly reason to demand equality unless $\Omega$ is a contour set. Equality means regular measure.

We have examples with non-Baire measurable open sets.

Same definition is

$$\mu(\Omega) = \sup_{\substack{0 \le f \le 1, \\ f \text{ carried} \\ \text{by } \Omega}} E(f)$$

Everything is the same except we get more null sets.

There are more Borel functions than Baire functions. Measure agrees on Baire sets.

~~By squeezing before, it follows that now fns are squeezed~~

Borel merely adds null sets.

Baire sets - useful in measure theory

Borel sets - " " topology

Proof of Egoroff:

Given $\sigma$-algebra of functions and measure. Suppose $f_n \to f$. a.e. does not matter. Claim $\exists$ set of measure $< \epsilon$ outside of which convergence is uniform.

Call $A_n(\epsilon) = \{x \mid |f_{n+p}(x) - f_{n+q}(x)| > \epsilon, \text{ some } p \text{ and } q\}$

$A_n(\epsilon)$ measure = union of measurable sets.

Fixed $\epsilon$: $A_n(\epsilon) \downarrow \phi$ by everywhere convergence

Hence $\mu(A_n(\epsilon)) \to 0$ (finiteness used) (wrong more generally)

Take $\epsilon_k \downarrow 0$. Find $n_k$ such that

$$\mu(A_{n_k}(\epsilon_k)) < \frac{\epsilon}{17^k}$$

Set $A = \cup A_{n_k}(\epsilon_k)$

Claim convergence is uniform outside A. For $\epsilon_p$

$$|f_{n+p}(x) - f_{n+q}(x)| < \epsilon_p \quad \text{outside } A, \text{ all } p \text{ and } q$$

for $n$ suff. large. Take $n_p$ as $n$

$$A \supset A_{n_p}(\epsilon_p)$$

Hence uniform Cauchy criterion. Q.E.D.

8

Radon-Nikodym idea

$\mu$ and $\nu$ two finite measures

$\mu-\nu$ is signed measure

Claim $\mu-\nu = \alpha^+ - \alpha^-$, carried by two different sets

If $0 \le \mu \le 1$

$0 \le \nu \le 1$, can decompose $\mu-\nu$. $X = X_1 \cup X_2$

on $X_1$, $\mu \ge \nu$

on $X_2$, $\mu \le \nu$

Do with $X_{11}, X_{12}$ with $\mu - 2\nu$ on one side $\mu - \frac{1}{2}\nu$ on other half

Get all diadic ratios.

A: For each $B \subset A$, $a\,\nu(B) > \mu(B) > b\,\nu(B)$

Make ratio lie between $n\epsilon$ and $(n+1)\epsilon$, Split space into layers

$17\epsilon$ $16\epsilon$ $A_{16}$: $B \subset A_{16}$ implies

0 $\underbrace{16\epsilon}_{\alpha_{16}}\,\nu(B) \le \mu(B) \le \underbrace{17\epsilon}_{\alpha_{17}}\,\nu(B)$

Can approx $\mu$ by $\nu$ on that set

On sets take $\sum \alpha_a X_{A_a}$, integral is bigger.

Squeeze $\mu$ between integrals of two step fcns. Essentially unique. So

$$\int_A \underline{\sigma}(x)\,\nu(dx) \le \mu(A) \le \int_A \overline{\sigma}(x)\,\nu(dx)$$

and $\mu(A) = \int f(x)\,\nu(dx)$ true for all sets that correspond to layers. All that has to be done is look at layers for ratios 0 and $\infty$. 0 gives set of measure zero. Look at top set.

$B \subset A_n$: $\mu(B) \geq n\,\nu(B)$

As $n \to \infty$, either $\mu(A_n) \to 0$, which is fine, or

$\mu(A_n) \to \alpha > 0$ while $A_n \downarrow \Pi$, $\nu(\Pi) = 0$

This gives singular component. We must therefore exclude $A \subset \Pi$. This is precisely absolute cont. condition

Looking at difference ratios is key idea. Problems are in sets of measure zero and in checking the singular part.

Halmos deals with $\frac{\mu}{\mu+\nu}$, rather than $\mu/\nu$, to keep things bounded.

Differentiation problem: $\frac{1}{h}\int_x^{x+h} f(x)\,dx$ generalized

If in a space we can take a diadic grid (Euclidean space, e.g.), if to every $x$ $\exists A_n \downarrow x$, then if $\overline{\lim} \frac{\mu(A_n)}{\nu(A_n)} > a$ for all $x \in B$, then $\mu(B) \geq a\,\nu(B)$. Same for lower limits (cover $B$ by such sets). Take $r_1$ and $r_2$ rational $r_1 > r_2$. If $\overline{\lim} \geq r_1$ and $\underline{\lim} \leq r_2$, then that observation gives contradiction unless both $\mu$ and $\nu$ are zero on set. Take all pairs of rationals, throw out this null set. So except for fixed $E$, $\nu(E) = 0$, then for $x \in E'$, $\lim_{A_n \downarrow x} \frac{\mu(A_n)}{\nu(A_n)}$ converges, say to $f(x)$.

Then $\mu(A) = \int_A f(x)\,\nu(dx)$ with respect to one fixed grid. Two grids give $f$ alike a.e. But want arbitrary open sets converging to work. This is impossibly hard in dim $> 1$.

For disks $\frac{1}{|\Omega_\epsilon|}\iint_{\Omega_\epsilon} f(x,y)\,dx\,dy$ and "figures that remain uniformly thick" (ellipses, squares, nice rectangles) can differentiate. But can't for arbitrary open sets.

Fubini:

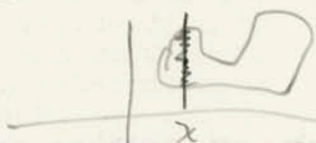
Given X and Y, form (X, Y)

For $\Sigma_X$, $\Sigma_Y$, $\mu$, and $\nu$, form rectangles

$$A \times B, \quad A \in \Sigma_X, \quad B \in \Sigma_Y$$

Set measure $= \mu(A) \times \nu(B)$

Extend to smallest σ-algebra containing rectangles. Set in product is good if for every x, section at x is in $\Sigma_Y$, and symmetrically. Good sets form monotone class, contain σ-algebra. Take




$$\nu(A_x) = \text{fcn of } x$$

This is measurable now.

$$\int_A \nu(A_x)\,\mu(dx) = \text{set fcn over } A = \text{measure}$$

$$= \mu(A)\,\nu(B) \text{ for rectangles}$$

So extends to smallest σ-algebra. If

$$\omega(A) = \int_A \nu(A_x)\,\mu(dx), \text{ then}$$

$$\int f(p)\,\omega(dp) = \ldots \text{ Fubini}$$